HARDY-RAMANUJAN
JOURNAL
(A Journal devoted to primes, diophantine equations, transcendental numbers and other questions on 1,2,3,4,5,...)

VOLUME 28
2005
Date of issue: 22.12.2005
(To be put on the internet around this time)

EDITORS:

R.BALASUBRAMANIAN AND K.RAMACHANDRA
Further Variations on the Six Exponentials Theorem

Michel Waldschmidt

Abstract. Let \( \mathcal{L} \) denote the set of linear combinations, with algebraic coefficients, of 1 and logarithms of algebraic numbers. The Strong Six Exponentials Theorem of D. Roy gives sufficient conditions for a \( 2 \times 3 \) matrix

\[
M = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23}
\end{pmatrix}
\]

whose entries are in \( \mathcal{L} \) to have rank 2.

Here we give sufficient conditions so that one at least of the three \( 2 \times 2 \) determinants

\[
\begin{vmatrix}
\lambda_{11} & \lambda_{12} \\
\lambda_{21} & \lambda_{22}
\end{vmatrix}, \quad \begin{vmatrix}
\lambda_{12} & \lambda_{13} \\
\lambda_{22} & \lambda_{23}
\end{vmatrix}, \quad \begin{vmatrix}
\lambda_{13} & \lambda_{11} \\
\lambda_{23} & \lambda_{21}
\end{vmatrix}
\]

is not in \( \mathcal{L} \).

1. Main result

We denote by \( \mathbb{Q} \) the field of rational numbers, by \( \overline{\mathbb{Q}} \) the field of algebraic numbers (algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \)), by \( \mathcal{L} \) the \( \mathbb{Q} \)-vector space of logarithms of algebraic numbers:

\[
\mathcal{L} = \{ \lambda \in \mathbb{C} ; \ e^\lambda \in \overline{\mathbb{Q}}^* \} = \{ \log \alpha ; \ \alpha \in \overline{\mathbb{Q}}^* \} = \exp^{-1}(\overline{\mathbb{Q}}^*)
\]

and by \( \widetilde{\mathcal{L}} \) the \( \mathbb{Q} \)-vector subspace of \( \mathbb{C} \) spanned by \( \{1\} \cup \mathcal{L} \). Hence \( \widetilde{\mathcal{L}} \) is the set of linear combinations of 1 and logarithms of algebraic numbers with algebraic coefficients:

\[
\widetilde{\mathcal{L}} = \left\{ \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n ; \quad n \geq 0, \ (\alpha_1, \ldots, \alpha_n) \in (\overline{\mathbb{Q}}^*)^n, \ (\beta_0, \beta_1, \ldots, \beta_n) \in \overline{\mathbb{Q}}^{n+1} \right\}
\]

Here is the so-called strong six exponentials Theorem of D. Roy ( ([5] Corollary 2 §4 p. 38; see also [7] Corollary 11.16):

---

**Key words and phrases.** Transcendental numbers, logarithms of algebraic numbers, four exponentials Conjecture, six exponentials Theorem, algebraic independence.

Acknowledgements: A suggestion by D. Roy in Banff in November 2004 turned out to be a key point in the proof of the main result. Thanks also to him and to Guy Diaz for their comments on previous versions of this text.
THEOREM 1.1. Let $M$ be a $2 \times 3$ matrix with entries in $\tilde{\mathbb{C}}$:

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}.$$ 

Assume that the two rows of $M$ are linearly independent over $\mathbb{Q}$ and also that the three columns are linearly independent over $\mathbb{Q}$. Then $M$ has rank 2.

Consider the three $2 \times 2$ determinants

$$\Delta_1 = \Lambda_{12}\Lambda_{23} - \Lambda_{13}\Lambda_{22}, \ \Delta_2 = \Lambda_{13}\Lambda_{21} - \Lambda_{11}\Lambda_{23}, \ \Delta_3 = \Lambda_{11}\Lambda_{22} - \Lambda_{12}\Lambda_{21}.$$ 

From the relation

$$\Delta_1 \begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \end{pmatrix} + \Delta_2 \begin{pmatrix} \Lambda_{12} \\ \Lambda_{22} \end{pmatrix} + \Delta_3 \begin{pmatrix} \Lambda_{13} \\ \Lambda_{23} \end{pmatrix} = 0,$$

it follows from the assumptions of Theorem 1.1 that one at least of the three numbers $\Delta_1, \Delta_2, \Delta_3$ is transcendental. We want to prove that one at least of these three numbers is not in $\tilde{\mathbb{C}}$.

If the five rows of the matrix $\begin{pmatrix} M \\ I_3 \end{pmatrix}$ (where $I_3$ is the $3 \times 3$ identity matrix) are linearly dependent over $\mathbb{Q}$, which means that there exists $(\gamma_1, \gamma_2) \in \mathbb{Q}^2 \setminus \{0\}$ such that the three numbers

$$\delta_j = \gamma_1 \Lambda_{1j} + \gamma_2 \Lambda_{2j} \quad (j = 1, 2, 3)$$

are algebraic, then the three numbers $\Delta_1, \Delta_2, \Delta_3$ are in $\tilde{\mathbb{C}}$. Indeed, if $(j, h, k)$ denotes any of the triples $(1, 2, 3), (2, 3, 1), (3, 1, 2)$, then

$$\gamma_1 \Delta_j = \delta_h \Lambda_{2k} - \delta_k \Lambda_{2h} \quad \text{and} \quad \gamma_2 \Delta_j = \delta_h \Lambda_{1k} - \delta_k \Lambda_{1h}.$$ 

Here is the main result of this paper.

THEOREM 1.2. Let $M$ be a $2 \times 3$ matrix with entries in $\tilde{\mathbb{C}}$:

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}.$$ 

Assume that the five rows of the matrix

$$\begin{pmatrix} M \\ I_3 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are linearly dependent over $\mathbb{Q}$, which means that there exists $(\gamma_1, \gamma_2) \in \mathbb{Q}^2 \setminus \{0\}$ such that the three numbers $\Delta_1, \Delta_2, \Delta_3$ are in $\tilde{\mathbb{C}}$. Indeed, if $(j, h, k)$ denotes any of the triples $(1, 2, 3), (2, 3, 1), (3, 1, 2)$, then

$$\gamma_1 \Delta_j = \delta_h \Lambda_{2k} - \delta_k \Lambda_{2h} \quad \text{and} \quad \gamma_2 \Delta_j = \delta_h \Lambda_{1k} - \delta_k \Lambda_{1h}.$$ 

Here is the main result of this paper.
are linearly independent over $\mathbb{Q}$ and that the five columns of the matrix

$$(I_2, M) = \begin{pmatrix} 1 & 0 & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ 0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}$$

are linearly independent over $\mathbb{Q}$. Then one at least of the three numbers

$$\Delta_1 = \begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & \Lambda_{23} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \Lambda_{13} & \Lambda_{11} \\ \Lambda_{23} & \Lambda_{21} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}$$

is not in $\widetilde{\mathcal{L}}$.

If $M$ is a $d \times \ell$ matrix of rank 1, with $d \geq 2$ and $\ell \geq 2$, whose columns are $\mathbb{Q}$-linearly independent, then the $d + \ell$ columns of the matrix $(I_d \ M)$ are also $\mathbb{Q}$-linearly independent. Hence on the one hand Theorem 1.2 generalizes Theorem 1.1. On the other hand, as noticed by G. Diaz, when one of the six numbers $\Lambda_{ij}$ is algebraic, Theorem 1.2 reduces to the next consequence of Theorem 1.1 (further related results are given in [1] and [8]).

**Corollary 1.3.** Let $\Lambda_1, \Lambda_2, \Lambda_3$ be three elements of $\widetilde{\mathcal{L}}$. Assume that $\Lambda_1$ is transcendental and that the three numbers $1, \Lambda_2, \Lambda_3$ are $\mathbb{Q}$-linearly independent. Then one at least of the two numbers $\Lambda_1 \Lambda_2, \Lambda_1 \Lambda_3$ is not in $\widetilde{\mathcal{L}}$.

The simple example

$$M = \begin{pmatrix} 0 & \Lambda_2 & \Lambda_3 \\ \Lambda_1 & 0 & 0 \end{pmatrix}$$

shows that the assumptions of Theorem 1.2 are not sufficient to ensure that none of the three determinants is in $\widetilde{\mathcal{L}}$.

Here is a simple result which follows from Theorem 1.2: Let $\Lambda_1, \Lambda_2, \Lambda_3$ be three elements in $\widetilde{\mathcal{L}}$ such that $1, \Lambda_1, \Lambda_2, \Lambda_3$ are linearly independent over $\mathbb{Q}$. Then one at least of the three numbers

$$\Lambda_1^2 - \Lambda_2 \Lambda_3, \quad \Lambda_2^2 - \Lambda_3 \Lambda_1, \quad \Lambda_3^2 - \Lambda_1 \Lambda_2$$

is not in $\widetilde{\mathcal{L}}$.

In §3 we shall deduce from Theorem 1.2 the following corollary.

**Corollary 1.4.** Let $M$ be a $2 \times 3$ matrix with entries in $\mathcal{L}$:

$$M = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{pmatrix}.$$

Assume that the two rows of $M$ are linearly independent over $\mathbb{Q}$ and also that the three columns of $M$ are linearly independent over $\mathbb{Q}$. Then one at
least of the three numbers

\[(1.5) \quad \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}, \quad \lambda_{12}\lambda_{23} - \lambda_{13}\lambda_{22}, \quad \lambda_{13}\lambda_{21} - \lambda_{11}\lambda_{23}\]

is not in \(\tilde{\mathcal{L}}\).

The six exponentials Theorem of S. Lang ([3], Chap. II § 1) and K. Ramachandra ([4] II § 4) states that, under the assumptions of Corollary 1.4, one at least of the three numbers (1.5) is not zero.

It is expected that a result similar to Theorem 1.2 holds when \(M\) is replaced by a \(2 \times 2\) matrix:

**Conjecture 1.6.** Let \(M\) be a \(2 \times 2\) matrix with entries in \(\tilde{\mathcal{L}}\):

\[
M = \begin{pmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{pmatrix}
\]

Assume that the four rows of the matrix

\[
\begin{pmatrix} M \\ I_2 \end{pmatrix} = \begin{pmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22} \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]

are linearly independent over \(\overline{\mathbb{Q}}\) and that the four columns of the matrix

\[
(\begin{pmatrix} I_2 \\ M \end{pmatrix} = \begin{pmatrix}
1 & 0 & \Lambda_{11} & \Lambda_{12} \\
0 & 1 & \Lambda_{21} & \Lambda_{22}
\end{pmatrix}
\]

are linearly independent over \(\overline{\mathbb{Q}}\). Then the number

\[
\Delta = \begin{vmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{21} & \Lambda_{22}
\end{vmatrix}
\]

is not in \(\tilde{\mathcal{L}}\).

Conjecture 1.6 follows from the conjecture (see for instance [3], Historical Note of Chapter III, [2], Chap. 6 p. 259 and [7], Conjecture 1.15 and [8] Conjecture 1.1) that \(\mathbb{Q}\)-linearly independent logarithms of algebraic numbers are algebraically independent.

2. A consequence of the Linear Subgroup Theorem

Let \(n\) be a positive integer and \(Y\) a \(\overline{\mathbb{Q}}\)-vector subspace of \(\mathbb{C}^n\). We define

\[
\mu(Y, \mathbb{C}^n) = \min_{V \subset \mathbb{C}^n} \frac{\dim \mathbb{Q}(Y/Y \cap V)}{\dim_{\mathbb{C}}(\mathbb{C}^n/V)},
\]

where \(V\) runs over the set of \(\mathbb{C}\)-vector subspaces of \(\mathbb{C}^n\) with \(V \neq \mathbb{C}^n\).
For \( x = (x_1, \ldots, x_n) \in \mathbb{C}^n \) and \( y = (y_1, \ldots, y_n) \in \mathbb{C}^n \) we denote by \( x \cdot y \) the scalar product
\[
x \cdot y = x_1 y_1 + \cdots + x_n y_n.
\]
For \( X \) and \( Y \) two subsets of \( \mathbb{C}^n \), we denote by \( X \cdot Y \) the set of scalar products \( x \cdot y \) where \( x \) ranges over the set \( X \) and \( y \) over \( Y \).

**Theorem 2.1.** Let \( X \) and \( Y \) be two \( \overline{\mathbb{Q}} \)-vector subspaces of \( \mathbb{C}^n \). Assume \( X \) has dimension \( d \) with \( d > n \). Assume further
\[
\mu(Y, \mathbb{C}^n) > \frac{d}{d-n}.
\]
Then the set \( X \cdot Y \) is not contained in \( \mathcal{L} \).

**Proof.** This is essentially Proposition 6.1 of [6], where \( \mathbb{Q} \) is replaced by \( \overline{\mathbb{Q}} \) and the \( \mathbb{Q} \)-vector space \( \mathcal{L} \) by the \( \overline{\mathbb{Q}} \)-vector space \( \mathcal{L} \). Henceforth the proof runs as follows.

Like in Lemma 5.2 of [6], one checks that if \( X \) and \( Y \) are two vector subspaces of \( \mathbb{C}^n \) over \( \overline{\mathbb{Q}} \), of dimensions \( d \) and \( \ell \) respectively, then there exist a positive integer \( n' \leq n \) and two vector subspaces \( X' \) and \( Y' \) of \( \mathbb{C}^{n'} \), of dimensions \( d' \) and \( \ell' \) respectively, such that
\[
\mu(X', \mathbb{C}^{n'}) = \frac{d'}{n'} \geq \frac{d}{n}, \quad \mu(Y', \mathbb{C}^{n'}) = \frac{\ell'}{n'} \geq \mu(Y, \mathbb{C}^n)
\]
and
\[
X' \cdot Y' \subset X \cdot Y.
\]
(2.2)

This shows that for the proof of Theorem 2.1, there is no loss of generality to assume \( \mu(X, \mathbb{C}^n) = d/n \) and \( \mu(Y, \mathbb{C}^n) = \ell/n \). The assumption \( \mu(Y, \mathbb{C}^n) > d/(d-n) \) reduces to \( \ell d > n(\ell + d) \).

Following the argument of Lemma 5.4 in [6], one proves that if \( X \) and \( Y \) are two vector subspaces of \( \mathbb{C}^n \) over \( \overline{\mathbb{Q}} \), of dimensions \( d \) and \( \ell \) respectively, \( X_1 \) a subspace of \( X \) of dimension \( d_1 \) and \( Y_1 \) a subspace of \( Y \) of dimension \( \ell_1 \) such that \( X_1 \cdot Y_1 = \{0\} \), then
\[
(d - d_1)\mu(Y, \mathbb{C}^n) + (\ell - \ell_1)\mu(X, \mathbb{C}^n) \geq n\mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n).
\]
(2.3)

In Lemma 5.4 in [6] an extra assumption is required, namely
\[
\mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n) \geq \mu(X, \mathbb{C}^n) + \mu(Y, \mathbb{C}^n),
\]
but we do not need it here, since our assumption \( X_1 \cdot Y_1 = \{0\} \) is stronger than the assumption in Lemma 5.4 of [6] that \( X_1 \cdot Y_1 \) has rank \( \leq 1 \).
Next we introduce the coefficient \( \theta(M) \) attached to a \( d \times \ell \) matrix \( M \) with entries in \( \mathbb{C} \). It is defined as follows:

\[
\theta(M) = \min \frac{\ell'}{d'},
\]

where \((d', \ell')\) ranges over the set of pairs of integers satisfying \( 0 \leq \ell' \leq \ell, \ 1 \leq d' \leq d \), such that there exist a \( d \times d \) regular matrix \( P \) and a regular \( \ell \times \ell \) regular matrix \( Q \), both with entries in \( \mathbb{Q} \), with

\[
PMQ = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} d^* \\ d' \end{pmatrix} \begin{pmatrix} \ell^* \\ \ell' \end{pmatrix}
\]

From (2.3) with \( d_1 = d' \) and \( \ell_1 = \ell' \) it follows that if

\[
X = \mathbb{Q}x_1 + \cdots + \mathbb{Q}x_d \quad \text{and} \quad Y = \mathbb{Q}y_1 + \cdots + \mathbb{Q}y_\ell
\]

are again two vector subspaces of \( \mathbb{C}^n \) over \( \mathbb{Q} \), of dimensions \( d \) and \( \ell \) respectively, satisfying \( \mu(X, \mathbb{C}^n) = d/n \), then the matrix

\[(2.4) \quad M = (x_i \cdot y_j)_{1 \leq i \leq d, \ 1 \leq j \leq \ell} \]

has

\[
\theta(M) \geq \frac{n}{d} \cdot \mu(Y, \mathbb{C}^n).
\]

In particular if \( \mu(X, \mathbb{C}^n) = d/n \) and \( \mu(Y, \mathbb{C}^n) = \ell/n \), then \( \theta(M) = \ell/d \).

Finally Theorem 4 in [5] (which is Proposition 11.19 or Theorem 12.19 in [7]) shows that the rank \( r \) of a \( d \times \ell \) matrix \( M \) with entries in \( \mathbb{L} \) satisfies

\[
r \geq \frac{d\theta}{1 + \theta},
\]

where \( \theta = \theta(M) \). Using this result for the matrix \( M \) given by (2.4) whose rank \( r \) is \( \leq n \), one concludes that if \( \mu(X, \mathbb{C}^n) = d/n \) and \( \mu(Y, \mathbb{C}^n) = \ell/n \) with \( X \cdot Y \subset \mathbb{L} \), then

\[
n \geq \frac{\ell d}{\ell + d}.
\]

Theorem 2.1 follows. \( \square \)

**Remark.** Theorem 1.1 is equivalent with the case \( n = 1 \) of Theorem 2.1.
3. Proof of the main results

In this section we prove Theorem 1.2 and Corollary 1.4.

**Proof of Theorem 1.2.** Assume that the hypotheses of Theorem 1.2 are satisfied. Define elements $v_1, \ldots, v_5$ in $\mathbb{C}^2$ by

$$v_1 = e_1, \quad v_2 = e_2, \quad v_{2+j} = (A_{1j}, A_{2j}), \quad (j = 1, 2, 3),$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For $v = (x, y) \in \mathbb{C}^2$, set $v' = (-y, x)$, so that $v' \cdot v = 0$. Consider the $5 \times 5$ matrix

$$A = \langle v'_i \cdot v_j \rangle_{1 \leq i, j \leq 5}.$$

From its very definition, it is plain that $A$ has rank 2. Explicitly one has

$$A = \begin{pmatrix}
0 & 1 & A_{21} & A_{22} & A_{23} \\
-1 & 0 & -A_{11} & -A_{12} & -A_{13} \\
-A_{21} & A_{11} & 0 & \Delta_3 & -\Delta_2 \\
-A_{22} & A_{12} & -\Delta_3 & 0 & \Delta_1 \\
-A_{23} & A_{13} & \Delta_2 & -\Delta_1 & 0
\end{pmatrix}.$$

Let $X$ be the $\overline{\mathbb{Q}}$-vector space spanned by $v_1, \ldots, v_5$ in $\mathbb{C}^2$ and similarly let $Y$ be the subspace of $\mathbb{C}^2$ spanned by $v'_1, \ldots, v'_5$ over $\overline{\mathbb{Q}}$. We claim

$$\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) \geq 2. \tag{3.1}$$

The equality $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2)$ follows from the fact that the map $(x, y) \mapsto (-y, x)$ is an automorphism of $\mathbb{C}^2$.

Since the five columns of $(I_2 \ M)$ are linearly independent over $\overline{\mathbb{Q}}$, $\dim_{\overline{\mathbb{Q}}} X = 5$.

Let $V$ be a vector subspace of $\mathbb{C}^2$ of dimension 1 and let $t_1z_1 + t_2z_2 = 0$ be an equation of $V$ in $\mathbb{C}^2$, with $(t_1, t_2) \in \mathbb{C}^2 \setminus \{0\}$. Consider the linear map

$$p : \mathbb{C}^2 \to \mathbb{C}$$

$$(z_1, z_2) \mapsto t_1z_1 + t_2z_2$$

whose kernel is $V$. Since the five rows of

$$\begin{pmatrix} M \\ I_3 \end{pmatrix}$$

are $\overline{\mathbb{Q}}$-linearly independent,

$$\dim_{\overline{\mathbb{Q}}}(X \cap V) = \dim_{\overline{\mathbb{Q}}} p(X) \geq 2.$$

This completes the proof of (3.1).

From (3.1) we deduce that the hypothesis $\mu(Y, \mathbb{C}^2) > d/(d-n)$ of Theorem 2.1 is satisfied with $d = 5$ and $n = 2$, hence the set $X \cdot Y$ is not contained in $\mathbb{L}$. Consequently one at least of the three numbers $\Delta_1, \Delta_2, \Delta_3$ is not in $\mathbb{L}$.
This completes the proof of the Main Theorem 1.2. \hfill \Box

Remark. In (3.1) we may have equality: for instance if $\Lambda_{22} = \Lambda_{23} = 0$ then $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) = 2$.

However the proof of Theorem 2.1 shows that in the case $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) < 5/2$, Theorem 1.2 should follow from Theorem 1.1. Indeed after a change of variables rational over $\overline{\mathbb{Q}}$ one needs only to consider a matrix

$$M = \begin{pmatrix} 0 & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & 0 & 0 \end{pmatrix},$$

which is the situation of Corollary 1.3. If $X$ is the $\overline{\mathbb{Q}}$-subspace of $\mathbb{C}^2$ spanned by

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (0, \Lambda_{21}), \quad v_4 = (\Lambda_{12}, 0), \quad v_4 = (\Lambda_{13}, 0)$$

and $Y$ the subspace spanned by

$$v'_1 = (0, 1), \quad v'_2 = (-1, 0), \quad v'_3 = (-\Lambda_{21}, 0), \quad v'_4 = (0, \Lambda_{12}), \quad v'_4 = (0, \Lambda_{13}),$$

then

$$X' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\Lambda_{12} + \overline{\mathbb{Q}}\Lambda_{13} \quad \text{and} \quad Y' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\Lambda_{21}$$

are $\overline{\mathbb{Q}}$-subspaces of $\mathbb{C}$ satisfying (2.2). Here $\mu(X', \mathbb{C}) = 3 > d/n = 5/2$ and $\mu(Y', \mathbb{C}) = 2 = \mu(Y, \mathbb{C}^2)$.

Proof of Corollary 1.4. From Baker's Theorem it follows that if $Y_0$ is a $\mathbb{Q}$-vector subspace of $\mathcal{L}^n$ of dimension $\ell$, then the $\overline{\mathbb{Q}}$-vector subspace of $\overline{\mathbb{Q}}^n$ spanned by $\overline{\mathbb{Q}}^n \cup Y_0$ has dimension $\ell + n$ (see Exercise 1.5 (iii) of [7]). Taking firstly $n = 2$, $\ell = 3$, and secondly $n = 3$, $\ell = 2$, we deduce that the matrix $M$ of corollary 1.4 satisfies the assumptions of Theorem 1.2. Corollary 1.4 follows. \hfill \Box

4. Erratum to [8]

We take the opportunity of this paper to point out a mistake in the statement of Corollary 2.12 p. 347 of [8]: the assumption that $\Lambda_{21}$ is not zero and $\Lambda_{11}/\Lambda_{21}$ is transcendental should be replaced by the assumption that the three numbers $1, \Lambda_{11}$ and $\Lambda_{21}$ are linearly independent over the field of algebraic numbers. Otherwise a counterexample is obtained for instance with $\Lambda_{21} = 1$ and $\Lambda_{2j} = 0$ for $2 \leq j \leq 5$. 
References


2000 Mathematics Subject Classification. 11J81 11J86 11J89.
Email address: miw@math.jussieu.fr
URL: http://www.math.jussieu.fr/~miw/