December 12, 2024

SRM University AP **University Distinguished Lecture (UDL)**

Diophantine equations from Brahmagupta to Ramanujan and later

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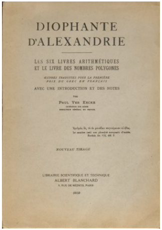
Abstract

A question, studied by the Greek mathematician Diophantus of Alexandria, is related with the so-called "house number problem", which was proposed to Srinivasa Ramanujan by Prasanta Chandra Mahalanobis in 1914.
The puzzle involves a street in the town of Louvain, Belgium, where the houses are supposedly numbered consecutively. Among these house numbers, one had the remarkable property that the sum of all the house numbers below it was exactly equal to the sum of all the house numbers above it.
Additionally, the mysterious house number was known to be greater than 50 but less than 500.

Ramanujan provided an immediate solution. His answer used early works of Indian mathematicians including Brahmagupta and Bhaskara II. The names of Pell and Fermat were later given by Euler to the relevant equations.

These questions also connect to the ISO 216 international standard for paper sizes, such as the A4 format. We will further quote open problems related to this topic.

Diophantus of Alexandria ($\sim 250 \pm 50$)



Diophantine equations
Diophantine approximation
Diophantine problems
Diophantine tuples

Main Theorem of Diophantine approximation : if a is a nonzero integer, then $|a| \geqslant 1$.

https://mathshistory.st-andrews.ac.uk/Biographies/Diophantus/

Rational Diophantine tuples

If x and y are two distinct elements among

$$\frac{1}{16}, \quad \frac{33}{16}, \quad \frac{17}{4}, \quad \frac{105}{16},$$

then $xy+1$ is a square.

$$\frac{1}{16}\cdot\frac{33}{16}+1=\left(\frac{17}{16}\right)^2, \quad \frac{1}{16}\cdot\frac{17}{4}+1=\left(\frac{9}{8}\right)^2,$$
$$\frac{1}{16}\cdot\frac{105}{16}+1=\left(\frac{19}{16}\right)^2, \quad \frac{33}{16}\cdot\frac{17}{4}+1=\left(\frac{25}{8}\right)^2,$$
$$\frac{33}{16}\cdot\frac{105}{16}+1=\left(\frac{61}{16}\right)^2, \quad \frac{17}{4}\cdot\frac{105}{16}+1=\left(\frac{43}{8}\right)^2.$$

Method of Diophantus of Alexandria

Four unknowns, six equations. The equation

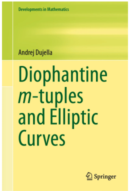
$$y^2 = 9x^2 + 24x + 13$$

has the solution

$$x = \frac{1}{16}, \quad y = \frac{61}{16}.$$

.

$$61^2 = 3\,721 = 9 + 24 \cdot 16 + 13 \cdot 16^2.$$



Andrej Dujella

Developments in Mathematics (DEVM, volume **79**) Springer, June 2024
DOI : https://doi.org/10.1007/978-3-031-56724-7

A Diophantine quadruple

Pierre de Fermat

1601–1665

Diophantine quadruple :

$(1, 3, 8, 120)$

$xy + 1$ is a square

$$1 \cdot 3 + 1 = 4 = 2^2, \quad 1 \cdot 8 + 1 = 9 = 3^2, \quad 1 \cdot 120 + 1 = 121 = 11^2,$$
$$3 \cdot 8 + 1 = 25 = 5^2, \quad 3 \cdot 120 + 1 = 361 = 19^2,$$
$$8 \cdot 120 + 1 = 961 = 31^2.$$

Rational Diophantine quintuple

Leonard Euler

1707–1783

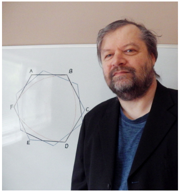
$$\left(1, 3, 8, 120, \frac{777480}{8288641}\right)$$

$$\frac{777480}{8288641} + 1 = \left(\frac{3011}{2879}\right)^2, \quad 3 \cdot \frac{777480}{8288641} + 1 = \left(\frac{3259}{2879}\right)^2,$$

$$8 \cdot \frac{777480}{8288641} + 1 = \left(\frac{3809}{2879}\right)^2,$$

$$120 \cdot \frac{777480}{8288641} + 1 = \left(\frac{10\,079}{2879}\right)^2.$$

Rational Diophantine sextuples



Philip Gibbs

Seven examples in
P.E. Gibbs,
Some Rational Diophantine Sextuples, (1999).
arxiv: math.NT/9902081
including

$$\frac{11}{192}, \quad \frac{35}{192}, \quad \frac{155}{27}, \quad \frac{512}{27}, \quad \frac{1235}{48}, \quad \frac{180\,873}{16}.$$

Five more examples in
P.E. Gibbs,
A generalised Stern-Brocot tree from regular Diophantine quadruples, (1999). arXiv: math.NT/9903035.

Diophantine quintuples

Bo He

Alain Togbé

Volker Ziegler

Bo He, Alain Togbé and Volker Ziegler. *There is no Diophantine quintuple.* Trans. Amer. Math. Soc. **371**, 6665–6709 (2019).

Diophantus of Alexandria : another problem

Find an integer n such that $10n+9$ and $5n+4$ are squares.

Solutions
$n=0$, $10n+9=9=3^2$, $5n+4=4=2^2$.

$n=28$, $10n+9=289=17^2$, $5n+4=144=12^2$.

$n=33\,292$,
$10n+9=332\,929=577^2$, $5n+4=166\,464=408^2$.

Next ones : $n=1\,130\,976$, $n=13\,051\,463\,040$.

Problem by Diophantus of Alexandria

$10n + 9$ and $5n + 4$ are squares :

$$x^2 = 10n + 9, \quad y^2 = 5n + 4$$

$$x^2 - 2y^2 = 1.$$

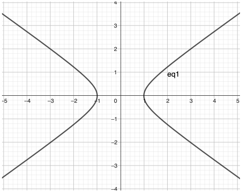
$n = 0$, $(x, y) = (3, 2)$, $x^2 = 9$, $y^2 = 4$.

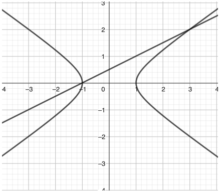
$n = 28$, $(x, y) = (17, 12)$, $x^2 = 289$, $y^2 = 144$, $2y^2 = 288$.

$n = 33\,292$, $(x, y) = (577, 408)$,
$$x^2 = 332\,929, \quad y^2 = 166\,464, \quad 2y^2 = 332\,928.$$

For $x = 99$, $y = 70$ we have $x^2 - 2y^2 = 9801 - 9800 = 1$ but this is not a solution to Diophantus problem : $x^2 = 980 \cdot 10 + 1$, $y^2 = 980 \cdot 5$.

Rational solutions to $x^2 - 2y^2 = 1$





$$y = t(x+1), t \in \mathbf{Q}$$

$$2t^2(x+1)^2 = x^2 - 1,$$
$$2t^2(x+1) = x - 1,$$

$$x = \frac{1+2t^2}{1-2t^2}, \quad y = \frac{2t}{1-2t^2}.$$

$$t = \frac{1}{2}, \quad (x,y) = (3,2).$$
$$t = \frac{2}{3}, \quad 1 - 2t^2 = \frac{1}{9}, \quad 1 + 2t^2 = \frac{17}{9}, \quad (x,y) = (17,12).$$

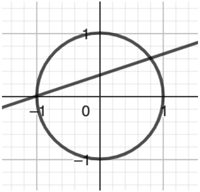
Pythagorean triples

Euclid

∼325 BC – ∼ 265 BC

Parametrisation of the circle : $x^2 + y^2 = 1$ rational points on the circle $y = t(x+1)$:

$$x = \frac{t^2-1}{t^2+1}, \quad y = \frac{2t}{t^2+1}.$$



Pythagoras equation $a^2 + b^2 = c^2$ (ref. : Hardy and Wright)
$m > n > 0$,

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.$$

Integer solutions to $x^2 - 2y^2 = 1$

$$x^2 - 2y^2 = 1.$$

Pell–Fermat equation

$$(x, y) = (3, 2)$$

$$(3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2}, \quad 17^2 - 2 \cdot 12^2 = 289 - 288 = 1.$$

$$(3 + 2\sqrt{2})^3 = 99 + 70\sqrt{2}, \quad 99^2 - 2 \cdot 70^2 = 9801 - 2 \cdot 4000 = 1.$$

$$(3+2\sqrt{2})^4 = 577+408\sqrt{2}, \quad 577^2 - 2 \cdot 408^2 = 232\,929 - 2 \cdot 166\,464 = 1.$$

An interesting street number

The puzzle itself was about a street in the town of Louvain in Belgium, where houses are numbered consecutively. One of the house numbers had the peculiar property that the total of the numbers lower than it was exactly equal to the total of the numbers above it. Furthermore, the mysterious house number was greater than 50 but less than 500.

Prasanta Chandra Mahalanobis

1893 – 1972

Srinivasa Ramanujan

1887 – 1920

http://mathshistory.st-andrews.ac.uk/Biographies/Mahalanobis.html

https://www.math.auckland.ac.nz/~butcher/miniature/miniature2.pdf

Street number : examples

Examples :

- House number 6 in a street with 8 houses :

$$1+2+3+4+5=15, \quad 7+8=15.$$

- House number 35 in a street with 49 houses. To compute

$$S := 1+2+3+\cdots+32+33+34$$

write

$$S = 34+33+32+\cdots+3+2+1$$

so that $2S = 34 \times 35$:

$$1+2+3+\cdots+34 = \frac{34 \times 35}{2} = 595.$$

On the other side of the house,

$$36+37+\cdots+49 = \frac{49 \times 50}{2} - \frac{35 \times 36}{2} = 1225 - 630 = 595.$$

Other solutions to the puzzle

- House number 1 in a street with 1 house.
- House number 0 in a street with 0 house.

Ramanujan : *if no banana is distributed to no student, will each student get a banana ?*

The puzzle requests the house number between 50 and 500.

Street number

Let m be the house number and n the number of houses :

$$1 + 2 + 3 + \cdots + (m-1) = (m+1) + (m+2) + \cdots + n.$$

$$\frac{m(m-1)}{2} = \frac{n(n+1)}{2} - \frac{m(m+1)}{2}.$$

This is $2m^2 = n(n+1)$. Complete the square on the right :

$$8m^2 = (2n+1)^2 - 1.$$

Set $x = 2n+1$, $y = 2m$. Then

$$x^2 - 2y^2 = 1.$$

Infinitely many solutions to the puzzle

Ramanujan said he has infinitely many solutions (but a single one between 50 and 500).
Sequence of balancing numbers (number of the house)
https://oeis.org/A001109

$$0, 1, 6, 35, \mathbf{204}, 1189, 6930, 40391, 235416, 1372105, 7997214 \ldots$$

This is a linear recurrence sequence $u_{n+1} = 6u_n - u_{n-1}$ with the initial conditions $u_0 = 0$, $u_1 = 1$.

The number of houses is https://oeis.org/A001108

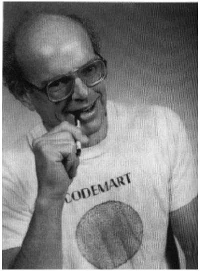
$$0, 1, 8, 49, \mathbf{288}, 1681, 9800, 57121, 332928, 1940449, \ldots$$

OEIS

The OEIS Foundation is supported by donations from users of the OEIS and by a grant from the Simons Foundation.

THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane



Neil J. A. Sloane's encyclopaedia

http://oeis.org/A001597

Brahmagupta (598 – 670)

Brāhmasphuṭasiddhānta (628) :

$$x^2 - 92y^2 = 1$$

Brahmagupta

The smallest solution is

$$x = 1151, \qquad y = 120.$$

Composition method : *samasa* – Brahmagupta identity

$$(a^2 - db^2)(x^2 - dy^2) = (ax + dby)^2 - d(ay + bx)^2.$$

http://mathworld.wolfram.com/BrahmaguptasProblem.html

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pell.html

Bhāskara II or Bhāskarāchārya (1114 - 1185)

Lilavati Ujjain (India)

(*Bijagaṇita*, 1150)

$$x^2 - 61y^2 = 1$$

$$x = 1\,766\,319\,049, \qquad y = 226\,153\,980.$$

Cyclic method (*Chakravala*) : produce a solution to Pell's equation $x^2 - dy^2 = 1$ starting from a solution to $a^2 - db^2 = k$ with a *small* k.

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pell.html

Narayaṇa Paṇḍit ∼ 1340 – ∼ 1400

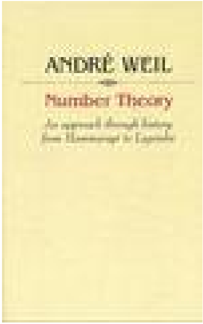
Author of *Gaṇita Kaumudi* on arithmetic in 1356.
Narayaṇa cows (*Tom Johnson*)

$$x^2 - 103y^2 = 1$$

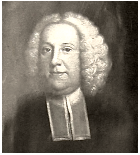
$$x = 227\,528, \qquad y = 22\,419.$$

Reference to Indian mathematics

André Weil
Number theory :
An approach through history. From Hammurapi to Legendre.
Birkhäuser Boston, Inc., Boston, Mass., (1984) 375 pp.
MR 85c:01004



Pell's equation $x^2 - dy^2 = \pm 1$



John Pell

1610 – 1685

It is often said that Euler mistakenly attributed Brouncker's work on this equation to Pell. However the equation appears in a book by Rahn which was certainly written with Pell's help : some say entirely written by Pell. Perhaps Euler knew what he was doing in naming the equation.

Johann Rahn (1622 - 1676) was a Swiss mathematician who was the first to use the symbol $\div$ for division.

https://mathshistory.st-andrews.ac.uk/Biographies/Pell/
https://fr.wikipedia.org/wiki/John_Pell

On the equation $x^2 - dy^2 = \pm 1$: history

Lord William Brouncker

1620–1684

Pierre de Fermat

1601–1665

Correspondence from Pierre de Fermat to Brouncker.

1657 : letter of Fermat to Frenicle de Bessy (1604–1674).

https://mathshistory.st-andrews.ac.uk/Biographies/

Correspondence from Fermat to Lord Brouncker

"*pour ne vous donner pas trop de peine*" (Fermat)
" *to make it not too difficult*"

$$x^2 - dy^2 = 1, \text{ with } d = 61 \text{ and } d = 109.$$

Solutions respectively :

$$(1\,766\,319\,049\ ,\ 226\,153\,980)$$
$$(158\,070\,671\,986\,249,\ 15\,140\,424\,455\,100)$$

$$158\,070\,671\,986\,249 \ + \ 15\,140\,424\,455\,100\sqrt{109} = \left(\frac{261 + 25\sqrt{109}}{2}\right)^6.$$

History (continued)

Leonard Euler
1707–1783

Joseph–Louis Lagrange
1736–1813

L. Euler : *Book of algebra in 1770 + continued fractions*

The complete theory of the equation $x^2 - dy^2 = \pm 1$ was worked out by Lagrange.

https://mathshistory.st-andrews.ac.uk/Biographies/

Solution of the equation $x^2 - dy^2 = \pm 1$

Let d be a positive integer, not a square. Then the equation $x^2 - dy^2 = \pm 1$ has infinitely many non negative solutions in integers (x, y).

There is a smallest positive *fundamental solution* (x_1, y_1) such that all non negative solutions are obtained by writing

$$x_\nu + y_\nu\sqrt{d} = \ (x_1 + y_1\sqrt{d})^\nu$$

with $\nu \geqslant 0$.

The trivial solution $(x, y) = (1, 0)$ is obtained with $\nu = 0$.

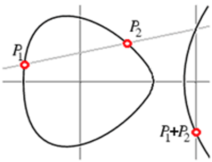
The set of solutions (x, y) in $\mathbf{Z} \times \mathbf{Z}$ is given by

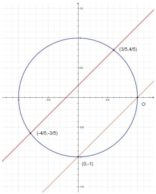
$$x_\nu + y_\nu\sqrt{d} = \pm(x_1 + y_1\sqrt{d})^\nu$$

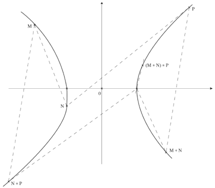
with $\nu \in \mathbf{Z}$. They form a group $\simeq \{\pm 1\} \times \mathbf{Z}$.

Group law on a conic

The curve $x^2 - dy^2 = 1$ is a conic, and on a conic there is a group law which can be described geometrically. The fact that it is associative is proved by using Pascal's Theorem.







Franz Lemmermeyer. Conics – a poor man's elliptic curves.
https://arxiv.org/pdf/math/0311306.pdf

Mahalanobis puzzle $x^2 - 2y^2 = 1$, $x = 2n+1,\ y = 2m$

Fundamental solution : $(x_1, y_1) = (3, 2)$.
Other solutions (x_ν, y_ν) with

$$x_\nu + y_\nu\sqrt{2} = (3 + 2\sqrt{2})^\nu.$$

- $\nu = 0$, trivial solution : $x = 1$, $y = 0$, $m = n = 0$.
- $\nu = 1$, $x_1 = 3$, $y_1 = 2$, $m = n = 1$.
- $\nu = 2$, $x_2 = 17$, $y_2 = 12$, $n = 8$, $m = 6$,

$$x_2 + y_2\sqrt{2} = (3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2}.$$

- $\nu = 3$, $x_3 = 99$, $y_3 = 70$, $n = 49$, $m = 35$,

$$x_3 + y_3\sqrt{2} = (3 + 2\sqrt{2})^3 = 99 + 70\sqrt{2}.$$

Diophantus problem

Find an integer n such that $10n+9$ and $5n+4$ are squares :

$$x^2 = 10n+9, \quad y^2 = 5n+4$$

$$x^2 - 2y^2 = 1$$

http://oeis.org/A001333

$$1,\ 3,\ 17,\ 99,\ 577,\ 3\,363,\ 19\,601,\ 114\,243,\ 665\,857,\ 3\,880\,899,\ \dots$$

$$x = 3, \quad 17, \quad 577, \quad 3\,363, \quad 114\,243, \dots$$

$$n = \frac{x^2-9}{10} = 0, \quad 28, \quad 33\,292, \quad 1\,130,976, \quad 1\,305\,146\,304\dots$$

Diophantine approximation

$3\,363^2 - 2 \cdot 2\,378^2 = 1$

$$\frac{3\,363}{2\,378} = \sqrt{2 + \frac{1}{2\,378^2}}.$$

$$\frac{3\,363}{2\,378} = \mathbf{1.414\,213}\,62\ldots, \qquad \sqrt{2} = \mathbf{1.414\,213}\,56\ldots$$

Diophantine approximation

For each $\epsilon > 0$, there exists $q_0 > 0$ such that, for $q \geqslant q_0$,

(*) $$\left|\sqrt{2} - \frac{p}{q}\right| > \frac{1-\epsilon}{2\sqrt{2}q^2}$$

and there exists infinitely many $p/q \in \mathbf{Q}$ such that

(**) $$\left|\sqrt{2} - \frac{p}{q}\right| < \frac{1+\epsilon}{2\sqrt{2}q^2}.$$

Sketch of proof. For q sufficiently large, let p be the nearest integer to $q\sqrt{2}$, so that

$$q\sqrt{2} - \frac{1}{2} < p < q\sqrt{2} + \frac{1}{2}.$$

Hence $p \simeq q\sqrt{2}$. From $|p^2 - 2q^2| \geqslant 1$ we deduce

$$|p - \sqrt{2}q| \geq \frac{1}{|p + \sqrt{2}q|},$$

which gives (*), while (**) follows when $p^2 - 2q^2 = \pm 1$.

Tablet YBC 7289 : 1800 – 1600 BC

Babylonian clay tablet, accurate sexagesimal approximation to $\sqrt{2}$ to the equivalent of six decimal digits.

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.\mathbf{414\,212}\,962\,962\,962\ldots$$

$$\sqrt{2} = 1.\mathbf{414\,213}\,562\,373\,095\,048\ldots$$

https://en.wikipedia.org/wiki/YBC_7289

A4 format 21×29.7

ISO 216 International standard https://en.wikipedia.org/wiki/ISO_216

$$\frac{297}{210} = \frac{99}{70} = 1.\mathbf{414\,28}5\,714\,285\,714\,285\ldots$$

$$\sqrt{2} = 1.\mathbf{414\,21}3\,562\,373\,095\,048\ldots$$

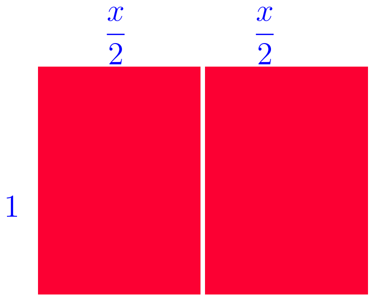
210

297

A, B, C formats

Large rectangle : sides x, 1 ; proportion $\frac{x}{1} = x$

Small rectangles : sides 1, $\frac{x}{2}$; proportion $\frac{1}{x/2} = \frac{2}{x}$

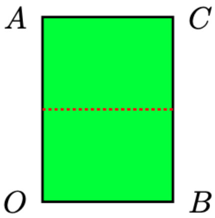


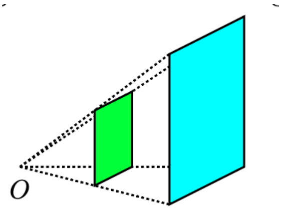
$$x = \frac{2}{x}, \qquad x^2 = 2.$$

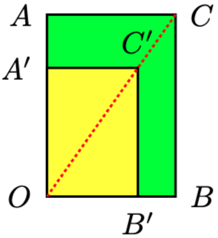
https://en.wikipedia.org/wiki/Paper_size

Rectangle format $\sqrt{2}$

The large rectangle and half of it are proportional.



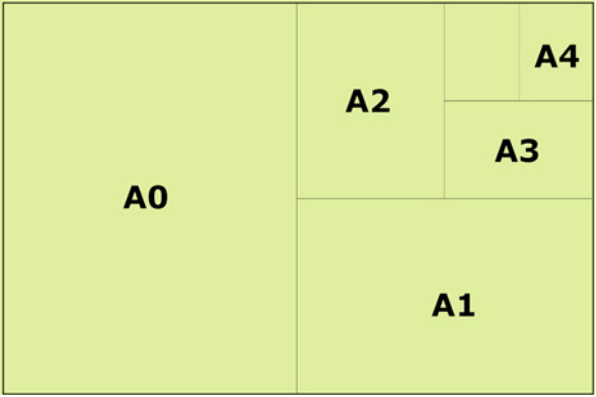




Reference : Paul Gérardin
http://www.lpsm.paris/pageperso/SemaineCollege-Avril08/web/doc/Coll08A4.pdf

A format

The number $\sqrt{2}$ is twice its inverse : $\sqrt{2} = 2/\sqrt{2}$.
Folding a rectangular piece of paper with sides in proportion $\sqrt{2}$ yields a new rectangular piece of paper with sides in proportion $\sqrt{2}$ again.



A0 is 118.8cm $\times 84$cm - area 1 m^2.

B and C formats

https://papersizes.io/

B0 is 1m $\times 1.414$m.
B7 (passeport) is 88mm $\times 125$mm.

C0 is 917mm $\times 1297$mm, approximately $\dfrac{1}{\sqrt[8]{2}} \times \sqrt[8]{8}$.

C6 : 114mm $\times 162$mm

enveloppe for a A6 paper 105mm $\times 148$mm

Xerox machine : enlarging and reducing

141%	119%	84%	71%
1.41	1.19	0.84	0.71
1.4142	1.1892	0.8409	0.7071
$\sqrt{2}$	$\sqrt[4]{2}$	$1/\sqrt[4]{2}$	$1/\sqrt{2}$

Paper format A0, A1, A2,... in cm

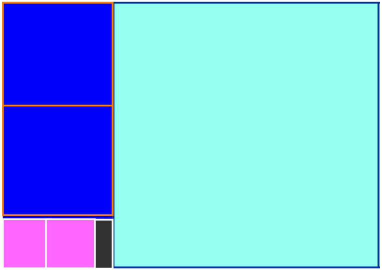
$$x_1 = 100\sqrt[4]{2} = 118.8, \qquad x_2 = \frac{100}{\sqrt[4]{2}} = 84.$$

$$\begin{array}{llll}
A0: & x_1 = 118.8 & x_2 = 84 \\
A1: & x_2 = 84 & \frac{x_1}{2} = 59.4 \\
A2: & \frac{x_1}{2} = 59.4 & \frac{x_2}{2} = 42 \\
A3: & \frac{x_2}{2} = 42 & \frac{x_1}{4} = 29.7 \\
A4: & \frac{x_1}{4} = 29.7 & \frac{x_2}{4} = 21 \\
A5: & \frac{x_2}{4} = 21 & \frac{x_1}{8} = 14.85
\end{array}$$

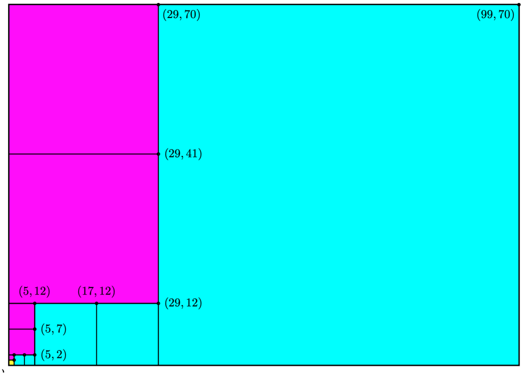
Rectangle with proportion $\sqrt{2}$

One square plus 2 rectangles with proportion $1+\sqrt{2}$:

$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}}, \qquad 1+\sqrt{2} = 2 + \frac{1}{1+\sqrt{2}}.$$



Irrationality of $\sqrt{2}$: geometric proof



$$\frac{99}{70} = 1 + \frac{29}{70},$$
$$\frac{70}{29} = 2 + \frac{12}{29},$$
$$\frac{29}{12} = 2 + \frac{5}{12},$$
$$\frac{12}{5} = 2 + \frac{2}{5},$$
$$\frac{5}{2} = 2 + \frac{1}{2}.$$

$$\frac{297}{210} = \frac{99}{70}.$$

Continued fraction of $\sqrt{2}$

The number

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,20\ \ldots$$

satisfies

$$\boxed{\sqrt{2}} = 1 + \cfrac{1}{1 + \boxed{\sqrt{2}}}.$$

Hence

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \boxed{\sqrt{2}}}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1,\ 2,\ 2,\ 2,\ 2,\ 2,\ \ldots] = [1,\ \overline{2}].$$

A4 format

$$\frac{297}{210} = 1 + \frac{29}{70},$$
$$\frac{70}{29} = 2 + \frac{12}{29},$$
$$\frac{29}{12} = 2 + \frac{5}{12},$$
$$\frac{12}{5} = 2 + \frac{2}{5},$$
$$\frac{5}{2} = 2 + \frac{1}{2}.$$

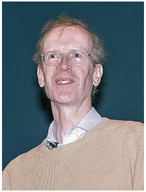
Hence

$$\frac{297}{210} = [1, 2, 2, 2, 2, 2].$$

Pierre de Fermat

Pierre de Fermat

1601–1665



Andrew Wiles

Proof of Fermat's last Theorem by Andrew Wiles (1993) : for $n \geqslant 3$, there is no positive integer solution (a, b, c) to

$$a^n + b^n = c^n.$$

Ramanujan – Nagell Equation

Srinivasa Ramanujan
1887 – 1920

Trygve Nagell
1895 – 1988

Ramanujan – Nagell Equation

$$x^2 + 7 = 2^n$$

$$\begin{aligned}
1^2 + 7 &= 2^3 &= 8 \\
3^2 + 7 &= 2^4 &= 16 \\
5^2 + 7 &= 2^5 &= 32 \\
11^2 + 7 &= 2^7 &= 128 \\
181^2 + 7 &= 2^{15} &= 32\,768
\end{aligned}$$

$x^2 + D = 2^n$

Nagell (1948) : for $D = 7$, no further solution

Apéry (1960) : for $D > 0$, $D \neq 7$, the equation $x^2 + D = 2^n$ has at most 2 solutions.

Roger Apéry
1916 – 1994

Examples with 2 solutions :

$$D = 23 : \qquad 3^2 + 23 = 32, \quad 45^2 + 23 = 2^{11} = 2\,048$$

$$D = 2^{\ell+1} - 1,\ \ell \geqslant 3 : \qquad (2^\ell - 1)^2 + 2^{\ell+1} - 1 = 2^{2\ell}$$

$x^2 + D = 2^n$

Beukers (1980) : at most one solution otherwise.

Frits Beukers

Mike Bennett

M. Bennett (1995) : considers the case $D < 0$.

Diophantine equations : early historical survey

Pierre Fermat (1601 ? – 1665)

Leonhard Euler (1707 – 1783)

Joseph Louis Lagrange (1736 – 1813)

XIXth Century : Adolf Hurwitz, Henri Poincaré





Hilbert's 8th Problem

David Hilbert
1862 – 1943

Second International Congress of Mathematicians in Paris. August 8, 1900

Twin primes,

Goldbach's Conjecture,

Riemann Hypothesis

http://www.maa.org/sites/default/files/pdf/upload$_-$library/22/Ford/Thiele1-24.pdf

Hilbert's tenth problem

D. Hilbert (1900) — *Problem :* to give an algorithm in order to decide whether a diophantine equation has an integer solution or not.

If we do not succeed in solving a mathematical problem, the reason frequently consists in our failure to recognize the more general standpoint from which the problem before us appears only as a single link in a chain of related problems. After finding this standpoint, not only is this problem frequently more accessible to our investigation, but at the same time we come into possession of a method which is applicable also to related problems.

Negative solution to Hilbert's 10th problem

Julia Robinson (1952)

Julia Robinson, Martin Davis, Hilary Putnam (1961)

Yuri Matijasevic (1970)

Remark : the analog for *rational points* of Hilbert's 10th problem is not yet solved :
Does there exist an algorithm in order to decide whether a Diophantine equation has a rational solution or not ?

Diophantine equations : historical survey

Thue (1908) : there are only finitely many integer solutions of

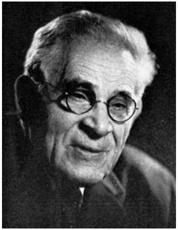
$$F(x, y) = m,$$

when F is homogeneous irreducible form over $\mathbf{Q}$ of degree $\geqslant 3$.

Mordell's Conjecture (1922) : rational points on algebraic curves

Siegel's Theorem (1929) : integral points on algebraic curves

Axel Thue
1863 - 1922

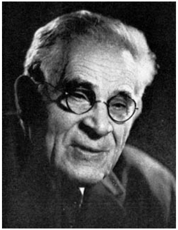


Louis Mordell
1888 – 1972

Carl Ludwig Siegel
1896 - 1981

Mordell's Conjecture, Faltings's Theorem

Mordell's Conjecture : 1922. Faltings's Theorem (1983). The set of rational points on a number field of a curve of genus $\geqslant 2$ is finite.



Louis Mordell
1888 – 1972

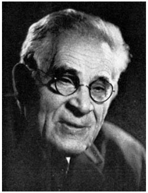
Gerd Faltings

The group of rational points on an elliptic curve

Conjecture (Henri Poincaré, 1901) : finitely many points are sufficient to deduce all rational points by the chord and tangent method.

Henri Poincaré

1854 – 1912



Louis Mordell

1888 – 1972

Theorem (Mordell, 1922). *If E is an elliptic curve over $\mathbf{Q}$, then the abelian group $E(\mathbf{Q})$ is finitely generated : there exists a nonnegative integer r (the Mordell-Weil* rank *of the curve over $\mathbf{Q}$) such that*

$$E(\mathbf{Q}) = E(\mathbf{Q})_{\text{tors}} \times \mathbf{Z}^r$$

Mordell–Weil Theorem

André Weil (1928) : generalization to number fields and abelian varieties :

If A is an Abelian variety over a number field K, then the abelian group $A(K)$ is finitely generated :

$$A(K) = A(K)_{\mathrm{tors}} \times \mathbf{Z}^r$$

with $r \geqslant 0$ while $A(K)_{\mathrm{tors}}$ is a finite group.

Jacques Hadamard

1865 - 1963

André Weil

1906 – 1998

Weil's thesis : 1928. Hadamard's comment.

Reference : ANTOINE CHAMBERT-LOIR. *La conjecture de Mordell : origines, approches, généralisations.* Séminaire Betty B., Septembre 2021 5e année, 2021–2022

A one million US$ open problem

Conjecture of Birch and Swinnerton-Dyer

Henry Peter Francis Swinnerton-Dyer (1927-2018)
and Bryan John Birch

B. Birch and H.P.F. Swinnerton-Dyer. *Notes on Elliptic Curves. II.* J. reine angew. Math. **218**, 79–108 (1965).

Clay Mathematics Institute. *The Birch and Swinnerton-Dyer Conjecture.* http://www.claymath.org/millennium/Birch_and_Swinnerton-Dyer_Conjecture/

December 12, 2024

SRM University AP **University Distinguished Lecture (UDL)**

Diophantine equations from Brahmagupta to Ramanujan and later

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