Fourth course: september 18, 2007. [6]

Let $n_0 \geq 0$, $n_1 \geq 0$ be two integers. Define $N = n_0 + n_1$ and

$$T(X) = (X - n_0 - 1)(X - n_0 - 2) \cdots (X - N).$$

Since T is monic of degree n_1 with integer coefficients, it follows from the differential equation of the exponential function

$$\delta(e^z) = ze^z$$

that there is a polynomial $B \in \mathbb{Z}[z]$, which is monic of degree n_1, such that $T(\delta)e^z = B(z)e^z$.

Set

$$A(z) = \sum_{k=0}^{n_0} T(k) \frac{z^k}{k!} \quad \text{and} \quad R(z) = \sum_{k \geq N+1} T(k) \frac{z^k}{k!} \cdot$$

Then

$$B(z)e^z = A(z) + R(z),$$

where A is a polynomial with rational coefficients of degree n_0 and leading coefficient

$$\frac{T(n_0)}{n_0!} = (-1)^{n_1} \frac{n_1!}{n_0!} \cdot$$

Also the analytic function R has a zero of multiplicity $\geq N + 1$ at the origin.

We can explicit these formulae for A and R. For $0 \leq k \leq n_0$ we have

$$\begin{aligned} T(k) &= (k - n_0 - 1)(k - n_0 - 2) \cdots (k - N) \\ &= (-1)^{n_1}(N - k) \cdots (n_0 + 2 - k)(n_0 + 1 - k) \\ &= (-1)^{n_1} \frac{(N - k)!}{(n_0 - k)!} \cdot \end{aligned}$$

For $k \geq N + 1$ we write in a similar way

$$T(k) = (k - n_0 - 1)(k - n_0 - 2) \cdots (k - N) = \frac{(k - n_0 - 1)!}{(k - N - 1)!} \cdot$$

Hence we have proved:

[6] Updated: October 12, 2007

Proposition 1.20 (Hermite's formulae for the exponential function)**.** *Let $n_0 \geq 0$, $n_1 \geq 0$ be two integers. Define $N = n_0 + n_1$. Set*

$$A(z) = (-1)^{n_1} \sum_{k=0}^{n_0} \frac{(N-k)!}{(n_0-k)!k!} \cdot z^k \quad \text{and} \quad R(z) = \sum_{k \geq N+1} \frac{(k-n_0-1)!}{(k-N-1)!} \cdot z^k.$$

Finally, define $B \in \mathbb{Z}[z]$ by the condition

$$(\delta - n_0 + 1)(\delta - n_0 + 2) \cdots (\delta - N)e^z = B(z)e^z.$$

Then

$$B(z)e^z = A(z) + R(z).$$

Further, B is a monic polynomial with integer coefficients of degree n_1, A is a polynomial with rational coefficients of degree n_0 and leading coefficient $(-1)^{n_1} n_1!/n_0!$, and the analytic function R has a zero of multiplicity $N+1$ at the origin.
Furthermore, if $n_1 \geq n_0$, then the coefficients of A are integers.

Proof. It remains only to check the last assertion on the integrality of the coefficients of A for $n_1 \geq n_0$. Indeed when $n_1 \geq n_0$ each coefficient of the polynomial A is an integral multiple of a binomial coefficient:

$$\frac{(N-k)!}{(n_0-k)!k!} = (N-k)(N-k-1)\cdots(n_0+1) \cdot \frac{n_0!}{(n_0-k)!k!}$$

for $0 \leq k \leq n_0$. Hence $A \in \mathbb{Z}[z]$. □

We now restrict to the case $n_0 = n_1$ and we set $n = n_0 = n_1$. We write also

$$T_n(z) = (z-n-1)(z-n-2)\cdots(z-2n)$$

and we denote by A_n, B_n and R_n the Hermite polynomials and the remainder in Hermite's Proposition 1.20.

Remark. *For $n_1 < n_0$ the leading coefficient of A is not an integer, but the polynomial $n_0!A$ always has integer coefficients.*

Lemma 1.21. *Let $z \in \mathbb{C}$. Then*

$$|R_n(z)| \leq \frac{|z|^{2n+1}}{n!} e^{|z|}.$$

In particular the sequence $(R_n(z))_{n\geq 0}$ tends to 0 as n tends to infinity.

Proof. We have

$$R_n(z) = \sum_{k \geq 2n+1} \frac{(k-n-1)!}{(k-2n-1)!k!} \cdot z^k = \sum_{\ell \geq 0} \frac{(\ell+n)!}{(\ell+2n+1)!} \cdot \frac{|z|^{\ell+2n+1}}{\ell!}.$$

The trivial upper bound

$$\prod_{j=n+1}^{n+\ell} j \leq \prod_{j=n+1}^{n+\ell} (j+n+1)$$

is equivalent to

$$\frac{(\ell+n)!}{(\ell+2n+1)!} \leq \frac{n!}{(2n+1)!},$$

hence

$$|R_n(z)| \leq \frac{n!|z|^{2n+1}}{(2n+1)!} \sum_{\ell \geq 0} \frac{|z|^\ell}{\ell!}.$$

We bound $n!/(2n+1)!$ by $n!$: Lemma 1.21 follows. $\square$

We are now able to complete the proof of the irrationality of e^r for $\in \mathbb{Q}$, $r \neq 0$.

Let $r = a/b$ be a non-zero rational number. Assume first r is positive. Set $s = e^r$ and replace z by $a = br$ in the previous formulae; we deduce

$$B_n(a)s^b - A_n(a) = R_n(a).$$

All coefficients in R_n are positive, hence $R_n(a) > 0$. Therefore $B_n(a)s^b - A_n(a) \neq 0$. Since $R_n(a)$ tends to 0 when n tends to infinity and since $B_n(a)$ and $A_n(a)$ are rational integers, we may use the implication (ii)⇒(i) in Lemma 1.6: we deduce that the number s^b is irrational. As we already saw this readily implies that $s = e^r$ and $s^{-1} = e^{-r}$ are irrational.

1.4.2 Irrationality of π

The proof of the irrationality of $\log s$ for s a positive rational number given in § 1.4.1 can be extended to the case $s = -1$ in such a way that one deduces the irrationality of the number π (this result was first proved by H. Lambert in 1761 [5], using continued fraction expansion for the tangent function).

Assume π is a rational number, $\pi = a/b$. Substitute $z = ia = i\pi b$ in the previous formulae. Notice that $e^z = (-1)^b$:

$$B_n(ia)(-1)^b - A_n(ia) = R_n(ia),$$

and that the two complex numbers $A_n(ia)$ and $B_n(ia)$ are in $\mathbb{Z}[i]$. The left hand side is in $\mathbb{Z}[i]$, the right hand side tends to 0 as n tends to infinity, hence both sides are 0.

In the proof of § 1.4.1 we used the positivity of the coefficients of R_n and we deduced that $R_n(a)$ was not 0 (this is the so-called "zero estimate" in transcendental number theory). Here we need another argument.

The last step of the proof of the irrationality of π is achieved by using two consecutive indices n and $n+1$. We eliminate e^z among the two relations

$$B_n(z)e^z - A_n(z) = R_n(z) \quad \text{and} \quad B_{n+1}(z)e^z - A_{n+1}(z) = R_{n+1}(z).$$

We deduce that the polynomial

$$\Delta_n = B_n A_{n+1} - B_{n+1} A_n \tag{1.22}$$

can be written

$$\Delta_n = -B_n R_{n+1} + B_{n+1} R_n. \tag{1.23}$$

As we have seen, the polynomial B_n is monic of degree n; the polynomial A_n also has degree n, its highest degree term is $(-1)^n z^n$. It follows from (1.22) that Δ_n is a polynomial of degree $2n+1$ and highest degree term $(-1)^n 2z^{2n+1}$. On the other hand since R_n has a zero of multiplicity at least $2n+1$, the relation (1.23) shows that it is the same for Δ_n. Consequently

$$\Delta_n(z) = (-1)^n 2z^{2n+1}.$$

We deduce that Δ_n does not vanish outside 0. From (1.23) we deduce that R_n and R_{n+1} have no common zero apart from 0. This completes the proof of the irrationality of π.

1.4.3 Hermite's integral formula for the remainder

For $h \geq 0$, the h-th derivative $D^h R(z)$ of the remainder in Proposition 2.8 is given by

$$D^h R(z) = \sum_{k \geq N+1} \frac{(k-n_0-1)!}{(k-N-1)!} \cdot \frac{z^{k-h}}{(k-h)!}.$$

In particular for $h = n_0 + 1$ the formula becomes

$$D^{n_0+1} R = \sum_{k \geq N+1} \frac{z^{k-n_0-1}}{(k-N-1)!} = z^{n_1} e^z. \tag{1.24}$$

This relations determines R since R has a zero of multiplicity $\geq n_0 + 1$ at the origin. When we restrict the operator of $D = d/dz$ to the functions vanishing at the origin, it has an inverse which is the operator J defined by

$$J(\varphi) = \int_0^z \varphi(t)dt.$$

Following [4], we can compute the iterates of J:

Lemma 1.25. *For $n \geq 0$,*

$$J^{n+1}\varphi = \frac{1}{n!} \int_0^z (z-t)^n \varphi(t)dt.$$

Proof. The formula is valid for $n = 0$. We first check it for $n = 1$. The derivative of the function

$$\int_0^z (z-t)\varphi(t)dt = z\int_0^z \varphi(t)dt - \int_0^z t\varphi(t)dt$$

is

$$\int_0^z \varphi(t)dt + z\varphi(z) - z\varphi(z) = \int_0^z \varphi(t)dt.$$

We now proceed by induction. The derivative of the function of z

$$\frac{1}{n!}\int_0^z (z-t)^n \varphi(t)dt = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} \cdot z^k \int_0^z t^{n-k}\varphi(t)dt$$

is

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} \left(kz^{k-1} \int_0^z t^{n-k}\varphi(t)dt + z^n \varphi(z) \right).$$

Since

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} = 0,$$

the right hand side is nothing else than

$$\sum_{k=1}^{n} \frac{(-1)^{n-k}}{(k-1)!(n-k)!} \cdot z^{k-1} \int_0^z t^{n-k}\varphi(t)dt = \frac{1}{(n-1)!}\int_0^z (z-t)^{n-1}\varphi(t)dt.$$

$\square$

From (1.24) and 1.25 it plainly follows:

Lemma 1.26. *The remainder $R(z)$ in Hermite's fomula with parameters n_0 and n_1 is given by*

$$R(z) = \frac{1}{n_0!}\int_0^z (z-t)^{n_0} t^{n_1} e^t dt.$$

1.4.4 Hermite's identity

The next formula is one of the many disguises of Hermite's identity.

Lemma 1.27. *Let f be a polynomial of degree $\leq N$. Define*

$$F = f + Df + D^2 + \cdots + D^N f.$$

Then for $z \in \mathbb{C}$

$$\int_0^z e^{-t} f(t)dt = F(0) - e^{-z}F(z).$$

We can also write the definition of F as

$$F = (1-D)^{-1} f \quad \text{where} \quad (1-D)^{-1} = \sum_{k \geq 0} D^k.$$

The series in the right hand side is infinite, but when we apply the operator to a polynomial only finitely many $D^k f$ are not 0: when f is a polynomial of degree $\leq N$ then $D^k f = 0$ for $k > N$.

Proof. More generally, if f is a complex function which is analytic at the origin and N is a positive integer, if we set

$$F = f + Df + D^2 + \cdots + D^N f,$$

then the derivative of $e^{-t}F(t)$ is $-e^{-t}f(t) + e^{-t}D^{N+1}f(t)$. $\square$

We shall come back to such formulae in section § 2.1.3.

2 Transcendence

2.1 Hermite's Method

In 1873 C. Hermite [2] proved that the number e is transcendental. In his paper he explains in a very clear way how he found his proof. He starts with an analogy between simultaneous diophantine approximation of real numbers on the one hand and analytic complex functions of one variable on the other. He first solves the analytic problem by constructing explicitly what is now called Padé approximants for the exponential function. In fact there are two types of such approximants, they are now called type I and type II, and what Hermite did in 1873 was to compute Padé approximants of type II. He also found those of type I in 1873 and studied them later in 1893. K. Mahler was the first in the mid's 1930 to relate the properties of the two types of Padé's approximants and to use those of type I in order to get a new proof of Hermite's transcendence Theorem (and also of the generalisation by Lindemann and Weierstraß as well as quantitative refinements). See [1] Chap. 2 § 3.

In the analogy with number theory, Padé approximants of type II are related with the simultaneous approximation of real numbers $\vartheta_1, \dots, \vartheta_m$ by rational numbers p_i/q with the same denominator q (one does not require that the fractions are irreducible), which means that we wish to bound from below

$$\max_{1 \le i \le m} \left| \vartheta_i - \frac{p_i}{q} \right|$$

in terms of q, while type I is related with the study of lower bounds for linear combinations

$$|a_0 + a_1\vartheta_1 + \cdots + a_m\vartheta_m|$$

when $a_0, \dots, a_m$ are rational integers, not all of which are 0, in terms of the number $\max_{0 \le i \le m} |a_i|$.

After Hermite's seminal work, F. Lindemann was able to extend the argument and to prove the transcendence of π (hence he solved the old greek problem of the quadrature of the circle: *it is not possible using ruler and compass to draw a square and a circle having the same area*). This extension led to the so-called Hermite-Lindemann's Theorem:

Theorem 2.1 (Hermite–Lindemann)**.** *Let α be a non zero complex algebraic number. Let $\log\alpha$ be any non-zero logarithm of α. Then $\log\alpha$ is transcendental.*

Equivalently, let β be a non-zero algebraic number. Then e^β is transcendental.

Recall that any non-zero complex number z has complex logarithms: these are the solutions $\ell \in \mathbb{C}$ of the equation $e^\ell = z$. If ℓ is one of them, then all solutions ℓ to this equation $e^\ell = z$ are $\ell + 2ik\pi$ with $k \in \mathbb{Z}$. The only non-zero complex of which 0 is a logarithm is 1.

The equivalence between both statements in Theorem 2.1 is easily seen by setting $e^\beta = \alpha$: one can phrase the result by saying that for any non-zero complex number β, one at least of the two numbers β, e^β is transcendental.

After the proofs by Hermite and Lindemann, a number of authors in the XIX-th century worked out variants of the argument. The main goal was apparently to get the shorter possible proof, and most often the reason for which it works is by no means so clear as in Hermite's original version. One can find in the literature such short proofs (see for instance [3]), the connexion with Hermite's arguments are most often not so transparent. So we shall come back to the origin and try to explain what is going on.

We concentrate now on Hermite's proof for the transcendence of e. The goal is to prove that for any positive integer m, the numbers $1, e, e^2, \dots, e^m$ are linearly independent over $\mathbb{Q}$.

2.1.1 Criterion of linear independence

We first state a criterion for linear independence. This is a generalisation (from personal notes of Michel Laurent after a course he gave in Marseille) of one of the previous criteria for irrationality, namely Lemma 1.18. Most often in mathematics there is sort of an entropy: when a statement provides a necessary and sufficient condition, and when one of the two implication is easy while the other requires more work, then it is the difficult part which is most useful. Here we have a counterexample to this claim (which does not belong to mathematics but rather to social science): in the criterion 2.2 below, one of the implications is easy while the other is deeper; but it turns out that it is the easy one which is required in transcendence proofs. So we state the statement and prove the easy part now, we postpone the reverse to a later section where we introduce some tools from geometry of numbers and give further consequences of these tools.

Let $\vartheta_1, \dots, \vartheta_m$ be real numbers and a_0, $a_1, \dots,$ a_m rational integers, not all of which are 0. Our goal is to prove that the number

$$L = a_0 + a_1\vartheta_1 + \cdots + a_m\vartheta_m$$

is not 0.

The idea is to approximate simultaneously $\vartheta_1, \dots, \vartheta_m$ by rational numbers $p_1/q, \dots, p_m/q$ with the same denominator $q > 0$.

Let $q, p_1, \dots, p_m$ be rational integers with $q > 0$. For $1 \le k \le m$ set

$$\epsilon_k = q\vartheta_k - p_k.$$

Then $qL = M + R$ with

$$M = a_0 q + a_1 p_1 + + \cdots + a_m q_m \in \mathbb{Z} \quad \text{and} \quad R = a_1\epsilon_1 + \cdots + a_m\epsilon_m \in \mathbb{R}.$$

If $M \neq 0$ and $|R| < 1$ we deduce $L \neq 0$.

One of the main difficulties is often to check $M \neq 0$. This question gives rise to the so-called *zero estimates* or *non-vanishing lemmas.* In the present situation, a solution is to construct not only one tuple $(q, p_1, \ldots, p_m)$ in $\mathbb{Z}^{m+1} \setminus \{0\}$, but $m+1$ such tuples which are linearly independent. This yields $m+1$ pairs (M_k, R_k), $k = 0, \ldots, m$ in place of a single pair (M, R), and from $(a_0, \ldots, a_m) \neq 0$ one deduces that one at least of $M_0, \ldots, M_m$ is not 0.

It turns out that nothing is lossed by using such arguments: existence of linearly independent simultaneous rational approximations for $\vartheta_1, \ldots, \vartheta_m$ are characteristic of linearly independent numbers $1, \vartheta_1, \ldots, \vartheta_m$. As we just said earlier, we shall use only the easy part of the next lemma 2.2, and we shall prove the converse later.

Lemma 2.2. *Let $\underline{\vartheta} = (\vartheta_1, \ldots, \vartheta_m) \in \mathbb{R}^m$. Then the following conditions are equivalent.*
(i) The numbers $1, \vartheta_1, \ldots, \vartheta_m$ are linearly independent over $\mathbb{Q}$.
(ii) For any $\epsilon > 0$ there exist $m+1$ linearly independent elements $\underline{b}_0, \underline{b}_1, \ldots, \underline{b}_m$ in $\mathbb{Z}^{m+1}$, say

$$\underline{b}_i = (q_i, p_{1i}, \ldots, p_{mi}), \quad (0 \leq i \leq m)$$

with $q_i > 0$, such that

$$\max_{1\leq k\leq m} \left| \vartheta_k - \frac{p_{ki}}{q_i} \right| \leq \frac{\epsilon}{q_i}, \quad (0 \leq i \leq m). \tag{2.3}$$

The condition on linear independence of the elements $\underline{b}_0, \underline{b}_1, \ldots, \underline{b}_m$ means that the determinant

$$\begin{vmatrix} q_0 & p_{10} & \cdots & p_{m0} \\ \vdots & \vdots & \ddots & \vdots \\ q_m & p_{1m} & \cdots & p_{mm} \end{vmatrix}$$

is not 0.

For $0 \leq i \leq m$, set

$$\underline{r}_i = \left(\frac{p_{1i}}{q_i}, \ldots, \frac{p_{mi}}{q_i} \right) \in \mathbb{Q}^m.$$

Further define, for $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$

$$|\underline{x}| = \max_{1\leq i\leq m} |x_i|.$$

Also for $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ set

$$\underline{x} - \underline{y} = (x_1 - y_1, \ldots, x_m - y_m),$$

so that

$$|\underline{x} - \underline{y}| = \max_{1 \le i \le m} |x_i - y_i|.$$

Then the relation (2.3) in Lemma 2.2 can be written

$$|\underline{\vartheta} - \underline{r}_i| \le \frac{\epsilon}{q_i}, \quad (0 \le i \le m).$$

We shall prove a more explicit version of (ii)⇒(i): we check that *any tuple* $(q, p_1, \ldots, p_m) \in \mathbb{Z}^{m+1}$ *producing a tuple* $(p_1/q, \ldots, p_m/q) \in \mathbb{Q}^m$ *of sufficiently good rational approximations to* $\underline{\vartheta}$ *satisfies the same linear dependence relations as* $1, \vartheta_1, \ldots, \vartheta_m$.

Lemma 2.4. *Let* $\vartheta_1, \ldots, \vartheta_m$ *be real numbers. Assume that the numbers* $1, \vartheta_1, \ldots, \vartheta_m$ *are linearly dependent over* $\mathbb{Q}$*: let* $a, b_1, \ldots, b_m$ *be rational integers, not all of which are zero, satisfying*

$$a + b_1\vartheta_1 + \cdots + b_m\vartheta_m = 0.$$

Let $\epsilon > 0$ *satisfy* $\sum_{k=1}^{m} |b_k] > 1/\epsilon$*. Assume further that* $(q, p_1, \ldots, p_m) \in \mathbb{Z}^{m+1}$ *satisfies* $q > 0$ *and*

$$\max_{1 \le k \le m} |q\vartheta_k - p_k| \le \epsilon.$$

Then

$$aq + b_1p_1 + \cdots + b_mp_m = 0.$$

Proof. In the relation

$$qa + \sum_{k=1}^{m} b_kp_k = -\sum_{k=1}^{m} b_k(q\vartheta_k - p_k),$$

the right hand side has absolute value less than 1 and the left hand side is a rational integer, so it is 0.

□

Proof of (ii)⇒(i) in Lemma 2.2. By assumption (ii) we have $m+1$ linearly independent elements $\underline{b}_i \in \mathbb{Z}^{m+1}$ such that the corresponding rational approximation satisfy the assumptions of Lemma 2.4. For each non-zero linear form

$$aX_0 + b_1X_1 + \cdots + b_mX_m = 0$$

one at least of the $L(\underline{b}_i)$ is not 0. Hence

$$a + b_1\vartheta_1 + \cdots + b_m\vartheta_m \ne 0.$$

□

2.1.2 Padé approximants

Henri Eugène Padé (1863–1953), who was a student of Charles Hermite (1822–1901), gave his name to the following objects.

Lemma 2.5. *Let $f_1, \ldots, f_m$ be analytic functions of one complex variable near the origin. Let $n_0, n_1, \ldots, n_m$ be non-negative integers. Set*

$$N = n_0 + n_1 + \cdots + n_m.$$

Then there exists a tuple $(Q, P_1, \ldots, P_m)$ of polynomials in $\mathbb{C}[X]$ satisfying the following properties:
(i) The polynomial Q is not zero, it has degree $\leq N - n_0$.
(ii) For $1 \leq \mu \leq m$, the polynomial P_μ has degree $\leq N - n_\mu$.
(iii) For $1 \leq \mu \leq m$, the function $x \mapsto Q(x)f_\mu(x) - P_\mu(x)$ has a zero at the origin of multiplicity $\geq N + 1$.

Definition. *A tuple $(Q, P_1, \ldots, P_m)$ of polynomials in $\mathbb{C}[X]$ satisfying the condition of Lemma 2.5 is called a* Padé system of the second type for $(f_1, \ldots, f_m)$ attached to the parameters $n_0, n_1, \ldots, n_m$.

Proof. The polynomial Q of Lemma 2.5 should have degree $\leq N - n_0$, so we have to find (or rather to prove the existence) its $N - n_0 + 1$ coefficients, not all being zero. We consider these coefficients as unknowns. The property we require is that for $1 \leq \mu \leq m$, the Taylor expansion at the origin of $Q(z)f_\mu(z)$ has zero coefficients for $z^{N-n_\mu+1}, z^{N-n_\mu+1}, \ldots, z^N$. If this property holds for $1 \leq \mu \leq m$, we shall define P_μ by truncating the Taylor series at the origin of $Q(z)f_\mu(z)$ at the rank z^{N-n_μ}, hence P_μ will have degree $\leq N - n_\mu$, while the remainder $Q(z)f_\mu(z) - P_\mu(z)$ will have a mutiplicity $\geq N + 1$ at the origin.

Now for each given μ the condition we stated amounts to require that our unknowns (the coefficients of Q) satisfy n_μ homogeneous linear relations, namely

$$\left(\frac{d}{dx}\right)^k [Q(x)f_\mu(x)]_{x=0} = 0 \quad \text{for} \quad N - n_\mu < k \leq N.$$

Therefore altogether we get $n_1 + \cdots + n_m = N - n_0$ homogeneous linear equations, and since the number $N - n_0 + 1$ of unknowns (the coefficients of Q) is larger, linear algebra tells us that a non-trivial solution exists.

□

There is no unicity, because of the homogeneity of the problem: the set of solutions (together with the trivial solution 0) is a vector space over $\mathbb{C}$, and Lemma 2.5 tells us that it has positive dimension. In the case where this dimension is 1 (which means that there is unicity up to a multiplicative factor), the system of approximants is called *perfect*. An example is with $m = 1$ and $f(z) = e^z$, as shown by Hermite's work.

Exercise 2.6. *Let $f_1, \ldots, f_m$ be analytic functions of one complex variable near the origin. Let $d_0, d_1, \ldots, d_m$ be non-negative integers. Set*

$$M = d_0 + d_1 + \cdots + d_m + m.$$

Show that there exists a tuple $(A_0, \ldots, A_m)$ of polynomials in $\mathbb{C}[X]$, not all of which are zero, where A_i has degree $\leq d_i$, such that the function

$$A_0 + A_1 f_1 + \cdots + A_m f_m$$

has a zero at the origin of multiplicity $\geq M$.
These are the Padé approximants of type I.

Most often it is not easy to find explicit solutions: we only know their existence. As we are going to show, Hermite succeeded to produce explicit solutions for the systems of Padé approximants of the functions $(e^x, e^{2x}, \ldots, e^{mx})$.

References

[1] N. I. Fel′dman & Y. V. Nesterenko – *Transcendental numbers*, in *Number Theory, IV*, Encyclopaedia Math. Sci., vol. **44**, Springer, Berlin, 1998, p. 1–345.

[2] C. Hermite – *Sur la fonction exponentielle*, C. R. Acad. Sci. Paris, **77** (1873), 18–24; 74–79; 226–233; 285–293; *Oeuvres*, Gauthier Villars (1905), III, 150–181. See also *Oeuvres* III, 127–130, 146–149, and *Correspondance Hermite-Stieltjes*, II, lettre 363, 291–295.

[3] I. Niven – *Irrational numbers*, Carus Math. Monographs **11** (1956).

[4] C.L. Siegel – *Transcendental Numbers*, Annals of Mathematics Studies, **16**. Princeton University Press, Princeton, N. J., 1949.