Sixth course: september 21, 2007. [8]

2.2.2 Diophantine approximation and applications

Diophantine approximation is the study of the approximation of real or complex numbers by rational or algebraic numbers. It has its early sources in astronomy, with the study of movement of the celestials bodies, and in the computations of π.

The number π occurs more or less explicitly in a number of ancient documents from different civilisations. In the Bible there is an implicit value 3. The Rhind Papyrus around 2000 BC gives an approximate value $2^8/3^4 = 3.1604\ldots$

In the early times in India, ancient Hindu and Jaina mathematicians considered this question. Sometimes between the 8th and the 4th century, the Indian sacred texts Sulvasūtras from Baudhāyana give $3,088$. Also in India, around 500 BC, Suryaprajnapati (a Jaina mathematician) gives $\sqrt{10} = 3.162\ldots$

The value of π was studied in ancient Greece (especially by Archimedes around 2500 BC), also in China where the approximation $355/113 = 3.1415929\ldots$ was known. In the Vth Century AC Aryabhaṭīya, Āryabhaṭa I had the approximation 3.1416 and he suggested that π might be irrational. One century later Bhāskara I suggests a negative solution to the problem of squaring the circle. In the XIIth century Bhāskarācārya (Bhāskara II) has the approximation $3927/1250 = 3.1416$.

It is remarkable that Madhava (1380–1420) knew a series which gave him 11 exact decimals 3.14159265359 (while Viète in 1579 had 9 decimals only). A number of other mathematicians in Europa studied this question (including Leibniz and Gregory).

Getting sharp rational approximations is now easy using the continued fraction expansion of $\pi = 3.1415926535898\ldots$ which starts with

$$\pi = [3;\ 7,\ 15,\ 1,\ 292,\ 1,\ 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 14,\ 2,\ 1,\ 1\ldots]$$

The sequence of rational approximations we get by truncating this expansion is

$$3,\ \frac{22}{7},\frac{333}{106},\frac{355}{113},\ \ldots$$

Computation of billions of decimals of π have been performed: it serves as a test for computers, and produces also good candidates for random sequences, even if proofs are not available that such sequences satisfy the required properties.

Another type of approximation for π is due to Ramanujan:

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.141\,592\,653\quad 805\ldots$$

[8] Updated: October 12, 2007

which is a root of $P(x) = 168\,125\,x^2 - 792\,225x + 829\,521$. Of course we know from Lindemann's Theorem that such estimate will not produce an exact value, since

$$\pi = 3.141\,592\,653\;\;589\dots$$

is not root of a polynomial with integer coefficients.

One recent (1997) formula for π produces efficiently its digits in base 16:

$$\pi = \sum_{n\geq 0}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right)2^{-4n}.$$

For computing a number with a sharp accuracy, one wishes to get many decimals (or binary digits) with a number of operations as small as possible. As we have seen for Diophantine questions, the cost is measured by the denominator q: one investigates how well ξ can be approximated in terms of q. So the notion of *complexity* is very different in these two points of view.

Diophantine approximation occurs in many different disguises (a very good reference here is [3]). It plays a crucial role in the question of small divisors and dynamical systems, introduced by H. Poincaré. See in particular [4]. In the study of the periods of Saturn orbits (Cassini divisions), Diophantine approximation is also there. It plays a role in the question of the stability of the solar system, in resonance in astronomy, in the problems of engrenages, in quasi-cristals, in the acoustic of concert halls, in calendars (bissextile years).

We give now an example of application of the question of rational approximations to $\log_2 3$ to musical scales.

The successive harmonics of a note of frequency n are the vibrations with frequencies $2n$, $3n$, $4n$, $5n$, ... with decreasing intensity. The successive octaves of a note of frequency n are vibrations with frequencies $2n$, $4n$, $8n$, $16n$...

Using octaves, one replaces each note by a note with frequency in a given interval, say $[n, 2n)$. The classical choice in Hertz is $[264, 528)$. For simplicity we take rather $[1, 2)$. Hence a note with frequency f is replaced by a note with frequency r with $1 \leq r < 2$, where

$$f = 2^a r, \qquad a = [\log_2 f] \in \mathbb{Z}, \quad r = 2^{\{\log_2 f\}} \in [1, 2).$$

For instance a note with frequency 3 (which is a harmonic of 1) is at the octave of a note with frequency $3/2$. The musical interval $[1, 3/2]$ is called *fifth*, the ratio of the endpoints of the interval is $3/2$.

The musical interval $[3/2, 2]$ is *the fourth*, with ratio $4/3$.

The successive fifths are the notes in the interval $[1, 2]$, which are at the octave of notes with frequency

$$1,\ 3,\ 9,\ 27,\ 81\dots$$

namely:

$$1,\ 3/2,\ 9/8,\ 27/16,\ 81/64\dots$$

We shall never come back to the initial value 1, since the Diophantine equation $3^a = 2^b$ has no solution in positive integers a, b. We cannot solve exactly the equation $2^a = 3^b$ in positive rational integers a and b, but we can look for powers of 2 which are close to powers of 3.

There are just three solutions to the equation $3^x - 2^y = \pm 1$ in positive integers x and y, namely $3-2=1$, $4-3=1$ and $9-8=1$. This question leads to the study of so-called *exponential Diophantine equations*, which include the Catalan's equation $x^p - y^q = 1$ where x,y, p and q are unknowns in $\mathbb{Z}$ all ≥ 2 (this was solved recently, the only solution is $3^2 - 2^3 = 1$, as suggested in 1844 by E. Catalan, the same year when Liouville produced the first examples of transcendental numbers). A generalisation of this question is a conjecture of Pillai, according to which *for any fixed positive $k \in \mathbb{Z}$ there are only finitely many x,y, p and q in $\mathbb{Z}$, all ≥ 2, with $x^p - y^q = k$.* It is easy to check that Pillai's conjecture is equivalent to the fact that *in the increasing sequence $(u_n)_{n\geq 1}$ of* perfect powers (namely integers of the form a^b with $a \geq 1$ and $b \geq 2$), *the difference between two consecutive terms $u_{n+1} - u_n$ tends to infinity.*

Instead of looking at Diophantine equations, one can consider rather the question of approximating 3^a by 2^b from another point of view. The fact that the equation $3^a = 2^b$ has no solution in positive integers a, b means that the logarithm in basis 2 of 3:

$$\log_2 3 = (\log 3)/\log 2 = 1.58496250072\ldots,$$

which is the solution x of the equation $2^x = 3$, is irrational. Powers of 2 which are close to powers of 3 correspond to rational approximations a/b to $\log_2 3$:

$$\log_2 3 \simeq a/b, \qquad 2^a \simeq 3^b.$$

Hence it is natural to consider the continued fraction expansion

$$\log_2 3 = [1; 1, 1, 2, 2, 3, 1, 5, \ \ldots]$$

The first approximations we obtain by truncating this expansion are

$$[1] = 1, \ [1; 1] = 2, \ [1; 1, 1] = \frac{3}{2}, \ [1; 1, 1, 2] = \frac{8}{5} = 1.6.$$

This last approximation suggest to consider $a = 3$ and $b = 5$:

$$2^8 = 256 \quad \text{is not too far from} \quad 3^5 = 243.$$

The approximation of $(3/2)^5 = 7.593\ldots$ by 2^3 means that 5 *fifths produces almost to* 3 *octaves.*

The next approximation is

$$[1; 1, 1, 2, 2] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{19}{12} = 1.5833\ldots$$

It is related to the fact that 2^{19} is close to 3^{12}:

$$2^{19} = 524\,288 \simeq 3^{12} = 531\,441, \quad (3/2)^{12} = 129.74\ldots \text{ is close to } 2^7 = 128.$$

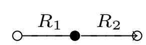
In music it means that *twelve fifths is a bit more than seven octaves*. The *comma of Pythagoras* is $3^{12}/2^{19} = 1,01364$. It produces an error of about 1.36%, which most people cannot ear.

A further remarkable Diophantine approximation is

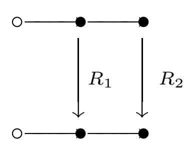
$$5^3 = 125 \simeq 2^7 = 128, \quad (5/4)^3 = 1.953 \simeq 2.$$

meaning that *three thirds* (ratio 5/4) *produce almost one octave.* This approximation can be written $2^{10} = 1024 \simeq 10^3$. It plays an important role in computers (kilo octets), of course, but also in acoustic: multiplying the intensity of a sound by 10 means adding 10 decibels. Multiplying the intensity by k, amounts to add d decibels with $10^d = k^{10}$. Since $2^{10} \simeq 10^3$, doubling the intensity, is close to adding 3 decibels.

A further example of application of continued fractions given in [3] deals with *electric networks*. The resistance of a network in series



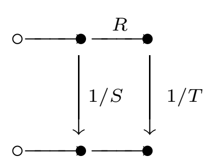
is the sum $R_1 + R_2$. The resistance R of the parallel network



satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}\cdot$$

The resistance U of the circuit



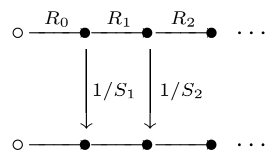
is given by

$$U = \cfrac{1}{S + \cfrac{1}{R + \cfrac{1}{T}}}\cdot$$

The resistance of the following network is given by a continued fraction

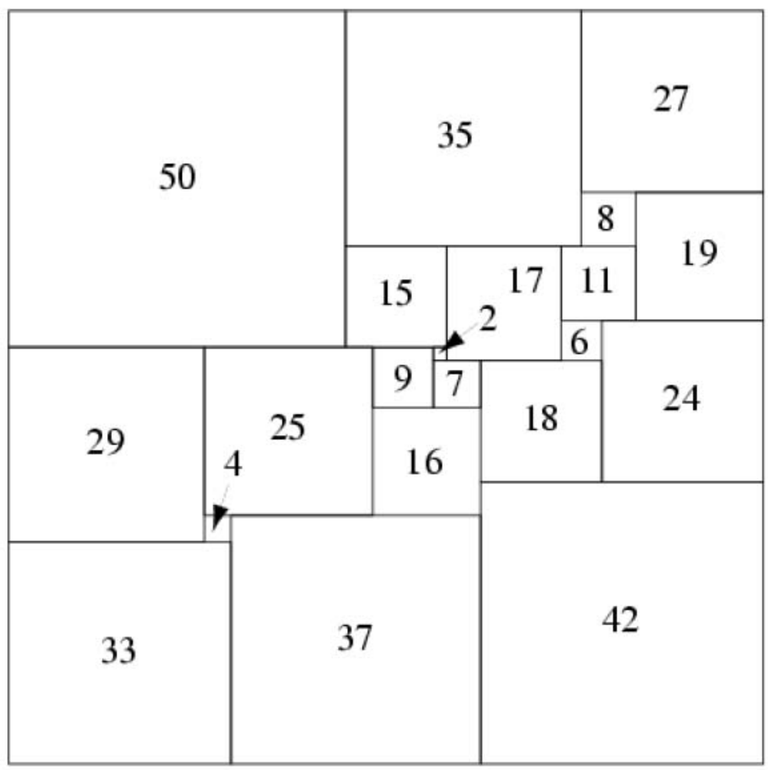
$$[R_0; S_1, R_1, S_2, R_2 \ldots]$$

for the circuit



For instance when $R_i = S_j = 1$ we get the quotients of consecutive Fibonacci numbers.

This fact provides a connexion between electric networks, and continued fractions, it has a surprising consequence on the problem of *decomposition of a square into squares* (squaring the square!): electric networks and continued fractions were used to find the first solution to the problem of decomposing a geometric integer square into distinct integer squares.



21-*square perfect square*

There is a unique simple perfect square of order 21 (the lowest possible order), discovered in 1978 by A. J. W. Duijvestijn (Bouwkamp and Duijvestijn 1992). It is composed of 21 squares with total side length 112, and is illustrated above.

We conclude this list of applications of Diophantine questions with a connexion between a problem raised by K. Mahler in 1967 and theoretical computer science.

Mahler notices that an integer power of e is never an integer, since e is transcendental. He asks whether there exists an absolute constant $c > 0$ such that, for a and b positive integers,

$$|e^b - a| > a^{-c}?$$

This is not yet solved. Mahler's conjecture arises by considering the numbers $\log a - b_a$ for $a = 1, \dots, A$, where b_a is the nearest integer to $\log a$, for growing values of A, and assuming that these numbers are evenly distributed in the interval $(-1/2, 1/2)$. Instead we could consider the numbers $e^b - a_b$ for $b = 1, \dots, B$, where a_b is the nearest integer to e^b , for growing values of B, and assume that these numbers are evenly distributed in the interval $(-1/2, 1/2)$. For this reason I suggested that Mahler's conjecture may not be the best possible estimate and that the following stronger estimate would be valid:

$$|e^b - a| > b^{-c}.$$

But this is not true, as pointed out to me by Iam Ho on September 27, 2007: if a denotes the integral part of e^b, then we have

$$0 < e^b - a < 1, \qquad 0 < a(b - \log a) < e^b - a < e^b(b - \log a),$$

hence

$$0 < b - \log a < \frac{e^b - a}{a} < \frac{1}{a}\cdot$$

The question of a lower bound for $|e^b - a|$ was considered first by K. Mahler (1953, 1967), then by M. Mignotte (1974), and more recently by F. Wielonsky (1997). The sharpest known estimate on Mahler's problem is

$$|e^b - a| > b^{-20b}.$$

In a joint work with Yu.V. Nesterenko [2] in 1996, we considered an extension of this question when a and b are rational numbers. A refinement of our estimate has been obtained by S. Khemira in 2005 and is currently being sharpened in a joint work of S. Khemira and P. Voutier.

Define $H(p/q) = \max\{|p|,\, q\}$. Then for a and b in $\mathbb{Q}$ with $b \neq 0$, the estimate is

$$|e^b - a| \geq \exp\{-1,3 \cdot 10^5 (\log A)(\log B)\}$$

where $A = \max\{H(a),\ A_0\}$, $B = \max\{H(b),\ 2\}$. The numerical value of the absolute constant A_0 will be explicitly computed.

There is a connexion with the question of *exact rounding of the elementary functions* in theoretical computer science. A reference to the *Arénaire project in Computer Arithmetic* is
http://www.ens-lyon.fr/LIP/Arenaire/

This team works on validated scientific computing: arithmetic. reliability, accuracy, and speed. Their goal is to improve the available arithmetic on computers, processors, dedicated or embedded chips, and they want to achieve more accurate results or getting them more quickly. This has implication in power consumption as well as reliability of numerical software.

Further applications of Diophantine Approximation include (see [1]): equidistribution modulo 1, discrepancy, numerical integration, interpolation, approximate solutions to integral and differential equations.

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