Special Year in Number Theory and Combinatorics 2004-05
The University of Florida Mathematics Department
France-Florida Research Institute (FFRI)

Elliptic Functions and Transcendence

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Transcendental numbers form a fascinating subject: so little is known about the nature of analytic constants that more research is needed in this area. Even if one is interested only in numbers related to the classical exponential function, like π and e^{π}, one finds that elliptic functions are required to prove transcendence results and get a better understanding of the situation. We will first review the historical development of the theory, which started in the first part of the 19th century in parallel with the development of the theory related to values of the exponential function. Next we will deal with more recent results. A number of conjectures will be stated that show that we are very far from a satisfactory state of the art.

Exponential Function and Transcendance

Hermite (1873): *the number e is transcendental.*

This means that for any non-zero polynomial $P \in \mathbb{Z}[X]$, the number $P(e)$ is not zero.

We denote by $\overline{\mathbb{Q}}$ the set of algebraic numbers.

Hermite's Theorem is: $e \not\in \overline{\mathbb{Q}}$.

Exponential Function and Transcendance

Hermite (1873): *the number e is transcendental.*

Lindemann (1881): *the number π is transcendental.*

Theorem of Hermite-Lindemann:

For $\alpha \in \overline{\mathbb{Q}}^{\times}$ any non-zero logarithm $\log \alpha$ of α is transcendental.

For any $\beta \in \overline{\mathbb{Q}}^{\times}$, the number e^{β} is transcendental.

The Exponential function

$$\frac{d}{dz}e^z = e^z, \qquad e^{z_1+z_2} = e^{z_1}e^{z_2}$$

$$\begin{array}{rccc} \exp : & \mathbb{C} & \longrightarrow & \mathbb{C}^\times \\ & z & \longmapsto & e^z \end{array}$$

$$\ker \exp = 2i\pi\mathbb{Z}.$$

$z \mapsto e^z$ is the exponential function of the multiplicative group $\mathbb{G}_m$.

The exponential function of the additive group $\mathbb{G}_a$ is

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & z \end{array}$$

Elliptic curves

$$E = \{(t:x:y)\ ;\ y^2t = 4x^3 - g_2xt^2 - g_3t^3\} \subset \mathbb{P}_2.$$

Elliptic functions

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

$$\wp(z_1+z_2) = R\big(\wp(z_1), \wp(z_2)\big)$$

$$\begin{array}{rccc} \exp_E : & \mathbb{C} & \longrightarrow & E(\mathbb{C}) \\ & z & \longmapsto & \big(1, \wp(z), \wp'(z)\big) \end{array}$$

$$\ker \exp_E = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Periods of an elliptic curve

The set of periods is a *lattice*:

$$\Omega = \{\omega \in \mathbb{C}\ ;\ \wp(z+\omega) = \wp(z)\} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

A pair (ω_1, ω_2) of fundamental periods is given by

$$\omega_i = \int_{e_i}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad (i = 1, 2)$$

where

$$4t^3 - g_2 t - g_3 = 4(t-e_1)(t-e_2)(t-e_3).$$

Modular invariant

$$j = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}$$

Set $\tau = \omega_2/\omega_1$, $q = e^{2i\pi\tau}$ and $J(e^{2i\pi\tau}) = j(\tau)$.

Then

$$\begin{aligned} J(q) &= q^{-1}\left(1 + 240\sum_{m=1}^{\infty} m^3 \frac{q^m}{1-q^m}\right)^3 \prod_{n=1}^{\infty}(1-q^n)^{-24} \\ &= \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + \cdots \end{aligned}$$

Complex multiplication

Let E be the elliptic curve attached to the Weierstraß $\wp$ function. The ring of endomorphisms of E is either $\mathbb{Z}$ or else an order in an imaginary quadratic field k. The latter case arises iff the quotient $\tau = \omega_2/\omega_1$ of a pair of fundamental periods is a quadratic number: the curve E has *complex multiplication*. This means also that the two functions $\wp(z)$ and $\wp(\tau z)$ are algebraically independent. In this case the value $j(\tau)$ of the modular invariant j is an algebraic integer of degree the class number h of the quadratic field $k = \mathbb{Q}(\tau)$.

Complex multiplication (continued)

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with class number $h(d) = h$. There are h non-isomorphic elliptic curves $E_1, \ldots, E_h$ with ring of endomorphisms the ring of integers of K. The numbers $j(E_i)$ are conjugate algebraic integers of degree h, each of them generates the Hilbert class field H of K (maximal unramified abelian extension of K). The Galois group of H/K is isomorphic to the ideal class group of the ring of integers of K.

Elliptic analog of Lindemann's Theorem on the transcendence of π.

(*Transcendence of periods of elliptic functions.*)

Theorem (Siegel, 1932): *Assume the invariants* g_2 *and* g_3 *of* $\wp$ *are algebraic. Then one at least of the two numbers* ω_1, ω_2 *is transcendental.*

(Dirichlet's box principle - Thue-Siegel Lemma)

In the case of complex multiplication, it follows that *any non-zero period of* $\wp$ *is transcendental.*

Example 1: $g_2 = 4$, $g_3 = 0$, $j = 1728$

A pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4xt^2.$$

is given by

$$\omega_1 = \int_1^\infty \frac{dt}{\sqrt{t^3 - t}} = \frac{1}{2} B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}}$$

and

$$\omega_2 = i\omega_1.$$

Example 2: $g_2 = 0$, $g_3 = 4$, $j = 0$

A pair of fundamental periods of the elliptic curve

$$y^2t = 4x^3 - 4t^3.$$

is

$$\omega_1 = \int_1^\infty \frac{dt}{\sqrt{t^3-1}} = \frac{1}{3}B(1/6, 1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi}$$

and

$$\omega_2 = \varrho\omega_1$$

where $\varrho = e^{2i\pi/3}$.

Gamma and Beta functions

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t}$$
$$= e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$
$$= \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6\cdot 3\cdot 5\cdot \pi^2}$$

and

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m+n\varrho)^{-6} = \frac{\Gamma(1/3)^{18}}{2^8\pi^6}$$

Formula of Chowla and Selberg (1966): *periods of elliptic curves with complex multiplication as products of Gamma values.*

Consequence of Siegel's 1932 result:
both numbers

$$\Gamma(1/4)^4/\pi \quad \textit{and} \quad \Gamma(1/3)^3/\pi$$

are transcendental.

Ellipse:

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}\,dx$$

Transcendence of the perimeter of the lemniscate

$$(x^2+y^2)^2 = 2a^2(x^2-y^2)$$

Transcendence of values of hypergeometric series related to elliptic integrals.

Gauss hypergeometric series

$$ {}_2F_1\left(a,\ b\ ;\ c \;\middle|\; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} $$

where $(a)_n = a(a+1)\cdots(a+n-1).$

$$ \begin{aligned} K(z) &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}} \\ &= \frac{\pi}{2} \cdot {}_2F_1\left(1/2,\ 1/2\ ;\ 1 \;\middle|\; z^2\right). \end{aligned} $$

Next step:

Schneider (1934): *Each non-zero period* ω *is transcendental also in the non-CM case.*
i.e.: *a non-zero period of an elliptic integral of the first kind is transcendental.*

Elliptic integrals of the second kind
Quasi-periods of an elliptic curve

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The *Weierstraß canonical product* attached to this lattice is the entire function σ_Ω defined by

$$\sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}}.$$

It has a simple zero at any point of Ω.

Canonical products:

for $\mathbb{N} = \{0, 1, 2, \ldots\}$: $e^{-\gamma z}\Gamma(-z)^{-1}$

for $\mathbb{Z}$: $\pi^{-1}\sin(\pi z)$

for $\mathbb{Z} + \mathbb{Z}i$: $\sigma_{\mathbb{Z}[i]}(z)$

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}$$

The logarithmic derivative of the sigma function is *Weierstraß zeta function*

$$\frac{\sigma'}{\sigma} = \zeta$$

and the derivative of ζ is $-\wp$. The sign $-$ arises from the normalization

$$\wp(z) = \frac{1}{z^2} + \text{an analytic function near } 0.$$

The function ζ is therefore *quasiperiodic*: for each $\omega \in \Omega$ there is a $\eta = \eta(\omega)$ such that

$$\zeta(z+\omega) = \zeta(z) + \eta.$$

These numbers η are the *quasiperiods* of the elliptic curve.

When (ω_1, ω_2) is a pair of fundamental periods, set $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$.

Legendre relation:

$$\omega_2\eta_1 - \omega_1\eta_2 = 2i\pi.$$

Examples. For the curve $y^2t = 4x^3 - 4xt^2$ the quasiperiods attached to the above mentioned pair of fundamental periods are

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad \eta_2 = -i\eta_1$$

while for the curve $y^2t = 4x^3 - 4t^3$ they are

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \qquad \eta_2 = \varrho^2\eta_1.$$

Transcendence properties of quasi periods

Pólya, Popken, Mahler (1935)

Schneider (1934): *each of the numbers* $\eta(\omega)$ *with* $\omega \neq 0$ *is transcendental.*

Examples: The numbers

$$\Gamma(1/4)^4/\pi^3 \quad \text{and} \quad \Gamma(1/3)^3/\pi^2$$

are transcendental.

Schneider (1937): *each of the numbers*

$$2i\pi/\omega_1, \quad \eta_1/\omega_1, \quad \alpha\omega_1 + \beta\eta_1$$

is transcendental when α *and* β *are non-zero algebraic numbers.*

Schneider (1948): *for* a *and* b *in* $\mathbb{Q}$ *with* a, b *and* $a+b$ *not in* $\mathbb{Z}$, *the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental.

The proof involves Abelian integrals of higher genus, related with the Jacobian of a Fermat curve.

A. Baker (1969): *transcendence of linear combinations with algebraic coefficients of*

$$\omega_1, \quad \omega_2, \quad \eta_1 \quad \text{and} \quad \eta_2.$$

J. Coates (1971): *transcendence of linear combinations with algebraic coefficients of*

$$\omega_1, \quad \omega_2, \quad \eta_1, \quad \eta_2 \quad \text{and} \quad 2i\pi.$$

Further, *in the non-CM case, the three numbers*

$$\omega_1, \quad \omega_2 \quad \text{and} \quad 2i\pi$$

are $\overline{\mathbb{Q}}$*-linearly independent.*

D.W. Masser (1975) : *the six numbers*

$$\mathbf{1}, \quad \boldsymbol{\omega_1}, \quad \boldsymbol{\omega_2}, \quad \boldsymbol{\eta_1}, \quad \boldsymbol{\eta_2} \quad \text{and} \quad \mathbf{2}\boldsymbol{i\pi}$$

span a $\overline{\mathbb{Q}}$*-vector space of dimension* $\mathbf{6}$ *in the CM case,* $\mathbf{4}$ *in the non-CM case:*

$$\mathbf{dim}_{\overline{\mathbb{Q}}}\{\mathbf{1}, \boldsymbol{\omega_1}, \boldsymbol{\omega_2}, \boldsymbol{\eta_1}, \boldsymbol{\eta_2}, \mathbf{2}\boldsymbol{i\pi}\} = \mathbf{2} + \mathbf{2}\,\mathbf{dim}_{\overline{\mathbb{Q}}}\{\boldsymbol{\omega_1}, \boldsymbol{\omega_2}\}.$$

Also: *measures of linear independence.*

Remark: These statements deal with periods of elliptic integrals of the first or second kind. We shall see further results related with elliptic integrals of the third kind, and also with abelian integrals of any kind.

Elliptic analog of Hermite-Lindemann Theorem

Schneider (1934): *If $\wp$ is a Weierstraß elliptic function with algebraic invariants g_2, g_3 and if β is a non-zero algebraic number, then β is not a pole of $\wp$ and $\wp(\beta)$ is transcendental.*

More generally, *if a and b are two algebraic numbers with $(a, b) \neq (0, 0)$, then for any $u \in \mathbb{C} \setminus \Omega$ one at least of the two numbers*

$$\wp(u), \qquad au + b\zeta(u)$$

is transcendental.

Other results of Schneider 1934

1. *If* $\wp$ *and* $\wp^*$ *are two algebraically independent elliptic functions with algebraic invariants* g_2, g_3, g_2^*, g_3^*, *if* $t \in \mathbb{C}$ *is a pole neither of* $\wp$ *nor of* $\wp^*$, *then one at least of the two numbers* $\wp(t)$ *and* $\wp^*(t)$ *is transcendental.*

2. *If* $\wp$ *is a Weierstraß elliptic functions with algebraic invariants* g_2, g_3, *for any* $t \in \mathbb{C} \setminus \Omega$ *one at least of the two numbers* $\wp(t)$, e^t *is transcendental.*

Corollary: Schneider's Theorem on the transcendence of the modular function

Let $\tau \in \mathcal{H}$ *be a complex number in the upper half plane* $\Im m(\tau) > 0$ *such that* $j(\tau)$ *is algebraic. Then* τ *is algebraic if and only if* τ *is imaginary quadratic (complex multiplication).*

Schneider's second problem:

Prove this result without using elliptic functions.

Sketch of proof of the corollary:

Assume that both $\tau \in \mathcal{H}$ and $j(\tau)$ are algebraic. There exists an elliptic function with algebraic invariants g_2, g_3 and periods ω_1, ω_2 such that

$$\tau = \frac{\omega_2}{\omega_1} \quad \text{and} \quad j(\tau) = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}.$$

Set $\wp^*(z) = \tau^2 \wp(\tau z)$. Then $\wp^*$ is a Weieirstraß function with algebraic invariants g_2^*, g_3^*. For $u = \omega_1/2$ the two numbers $\wp(u)$ and $\wp^*(u)$ are algebraic. Hence the two functions $\wp(z)$ and $\wp^*(z)$ are algebraically dependent. It follows that the corresponding elliptic curve has non trivial endomorphisms, therefore τ is quadratic.

Gel'fond and Schneider, 1934.
Solution of Hilbert's seventh problem
on the transcendence of α^β

For α and β algebraic numbers with $\alpha \neq 0$ and $\beta \notin \mathbb{Q}$ and for any choice of $\log \alpha \neq 0$, the number

$$\alpha^\beta = \exp(\beta \log \alpha)$$

is transcendental.

The two algebraically independent functions e^z and $e^{\beta z}$ cannot take algebraic values at the point $\log \alpha$.

Example: Transcendence of the number

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743.999\,999\,999\,999\,2\ldots.$$

Remark. For

$$\tau = \frac{1+i\sqrt{163}}{2}, \quad q = e^{2i\pi\tau} = -e^{-\pi\sqrt{163}}$$

we have $j(\tau) = -640\,320^3$ and

$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

Equivalent statement to Gel'fond-Schneider Theorem:

Let $\log \alpha_1, \log \alpha_2$ *be two non-zero logarithms of algebraic numbers. Assume that the quotient* $(\log \alpha_1)/(\log \alpha_2)$ *is irrational. Then this quotient is transcendental.*

Baker's Theorem (1966): linear independence of logarithms of algebraic numbers.

Theorem. *Let* $\log \alpha_1, \ldots, \log \alpha_n$ *be* $\mathbb{Q}$*- linearly independent logarithms of algebraic numbers. Then the numbers* $1, \log \alpha_1, \ldots, \log \alpha_n$ *are linearly independent over the field* $\overline{\mathbb{Q}}$*.*

Elliptic analog: Masser (1974) in the CM case.

Bertrand-Masser in general case. New proof of Baker's Theorem using functions of several variables (Cartesian products, due to Schneider (1949), before Bombieri's solution of Nagata's Conjecture in 1970).

Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Let $u_1, \ldots, u_n$ in $\mathbb{C}$ be linearly independent over $\mathrm{End}(E)$. Assume, for $1 \le i \le n$, that either $u_i \in \Omega$ or else $\wp(u_i) \in \overline{\mathbb{Q}}$. Then the numbers $1, u_1, \ldots, u_n$ are linearly independent over the field $\overline{\mathbb{Q}}$.

New proof by Wüstholz (1987) – extends to abelian varieties and integrals. Also covers elliptic (as well as abelian) integrals of the third kind. General linear independence theorem for commutative algebraic groups extending Baker's Theorem.

Further results by Wolfart and Wüstholz on the values on Beta and Gamma functions: linear independence over the field of rational numbers of values of the Beta function at rational points (a, b).

Yields the transcendence of the values at algebraic points of hypergeometric functions with rational parameters.

Elliptic integrals of the third kind

Quasiperiodic relation for Weierstraß sigma function

$$\sigma(z+\omega_i) = -\sigma(z)e^{\eta_i(z+\omega_i/2)} \quad (i=1,2).$$

Hence (J-P. Serre, 1979) the function

$$F_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)}e^{-z\zeta(u)}$$

satisfies

$$F_u(z+\omega_i) = F_u(z)e^{\eta_i u - \omega_i\zeta(u)}.$$

Theorem (1979). *Assume* g_2, g_3, $\wp(u_1)$, $\wp(u_2)$, β *are algebraic and* $\mathbb{Z}u_1 \cap \Omega = \{0\}$. *Then the number*

$$\frac{\sigma(u_1+u_2)}{\sigma(u_1)\sigma(u_2)} e^{\left(\beta-\zeta(u_1)\right)u_2}$$

is transcendental.

Corollary. *Transcendence of periods of elliptic integrals of the third kind:*

$$e^{\omega\zeta(u)-\eta u+\beta\omega}.$$

Four exponentials Conjecture

and six exponentials Theorem

Ramanujan: highly composite numbers. *Let* t *be a real number such that* 2^t *and* 3^t *are integers. Does it follow that* t *is a positive integer?*

Alaoglu and Erdös.

Siegel, Selberg, Lang, Ramachandra:

Theorem: *If the three numbers* 2^t, 3^t *and* 5^t *are integers, then* t *is a rational number (hence a positive integer).*

Set $2^t = a$ and $3^t = b$. Then the determinant

$$\begin{vmatrix} \log 2 & \log 3 \\ \log a & \log b \end{vmatrix}$$

vanishes.

Four exponentials Conjecture. *Let*

$$\begin{pmatrix} \log \alpha_1 & \log \alpha_2 \\ \log \beta_1 & \log \beta_2 \end{pmatrix}$$

be a 2×2 *matrix whose entries are logarithms of algebraic numbers. Assume the two columns are* $\mathbb{Q}$ *-linearly independent and the two rows are also* $\mathbb{Q}$ *-linearly independent. Then the matrix is regular.*

Six exponentials Theorem

Theorem (Siegel, Lang, Ramachandra). *Let*

$$\begin{pmatrix} \log \alpha_1 & \log \alpha_2 & \log \alpha_3 \\ \log \beta_1 & \log \beta_2 & \log \beta_3 \end{pmatrix}$$

be a 2 *by* 3 *matrix whose entries are logarithms of algebraic numbers. Assume the three columns are linearly independent over* $\mathbb{Q}$ *and the two rows are also linearly independent over* $\mathbb{Q}$*. Then the matrix has rank* 2*.*

Main Conjecture for usual logarithms of algebraic numbers:

$\mathbb{Q}$ -linearly independent logarithms of algebraic numbers are algebraically independent.

Elliptic analog of the Main Conjecture:

Let $u_1, \ldots, u_n$ be complex numbers which are linearly independent over the field of endomorphisms of E. For $1 \leq i \leq n$ assume that either u_i is a pole of $\wp$ or else $\wp(u_i)$ is algebraic. Then $u_1, \ldots, u_n$ are algebraically independent.

Further conjectures: A. Grothendieck, Y. André, C. Bertolin.

Ramachandra (1968): elliptic analogs of the six exponentials theorem

Let $\boldsymbol{E}$ *be an elliptic curve with complex multiplication. Let*

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

be a 2×3 *matrix whose entries are elliptic logarithms of algebraic numbers:* $\wp(u_i)$ *and* $\wp(v_i)$ *are algebraic. Assume the three columns are linearly independent over* $\mathrm{End}(\boldsymbol{E})$ *and the two rows are also linearly independent over* $\mathrm{End}(\boldsymbol{E})$*. Then the matrix has rank* 2*.*

Chudnovskii (1978)

Theorem 1. *Two at least of the numbers*

$$g_2,\ g_3,\ \omega_1,\ \omega_2,\ \eta_1,\ \eta_2$$

are algebraically independent.

Theorem 2. *Assume* g_2 *and* g_3 *are algebraic. Let* ω *be a non-zero period of* $\wp$, *set* $\eta = \eta(\omega)$, *and let* $u \in \mathbb{C} \setminus \{\mathbb{Q}\omega \cup \Omega\}$ *be such that* $\wp(u) \in \overline{\mathbb{Q}}$. *Then*

$$\zeta(u) - \frac{\eta}{\omega}u, \quad \frac{\eta}{\omega}$$

are algebraically independent.

Chudnovskii (1978 continued)

Corollary: *Let* ω *be a non zero period of* $\wp$ *and* $\eta = \eta(\omega)$*. If* g_2 *and* g_3 *are algebraic then the two numbers* π/ω *and* η/ω *are algebraically independent.*

Corollary: *Assume* g_2 *and* g_3 *are algebraic, and the elliptic curve has complex multiplication. Then the two numbers* ω_1*,* π *are algebraically independent.*

Corollary: π *and* $\Gamma(1/4)$ *are algebraically independent. Also* π *and* $\Gamma(1/3)$ *are algebraically independent.*

Conjecture of Lang (1971). *If $j(\tau)$ is algebraic with $j'(\tau) \neq 0$, then $j'(\tau)$ is transcendental.*

Amounts to the transcendence of ω^2/π since

$$j'(\tau) = 18\frac{\omega_1^2}{2i\pi}\cdot\frac{g_2}{g_3}j(\tau).$$

True in CM case:

Corollary. *If $\tau \in \mathcal{H}$ is quadratic and $j'(\tau) \neq 0$, then π and $j'(\tau)$ are algebraic independent.*

Chudnovskii's method yields:

Theorem (K.G. Vasil'ev, P. Grinspan). *Two at least of the three numbers* π, $\Gamma(1/5)$ *and* $\Gamma(2/5)$ *are algebraically independent.*

The proof involves the Jacobian of the Fermat curve

$$X^5 + Y^5 = Z^5$$

which is an Abelian variety of dimension 2.

Philippon, Wüstholz (1982): elliptic analog of Lindemann Weierstraß Theorem on the algebraic independence of $e^{\alpha_1}, \ldots, e^{\alpha_n}$:

Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3 and complex multiplication. Let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers which are linearly independent over the field of endomorphisms of E. Then the numbers $\wp(\alpha_1), \ldots, \wp(\alpha_n)$ are algebraically independent.

Open in the non-CM case - partial results towards a proof of:

? At least $n/2$ of these numbers are algebraically independent.

Mahler-Manin problem on $J(q)$.

$$J(e^{2i\pi\tau}) = j(\tau)$$

$$J(q) = \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + \cdots$$

Theorem (K. Barré, G. Diaz, F. Gramain, G. Philibert, 1996**).** *Let $q \in \mathbb{C}$, $0 < |q| < 1$. If q is algebraic, then $J(q)$ is transcendental.*

First transcendence proof using modular functions.

p-*adic elliptic functions*

D. Bertrand (1977) algebraic values of p-adic elliptic functions: linear independence of elliptic logarithms in the CM case.
Non vanishing of the height on elliptic curve.

Consequence of the solution of Manin's problem: Greenberg, zeroes of p-adic L functions.
Application to the solution of the main Conjecture for Selmer group of the square symetric of an elliptic curve with multiplicative reduction at p by Hida, Tilouine and Urban.

Ramanujan Functions

$$
\begin{aligned}
P(q) &= 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \\
Q(q) &= 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \\
R(q) &= 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}.
\end{aligned}
$$

Eisenstein Series

Bernoulli numbers :

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{z^{2k}}{(2k)!},$$

$$B_1 = 1/6, \quad B_2 = 1/30 \quad B_3 = 1/42.$$

$$E_{2k}(z) = 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{\infty} \frac{n^{2k-1} z^n}{1-z^n},$$

$$P(z) = E_2(z), \quad Q(z) = E_4(z), \quad R(z) = E_6(z).$$

Connection with the modular invariant J :

$$\Delta = 12^{-3}(Q^3 - R^2) = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

$$J = Q^3/\Delta.$$

Special values

$$\tau = i, \quad q = e^{-2\pi}, \quad \omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\ldots$$

$$P(e^{-2\pi}) = \frac{3}{\pi}, \quad Q(e^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4,$$

$$R(e^{-2\pi}) = 0, \quad \Delta(e^{-2\pi}) = \frac{1}{2^6}\left(\frac{\omega_1}{\pi}\right)^{12}.$$

$$\tau = \varrho, \quad q = -e^{-\pi\sqrt{3}}, \quad \omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots$$

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi}, \quad Q(-e^{-\pi\sqrt{3}}) = 0,$$

$$R(-e^{-\pi\sqrt{3}}) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6, \quad \Delta(-e^{-\pi\sqrt{3}}) = -\frac{27}{256}\left(\frac{\omega_1}{\pi}\right)^{12}.$$

Corollary of the transcendence of $J(q)$:

Let $\log\alpha$ be a logarithm of a non-zero algebraic number. Let $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ be a lattice with algebraic invariants g_2, g_3. Then the determinant

$$\begin{vmatrix} 2i\pi & \log\alpha \\ \omega_1 & \omega_2 \end{vmatrix}$$

does not vanish.

This is a mixed analog of the four exponentials conjecture.

Four exponentials conjecture for the product of an elliptic curve by the multiplicative group

Conjecture. *Let* $\wp$ *be a Weierstraß elliptic function with algebraic invariants* g_2, g_3. *Let* u_1 *and* u_2 *be complex numbers such that for* $i = 1$ *and* $i = 2$, *either* $u_i \in \Omega$ *or else* $\wp(u_i) \in \overline{\mathbb{Q}}$. *Let* $\log \alpha_1$ *and* $\log \alpha_2$ *be two logarithms of algebraic numbers. Assume further that the two rows of the matrix*

$$\begin{pmatrix} u_1 & \log \alpha_1 \\ u_2 & \log \alpha_2 \end{pmatrix}$$

are linearly independent over $\mathbb{Q}$. *Then the determinant of* M *does not vanish.*

Open Problems (G. Diaz)

1. *For any* $z \in \mathbb{C}$ *with* $|z| = 1$ *and* $z \neq \pm 1$, *the number* $e^{2i\pi z}$ *is transcendental.*

2. *If* q *is an algebraic number with* $0 < |q| < 1$ *such that* $J(q) \in [0, 1728]$, *then* $q \in \mathbb{R}$.

3. *The function* J *is injective on the set of algebraic numbers* α *with* $0 < |\alpha| < 1$.

Remark (G. Diaz). The third conjecture implies the two first ones, and follows from the four exponentials Conjecture. Also follows from the next Conjecture of D. Bertrand.

Conjecture (D. Bertrand). – *If* α_1 *and* α_2 *are two multiplicatively independent algebraic numbers in the domain*

$$\{z \in \mathbb{C}\,;\, 0 < |z| < 1\},$$

then the two numbers $J(\alpha_1)$ *and* $J(\alpha_2)$ *are algebraically independent.*

Implies the special case of the four exponentials Conjecture where two of the algebraic numbers are roots of unity and the two others have modulus $\neq 1$.

Further analog of the four exponentials Conjecture

Question of Yu. V. Manin:
Let $\log \alpha_1$ *and* $\log \alpha_2$ *be two non-zero logarithms of algebraic numbers and let* $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *be a lattice with algebraic invariants* g_2 *and* g_3*. Then is-it true that*

$$\frac{\omega_1}{\omega_2} \neq \frac{\log \alpha_1}{\log \alpha_2}?$$

Analog of Schneider's second problem: *Prove the transcendence of* $J(q)$ *by means of elliptic functions.*

Bertrand's remark:

Chudnovskii's 1978 result

Two at least of the numbers $g_2,\ g_3,\ \omega/\pi,\ \eta/\pi$ *are algebraically independent*

can be rephrased:

For any $q \in \mathbb{C}$ *with* $0 < |q| < 1$, *two at least of the numbers* $P(q),\ Q(q),\ R(q)$ *are algebraically independent.*

Theorem (Nesterenko, 1996). *For any* $q \in \mathbb{C}$ *with* $0 < |q| < 1$*, three at least of the four numbers*

$$q, \quad P(q), \quad Q(q), \quad R(q)$$

are algebraically independent.

Tools: The functions P, Q, R are algebraicaly independent over $\mathbb{C}(q)$ (K. Mahler) and satisfy a system of differential equations for $D = q\, d/dq$:

$$12\frac{DP}{P} = P - \frac{Q}{P}, \qquad 3\frac{DQ}{Q} = P - \frac{R}{Q}, \qquad 2\frac{DR}{R} = P - \frac{Q^2}{R}.$$

Corollary. *The three numbers*

$$\pi, \quad e^{\pi}, \quad \Gamma(1/4)$$

are algebraically independent.

Corollary. *The three numbers*

$$\pi, \quad e^{\pi\sqrt{3}}, \quad \Gamma(1/3)$$

are algebraically independent.

The number

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}$$

is transcendental.

(P. Bundschuh): *the number*

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}}$$

is transcendental.

$$
\begin{aligned}
\theta_2(q) &= 2q^{1/4}\sum_{n\geq 0} q^{n(n+1)}\\
&= 2q^{1/4}\prod_{n=1}^{\infty}(1-q^{4n})(1+q^{2n}),
\end{aligned}
$$

$$
\begin{aligned}
\theta_3(q) &= \sum_{n\in\mathbb{Z}} q^{n^2}\\
&= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2,
\end{aligned}
$$

$$
\begin{aligned}
\theta_4(q) &= \theta_3(-q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\\
&= \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2.
\end{aligned}
$$

Corollary (values of Jacobi theta series) *Let* i, j *and* $k \in \{2,3,4\}$ *with* $i \neq j$. *Let* $q \in \mathbb{C}$ *satisfy* $0 < |q| < 1$. *Then each of the fields*

$$\mathbb{Q}\big(q, \theta_i(q), \theta_j(q), D\theta_k(q)\big)$$

and

$$\mathbb{Q}\big(q, \theta_k(q), D\theta_k(q), D^2\theta_k(q)\big)$$

has transcendence degree ≥ 3 *over* $\mathbb{Q}$.

`Example.` *For algebraic* $q \in \mathbb{C}$ *with* $0 < |q| < 1$, *the number*

$$\theta_3(q) = \sum_{n \in \mathbb{Z}} q^{n^2}$$

is transcendental.

Corollary. *Rogers-Ramanujan continued fraction:*

$$RR(\alpha) = 1 + \cfrac{\alpha}{1 + \cfrac{\alpha^2}{1 + \cfrac{\alpha^3}{1 + \ddots}}}$$

is transcendental for any algebraic α *with* $0 < |\alpha| < 1$.

Corollary. *Let* $(F_n)_{n\geq 0}$ *be the Fibonacci sequence:*

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}.$$

Then the number

$$\sum_{n=1}^{\infty} \frac{1}{F_n^2}$$

is transcendental.

Further open problems

Algebraic independence of the three numbers

$$\pi, \quad \Gamma(1/3), \quad \Gamma(1/4).$$

Algebraic independence of at least three numbers among

$$\pi, \quad \Gamma(1/5), \quad \Gamma(2/5), \quad e^{\pi\sqrt{5}}.$$

Standard relations among Gamma values

(Translation): $\Gamma(a+1) = a\Gamma(a)$

(Reflexion): $\Gamma(a)\Gamma(1-a) = \dfrac{\pi}{\sin(\pi a)}$

(Multiplication): For any non-negative integer n,

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

Conjecture (D. Rohrlich) *Any multiplicative relation*

$$\pi^{b/2} \prod_{a\in\mathbb{Q}} \Gamma(a)^{m_a} \in \overline{\mathbb{Q}}$$

with b *and* m_a *in* $\mathbb{Z}$ *is a consequence of* (in the ideal generated by) *the standard relations.*

Conjecture (S. Lang) *Any algebraic dependence relation among* $(2\pi)^{-1/2}\Gamma(a)$ *with* $a \in \mathbb{Q}$ *is a consequence of* (in the ideal generated by) *the standard relations* (universal odd distribution).

Diophantine approximation

Transcendence measures for $\Gamma(1/4)$

(P. Philippon, S. Bruiltet)

For $P \in \mathbb{Z}[X, Y]$ with degree d and height H,

$$\log|P(\pi, \Gamma(1/4)| > 10^{326}\Big((\log H + d\log(d+1)\Big) \cdot d^2\Big(\log(d+1)\Big)^2$$

Corollary. $\Gamma(1/4)$ *is not a Liouville number:*

$$\left|\Gamma(1/4) - \frac{p}{q}\right| > \frac{1}{q^{10^{330}}}.$$

Lower bounds for linear combinations of elliptic logarithms: Baker, Coates, Anderson . . . in the CM case, Philippon-Waldschmidt in the general case, refinements by N. Hirata Kohno, S. David, É. Gaudron - use Arakhelov's Theory (J-B. Bost: *slopes inequalities*).
Motivation: method of S. Lang for solving Diophantine equations (integer points on elliptic curves).

Isogeny Theorem: effective results (D.W. Masser and G.Wüstholz).

Mazur's density conjecture : *density of rational points on varieties.*

Conclusion

The proof of the algebraic independence of π and e^{π} requires elliptic and modular functions. Higher dimensional objects (Abelian varieties, motives) should be involved now.

1976 Chudnovskii - algebraic independence of π and $\Gamma(1/4)$

1996 Nesterenko- algebraic independence of π, e^{π} and $\Gamma(1/4)$

?? Algebraic independence of e, π, e^{π} and $\Gamma(1/4)$?