UNIVERSITY OF NATURAL SCIENCES HO CHI MINH City
http://www.hcmuns.edu.vn/ September 12 – October 4, 2007

Introduction to Diophantine methods: irrationality and transcendence

Michel Waldschmidt,
Professeur, Université P. et M. Curie (Paris VI)
http://www.math.jussieu.fr/∼miw/coursHCMUNS2007.html

Diophantine approximation is a chapter in number theory which has witnessed outstanding progress together with a number of deep applications during the recent years. The proofs have long been considered as technically difficult. However, we understand better now the underlying ideas, hence it becomes possible to introduce the basic methods and the fundamental tools in a more clear way.

We start with irrationality proofs. Historically, the first ones concerned irrational algebraic numbers, like the square roots of non square positive integers. Next, the theory of continued fraction expansion provided a very useful tool. Among the first proofs of irrationality for numbers which are now known to be transcendental are the ones by H. Lambert and L. Euler, in the XVIIIth century, for the numbers e and π. Later, in 1815, J. Fourier gave a simple proof for the irrationality of e.

We first give this proof by Fourier and explain how J. Liouville extended it in 1840 (four years before his outstanding achievement, where he produced the first examples of transcendental numbers). Such arguments are very nice but quite limited, as we shall see. Next we explain how C. Hermite was able in 1873 to go much further by proving the transcendence of the number e. We introduce these new ideas of Hermite in several steps: first we prove the irrationality of e^r for rational $r \neq 0$ as well as the irrationality of π. Next we relate these simple proofs with Hermite's integral formula, following C.L. Siegel (1929 and 1949). Hermite's arguments led to the theory of Padé Approximants. They also enable Lindemann to settle the problem of the quadrature of the circle in 1882, by proving the transcendence of π.

One of the next important steps in transcendental number theory came with the solution by A.O. Gel'fond and Th. Schneider of the seventh of the 23 problems raised by D. Hilbert at the International Congress of Mathematicians in Paris in 1900: *for algebraic α and β with $\alpha \neq 0$, $\alpha \neq 1$ and β irrational, the number α^β is transcendental.* An example is $2^{\sqrt{2}}$, another less obvious example is e^π. The proofs of Gel'fond and Schneider came after the study, by G. Pólya, in 1914, of integer valued entire functions, using interpolation formulae going back to Hermite. We introduce these formulae as well as some variants for meromorphic functions due to R. Lagrange (1935) and recently rehabilitated by T. Rivoal (2006) [10].

The end of the course will be devoted to a survey of the most recent irrationality and transcendence results, including results of algebraic independence. We shall also introduce the main conjectures on this topic.

First course: september 12, 2007. [1]

We denote by $\mathbb{Z}$ the ring of rational integers, by $\mathbb{Q}$ the field of rational numbers, by $\mathbb{R}$ the field of real numbers and by $\mathbb{C}$ the field of complex numbers. Given a real number, we want to know whether it is rational or not, that means whether he belongs to $\mathbb{Q}$ or not. The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ has no nice algebraic properties: it is not stable by addition nor by multiplication.

Irrationality is the first step, the second one is transcendence. Given a complex number, one wants to know whether it is algebraic of not. The set of algebraic numbers, which is the set of roots of all non-zero polynomials with rational coefficients, is nothing else than the algebraic closure of $\mathbb{Q}$ into $\mathbb{C}$. We denote it by $\overline{\mathbb{Q}}$. The set of transcendental numbers is defined as $\mathbb{C} \setminus \overline{\mathbb{Q}}$. Since $\overline{\mathbb{Q}}$ is a field, the set of transcendental numbers is not stable by addition nor by multiplication.

1 Irrationality

1.1 Simple proofs of irrationality

The early history of irrationality goes back to the Greek mathematicians Hippasus of Metapontum (around 500 BC) and Theodorus of Cyrene, Eudoxus, Euclid. There are different early references in the Indian civilisation and the Sulba Sutras (around 800-500 BC).

Let us start with the irrationality of the number

$$\sqrt{2} = 1,414\,213\,562\,373\,095\,048\,801\,688\,724\,209\ \dots$$

One of the most well known proofs is to argue by contradiction as follows: assume $\sqrt{2}$ is rational and write it as a/b where a and b are relatively prime positive rational integers. Then $a^2 = 2b^2$. It follows that a is even. Write $a = 2a'$. From $2{a'}^2 = b^2$ one deduces that b also is even, contradicting the assumption that a and b were relatively prime.

There are variants of this proof - a number of them are in the nice booklet [9]. For instance using the relation

$$\sqrt{2} = \frac{2-\sqrt{2}}{\sqrt{2}-1}$$

[1]Updated: October 12, 2007

with $\sqrt{2} = a/b$ one deduces

$$\sqrt{2} = \frac{2b-a}{a-b}.$$

Now we have $1 < \sqrt{2} < 2$, hence $0 < a-b < b$, which shows that the denominator b of fraction $\sqrt{2} = a/b$ was not minimal.

This argument can be converted into a geometric proof: starting with an isosceles rectangle triangle with sides b and hypothenuse a, one constructs (using ruler and compass if one wishes) another similar triangle with smaller sides $a-b$ and hypothenuse $2b-a$. Such a proof of irrationality is reminiscent of the ancient Greek geometers constructions, and also of the infinite descent of Fermat.

A related but different geometric argument is to start with a rectangle having sides 1 and $1+\sqrt{2}$. We split it into two unit squares and a smaller rectangle. The length of this second rectangle is 1, its width is $\sqrt{2}-1$, hence its proportion is

$$\frac{1}{\sqrt{2}-1} = 1+\sqrt{2}.$$

Therefore the first and second rectangles have the same proportion. Now if we repeat the process and split the small rectangle into two squares (of sides $\sqrt{2}-1$) and a third tiny rectangle, the proportions of this third rectangle will again be $1+\sqrt{2}$. This means that the process will not end, each time we shall get two squares and a remaining smaller rectangle having the same proportion.

On the other hand if we start with a rectangle having integer side lengths, if we split it into several squares and if a small rectangle remains, then clearly the small rectangle while have integer side lengths. Therefore the process will not continue forever, it will stop when there is no remaining small rectangle. This proves again the irrationality of $\sqrt{2}$.

In algebraic terms the number $x = 1+\sqrt{2}$ satisfies

$$x = 2 + \frac{1}{x},$$

hence also

$$x = 2 + \cfrac{1}{2+\cfrac{1}{x}} = 2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{x}}} = \cdots,$$

which yields the *continued fraction expansion* of $1+\sqrt{2}$. Here is the definition of the continued fraction expansion of a real number.

Given a real number x, the Euclidean division in $\mathbb{R}$ of x by 1 yields a quotient $[x] \in \mathbb{Z}$ (the *integral part of* x) and a remainder $\{x\}$ in the interval $[0,1)$ (the *fractional part of* x) satisfying

$$x = [x] + \{x\}.$$

Set $a_0 = [x]$. Hence $a_0 \in \mathbb{Z}$. If x is an integer then $x = [x] = a_0$ and $\{x\} = 0$. In this case we just write $x = a_0$ with $a_0 \in \mathbb{Z}$. Otherwise we have $\{x\} > 0$ and we set $x_1 = 1/\{x\}$ and $a_1 = [x_1]$. Since $\{x\} < 1$ we have $x_1 > 1$ and $a_1 \geq 1$. Also

$$x = a_0 + \frac{1}{a_1 + \{x_1\}}.$$

Again, we consider two cases: if $x_1 \in \mathbb{Z}$ then $\{x_1\} = 0$, $x_1 = a_1$ and

$$x = a_0 + \frac{1}{a_1}$$

with two integers a_0 and a_1, with $a_1 \geq 2$ (recall $x_1 > 1$). Otherwise we can define $x_2 = 1/\{x_1\}$, $a_2 = [x_2]$ and go one step further:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \{x_2\}}}.$$

Inductively one obtains a relation

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, a_{n-1} + \cfrac{1}{a_n + \{x_n\}}}}}$$

with $0 \leq \{x_n\} < 1$. The connexion with the geometric proof of irrationality of $\sqrt{2}$ by means of rectangles and squares is now obvious: start with a positive real number x and consider a rectangle of sides 1 and x. Divide this rectangle into unit squares and a second rectangle. Then a_0 is the number of unit squares which occur, while the sides of the second rectangle are 1 and $\{x\}$. If x is not an integer, meaning $\{x\} > 0$, then we split the second rectangle into squares of sides $\{x\}$ plus a third rectangle. The number of squares is now a_1 and the third rectangle has sides $\{x\}$ and $1 - a_1\{x\}$. Going one in the same way, one checks that the number of squares we get at the n-th step is a_n.

This geometric point of view shows that the process stops after finitely many steps (meaning that some $\{x_n\}$ is zero, or equivalently that x_n is in $\mathbb{Z}$) if and only if x is rational.

For simplicity of notation we write

$$x = [a_0; a_1, \ldots, a_n] \quad \text{or} \quad x = [a_0; a_1, \ldots, a_n, \ldots]$$

depending on whether $x_n \in \mathbb{Z}$ for some n or not. This is the *continued fraction expansion of x*. Notice that any irrational number has a unique infinite continued fraction expansion, while for rational numbers, the above construction provides a unique well defined continued fraction which bears the restriction that the last a_n is ≥ 2. But we allow also the representation

$$[a_0; a_1, \ldots, a_n - 1, 1].$$

For instance $11/3 = [3; 1, 2] = [3; 1, 1, 1]$.

We need a further notation for ultimately periodic continued fraction. Assume that x is irrational and that for some integers n_0 and $r > 0$ its continued fraction expansion $[a_0; a_1, \dots, a_n, \dots]$ satisfies

$$a_{n+r} = a_n \quad \text{for any } n \geq n_0.$$

Then we write

$$x = [a_0; a_1, \dots, a_{n_0-1}, \overline{a_{n_0}, a_{n_0+1}, \dots, a_{n_0+r-1}}].$$

For instance

$$\sqrt{2} = [1; 2, 2, 2, \dots] = [1; \overline{2}].$$

References on continued fractions are [4, 11, 6, 7, 2]. An interesting remark [9] on the continued fraction expansion of $\sqrt{2}$ is to relate the A4 paper format 21×29.7 to the fraction expansion

$$\frac{297}{210} = \frac{99}{70} = [1; 2, 2, 2, 2, 2].$$

There is nothing special with the square root of 2: most of the previous argument extend to the proof of irrationality of $\sqrt{n}$ when n is a positive integer which is not the square of an integer. For instance a proof of the irrationality of $\sqrt{n}$ when n is not the square of an integer runs as follows. Write $\sqrt{n} = a/b$ where b is the smallest positive integer such that $b\sqrt{n}$ is an integer. Further, denote by m the integral part of $\sqrt{n}$: this means that m is the positive integer such that $m < \sqrt{n} < m + 1$. The strict inequality $m < \sqrt{n}$ is the assumption that n is not a square. From $0 < \sqrt{n} - m < 1$ one deduces

$$0 < (\sqrt{n} - m)b < b.$$

Now the number $b' = (\sqrt{n} - m)b$ is a positive rational integer, the product $b'\sqrt{n}$ is an integer and $b' < b$, which contradicts the choice of b minimal.

An easy variant of the argument yields the irrationality of $n^{1/k}$ when n and k are positive integers for which $n^{1/k}$ is not an integer.

The irrationality of $\sqrt{5}$ is equivalent to the irrationality of the *Golden ratio* $\Phi = (1 + \sqrt{5})/2$, root of the polynomial $X^2 - X - 1$, whose continued fraction expansion is

$$\Phi = [1; 1, 1, 1, 1, \dots] = [1, \overline{1}].$$

This expansion follows from the relation

$$\Phi = 1 + \frac{1}{\Phi} \cdot$$

The geometric irrationality proof using rectangles that we described above for $1 + \sqrt{2}$ works in a similar way for the Golden ratio: a rectangle of sides Φ and

1 splits into a square and a small rectangle of sides 1 and $\Phi - 1$, hence the first and the second rectangles have the same proportion

$$\Phi = \frac{1}{\Phi - 1}\cdot \tag{1.1}$$

As a consequence the process continues forever with one square and one smaller rectangle with the same proportion. Hence Φ and $\sqrt{5}$ are irrational numbers.

Another proof of the same result is to deduce from the equation (1.1) that a relation $\Phi = a/b$ with $0 < b < a$ yields

$$\Phi = \frac{b}{a-b},$$

hence a/b is not a rational fraction with minimal denominator.

Other numbers for which it is easy to prove the irrationality are quotients of logarithms: if m and n are positive integers such that $(\log m)/(\log n)$ is rational, say a/b, then $m^b = n^a$, which means that *m and n are multiplicatively dependent*. Recall that elements $x_1, \dots, x_r$ in an additive group are *linearly independent* if a relation $a_1x_1 + \cdots + a_rx_r = 0$ with rational integers $a_1, \dots, a_r$ implies $a_1 = \cdots = a_r = 0$. Similarly, elements $x_1, \dots, x_r$ in a multiplicative group are *multiplicatively independent* if a relation $x_1^{a_1} \cdots x_r^{a_r} = 1$ with rational integers $a_1, \dots, a_r$ implies $a_1 = \cdots = a_r = 0$. Therefore a quotient like $(\log 2)/\log 3$, and more generally $(\log m)/\log n$ where m and n are multiplicatively independent positive rational numbers, is irrational.

We have seen that *a real number is rational if and only if its continued fraction expansion is finite.* There is another criterion of irrationality using the b-adic expansion when b is an integer ≥ 2 (for $b = 10$ this is the decimal expansion, for $b = 2$ it is the diadic expansion). Indeed any real number x can be written

$$x = [x] + d_1b^{-1} + d_2b^{-2} + \cdots + d_nb^{-n} + \cdots$$

where the integers d_n (the digits of x) are in the range $0 \leq d_n < b$. Again there is unicity of such an expansion apart from the integer multiples of some b^{-n} which have two expansions, one where all sufficiently large digits vanish and one for which all sufficiently large digits are $b-1$. This is due to the equation

$$b^{-n} = \sum_{k=0}^{n} (b-1)b^{-n-k-1}.$$

Here is the irrationality criterion using such expaansions: fix an integer $b \geq 2$. Then *the real number x is rational if and only if the sequence of digits $(d_n)_{n\geq 1}$ of x in basis b is ultimately periodic.*

One might be tempted to conclude that it should be easy to decide whether a given real number is rational or not. However this is not the case with many constants from analysis, because most often one does not know any expansion, either in continued fraction or in any basis $b \geq 2$. And the fact is that for many such constants the answer is not known. For instance one does not know whether the *Euler–Mascheroni constant*

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)\\ &= 0,577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots\end{aligned}$$

is rational or not: one expects that it is an irrational number (and even a transcendental number - see later). Other formulas for the same number are

$$\begin{aligned}\gamma &= \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right)\\ &= \int_1^{\infty}\left(\frac{1}{[x]}-\frac{1}{x}\right)dx\\ &= -\int_0^1\int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}\cdot\end{aligned}$$

Recent papers on that question have been published by J. Sondow [13], they are inspired by F. Beukers' work on Apéry's proof of the irrationality of

$$\zeta(3)=\sum_{n\geq 1}\frac{1}{n^3}=1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ \ldots$$

in 1978. Recall that the values of the *Riemann zeta function*

$$\zeta(s)=\sum_{n\geq 1} n^{-s}$$

was considered by Euler for real s and by Riemann for complex s, the series being convergent for the real part of s greater than 1. Euler proved that the values $\zeta(2k)$ of this function at the even positive integers ($k\in\mathbb{Z}$, $k\geq 1$) are rational multiples of π^{2k}. For instance $\zeta(2)=\pi^2/6$. It is interesting to notice that Euler's proof relates the values $\zeta(2k)$ at the positive even integers with the values of the same function at the odd negative integers, namely $\zeta(1-2k)$. For Euler this involved divergent series, while Riemann defined $\zeta(s)$ for $s\in\mathbb{C}$, $s\neq 1$, by analytic continuation.

One might be tempted to guess that $\zeta(2k+1)/\pi^{2k+1}$ is a rational number when $k\geq 1$ is a positive integer. However the folklore conjecture is that this is not the case. In fact there are good reasons to conjecture that for any $k\geq 1$ and any non-zero polynomial $P\in\mathbb{Z}[X_0,X_1,\ldots,X_k]$, the number $P(\pi,\zeta(3),\zeta(5),\ldots,\zeta(2k+1))$ is not 0. But one does not know whether

$$\zeta(5)=\sum_{n\geq 1}\frac{1}{n^5}=1,036\,927\,755\,143\,369\,926\,331\,365\,486\,457\ldots$$

is irrational or not. And there is no proof so far that $\zeta(3)/\pi^3$ is irrational. According to T. Rivoal, among the numbers $\zeta(2n+1)$ with $n \geq 2$, infinitely many are irrational. And W. Zudilin proved that one at least of the four numbers

$$\zeta(5), \zeta(7), \zeta(9), \zeta(11)$$

is irrational. References with more information on this topic are given in the Bourbaki talk [5] by S. Fischler.

A related open question is the arithmetic nature of *Catalan's constant*

$$G = \sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2} = 0,915\,965\,594\,177\,219\,015\,0\ldots$$

Other open questions can be asked on the values of *Euler's Gamma fonction*

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n} = \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t}\cdot$$

As an example we do not know how to prove that the number

$$\Gamma(1/5)\cdots = 4,590\,843\,711\,998\,803\,053\,204\,758\,275\,929\,152\,0\ldots$$

is irrational.

The only rational values of z for which the answer is known (and in fact one knows the transcendence of the Gamma value in these cases) are

$$r \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\right\} \pmod 1.$$

The number $\Gamma(1/n)$ appears when one computes *periods* of the Fermat curve $X^n + Y^n = Z^n$, and this curve is simpler (in technical terms it has genus ≤ 1) for $n = 2, 3, 4$ and 6. For $n = 5$ the genus is 2 and this is related with the fact that one is not able so far to give the answer for $\Gamma(1/5)$.

The list of similar open problems is endless. For instance, is the number

$$e + \pi = 5,859\,874\,482\,048\,838\,473\,822\,930\,854\,632\ldots$$

rational or not? The answer is not yet known. And the same is true for any number in the following list

$$\log \pi,\ 2^\pi,\ 2^e,\ \pi^e,\ e^e.$$

1.2 Variation on a proof by Fourier (1815)

That e is not quadratic follows from the fact that the continued fraction expansion of e, which was known by L. Euler in 1737 [4] (see also [3]), is not periodic:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}}}} = [2;\, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$

Since this expansion is infinite we deduce that e is irrational. The fact that it is not ultimately periodic implies also that e is not a quadratic irrationality, as shown by Lagrange in 1770 – Euler knew already in 1737 that a number with an ultimately period continued fraction expansion is quadratic (see [4, 2, 11]).

The following easier and well known proof of the irrationality of e was given by J. Fourier in his course at the École Polytechnique in 1815. Later, in 1872 , C. Hermite proved that e is transcendental, while the work of F. Lindemann a dozen of years later led to a proof of the so-called Hermite–Lindemann Theorem: *for any nonzero algebraic number α the number e^α is transcendental.* However for this first section we study only weaker statements which are very easy to prove. We also show that Fourier's argument can be pushed a little bit further than what is usually done, as pointed out by J. Liouville in 1844.

1.2.1 Irrationality of e

We truncate the exponential series giving the value of e at some point N:

$$N!\, e - \sum_{n=0}^{N} \frac{N!}{n!} = \sum_{k \geq 1} \frac{N!}{(N+k)!} \cdot \tag{1.2}$$

The right hand side of (1.2) is a sum of positive numbers, hence is positive (not zero). From the lower bound (for the binomial coefficient)

$$\frac{(N+k)!}{N!k!} \geq N+1 \quad \text{for } k \geq 1,$$

one deduces

$$\sum_{k \geq 1} \frac{N!}{(N+k)!} < \frac{1}{N+1} \sum_{k \geq 1} \frac{N!}{(N+k)!} < \frac{1}{N+1} \sum_{k \geq 1} \frac{1}{k!} = \frac{e-1}{N+1} \cdot$$

Therefore the right hand side of (1.2) tends to 0 when N tends to infinity. In the left hand side, $N!$ and $\sum_{n=0}^{N} N!/n!$ are integers. It follows that $N!e$ is never an integer, hence e is an irrational number.

1.2.2 The number e is not quadratic

.

The fact that e is not a rational number implies that for each $m \geq 1$ the number $e^{1/m}$ is not rational. To prove that e^2 for instance is also irrational is not so easy (see the comment on this point in [1]).

The proof below is essentially the one given by J. Liouville in 1840 [8] which is quoted by Ch. Hermite ("ces travaux de l'illustre géomètre").

To prove that e does not satisfy a quadratic relation $ae^2 + be + c$ with a, b and c rational integers, not all zero, requires some new trick. Indeed if we just mimic the same argument we get

$$cN! + \sum_{n=0}^{N} (2^n a + b) \frac{N!}{n!} = -\sum_{k \geq 0} \left(2^{N+1+k} a + b\right) \frac{N!}{(N+1+k)!}.$$

The left hand side is a rational integer, but the right hand side tends to infinity (and not 0) with N, so we draw no conclusion.

Instead of this approach we write the quadratic relation as $ae + b + ce^{-1} = 0$. This time it works:

$$bN! + \sum_{n=0}^{N} (a + (-1)^n c) \frac{N!}{n!} = -\sum_{k \geq 0} \left(a + (-1)^{N+1+k} c\right) \frac{N!}{(N+1+k)!}.$$

Again the left hand side is a rational integer, but now the right hand side tends to 0 when N tends to infinity, which is what we expected. However we need a little more work to conclude: we do not yet get the desired conclusion, we only deduce that both sides vanish. Now let us look more closely to the series in the right hand side. Write the two first terms A_N for $k = 0$ and B_N for $k = 1$:

$$\sum_{k \geq 0} \left(a + (-1)^{N+1+k} c\right) \frac{N!}{(N+1+k)!} = A_N + B_N + C_N$$

with

$$\begin{aligned} A_N &= \left(a - (-1)^N c\right) \frac{1}{N+1} \\ B_N &= \left(a + (-1)^N c\right) \frac{1}{(N+1)(N+2)} \\ C_N &= \sum_{k \geq 2} \left(a + (-1)^{N+1+k} c\right) \frac{N!}{(N+1+k)!} \end{aligned}$$

The above proof that the sum $A_N + B_N + C_N$ tends to zero as N tends to infinity shows more: each of the three sequences

$$A_N, \quad (N+1)B_N, \quad (N+1)(N+2)C_N$$

tends to 0 as N tends to infinity. Hence, from the fact that the sum $A_N + B_N + C_N$ vanishes for sufficiently large N, it easily follows that for sufficiently large N, each of the three terms A_N, B_N and C_N vanishes, hence $a - (-1)^N c$ and $a + (-1)^N c$ vanish, therefore $a = c = 0$, and finally $b = 0$.

References

[1] M. Aigner& G.M. Ziegler – *Proofs from THE BOOK*, Springer (2001).

[2] Y. Bugeaud – *Approximation by algebraic numbers*, Cambridge Tracts in Mathematics, vol. 160, Cambridge University Press, Cambridge, 2004.

[3] H. Cohn – *A short proof of the simple continued fraction expansion of e*, Math Monthly **113** January 2006, 57–62 = MathArxiv NT/061660 v2 January 30 2006.

[4] L. Euler – *De fractionibus continuis dissertatio*, Commentarii Acad. Sci. Petropolitanae, 9 (1737), 1744, p. 98–137; Opera Omnia Ser. I vol. 14, Commentationes Analyticae, p. 187–215.

[5] S. Fischler – *Irrationalité de valeurs de zêta, (d'après Apéry, Rivoal, ...)*, Sém. Bourbaki 2002-2003, N° 910 (Novembre 2002). Astérisque **294** (2004), 27-62.
http://www.math.u-psud.fr/~fischler/publi.html

[6] A. Ya. Khinchin – *Continued fractions*, Dover Publications Inc., third edition (1997).

[7] H.W. Lenstra Jr– *Solving the Pell Equation*, Notices of the A.M.S. **49** (2) (2002) 182–192.

[8] J. Liouville – *Sur l'irrationalité du nombre $e = 2,718\ldots$*, J. Math. Pures Appl. (1) **5** (1840), p. 192.

[9] Benoît Rittaud – *Le fabuleux destin de $\sqrt{2}$*, Éditions Le Pommier (2006).

[10] T. Rivoal – *Applications arithmétiques de l'interpolation lagrangienne*, IJNT, to appear.

[11] W. M. Schmidt – *Diophantine approximation*, Lecture Notes in Mathematics, vol. 785, Springer-Verlag, Berlin, 1980.

[12] S.A. Shirali – *Continued fraction for e*, Resonance, vol. **5** N∘1, Jan. 2000, 14–28.
http://www.ias.ac.in/resonance/

[13] Jonathan Sondow – *Criterion for irrationality of Euler's constant*, (2002)
http://xxx.lanl.gov/pdf/math.NT/0209070
http://home.earthlink.net/~jsondow/