October 26 - 30, The Kerala School of Mathematics (KSoM), Kozhikode.
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On the number of integers represented by families of binary binomial forms (joint work with Etienne Fouvry)

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Abstract

In an earlier joint work with Etienne Fouvry, we investigated the number of integers represented by a cyclotomic form of degree larger than a given value. Last year for the ICCGNFRT2022 I explained how we extended this previous result of ours to more general families of binary forms with integer coefficients. At that time our method did not apply to families of irreducible forms having at least one real root. Recently we added one more ingredient in our work, namely a lower bound for linear forms in logarithms, which enables us to deal for instance with families of binary binomial forms $aX^d + bY^d$.

ICCGNFRT-2018

Cyclotomic binary forms :

$$X^k - Y^k = \prod_{n|k} \Phi_n(X,Y).$$

Number of integers represented by one of $\Phi_n(X,Y)$ with $\varphi(n) \geq 2$.

$$\#\{m \leq N,\ m = \Phi_n(x,y),\ \max\{|x|,|y|\} \geq 2,\ \varphi(n) \geq 2\}.$$

Asymptotic : $\leq CN(\log N)^{-1/2}$.

Fouvry, É., Levesque, C. & Waldschmidt, M., *Representation of integers by cyclotomic binary forms*, Acta Arith., (**184**) (2018), no. 1, 67–86.
https://doi.org/10.4064/aa171012-24-12
http://arxiv.org/abs/1701.01230

ICCGNFRT-2019

For $d \geq 4$, number of integers represented by one of $\Phi_n(X,Y)$ with $\varphi(n) \geq d$

$$\#\{m \leq N,\ m = \Phi_n(x,y),\ \max\{|x|,|y|\} \geq 2,\ \varphi(n) \geq d\}.$$

Asymptotic : $\leq CN^{2/d}$ if there exists n with $\varphi(n) = d$.

Étienne Fouvry & Michel Waldschmidt, *Sur la représentation des entiers par des formes cyclotomiques de grand degré.* Bull. Soc. Math. France, **148** 2 (2020), 253-282.
https://doi.org/10.24033/bsmf.2805
arXiv: 1909.01892 [math.NT]
Zbl 1455.11066 MR4124501

ICCGNFRT-2022.

Regular families of binary forms of degree ≥ 3.
Examples : cyclotomic forms, products of linear and quadratic binary forms, **positive definite** binary binomial forms.

Étienne Fouvry & Michel Waldschmidt, Number of integers represented by families of binary forms (I).
Acta Arithmetica, online First, June 2023.
https://doi.org/10.4064/aa220606-16-2
arXiv: 2206.03733 [math.NT]

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Binary binomial forms $aX^d + bY^d$.

Étienne Fouvry & Michel Waldschmidt, Number of integers represented by families of binary forms (II) : binomial forms. Acta Arithmetica, to appear.
arXiv: 2306.02462 [math.NT]

Cam Stewart – Stanley Yao Xiao : a single form

Let d, a and b in $\mathbb{Z}$, with $ab \neq 0$ and $d \geq 3$. Then as $N \to \infty$,

$$\#\{m \mid -N \leq m \leq N,\ m = ax^d + by^d\} = C_{a,b,d} N^{2/d} + O(N^\beta)$$

with $\beta < 2/d$ and $C_{a,b,d} = A_{F_{a,b,d}} W_{F_{a,b,d}}$, where

$$A_{F_{a,b,d}} = \iint_{|ax^d+by^d| \leq 1} dxdy$$

Cameron L. Stewart & Stanley Yao Xiao, *On the representation of integers by binary forms*,
Math. Ann. **375** (2019), no. 1-2, 133–163.
https://doi.org/10.1007/s00208-019-01855-y
Zbl 1464.11035 MR4000237

W_F

- If a/b is not a d–th power of a rational number, then

$$W_{F_{a,b,d}} = \begin{cases} 1 & \text{if } d \text{ is odd,} \\ 1/4 & \text{if } d \text{ is even.} \end{cases}$$

- If a/b is a d–th power of a rational number say $a/b = (A/B)^d$ then

$$W_{F_{a,b,d}} = \begin{cases} 1 - 1/(2|AB|) & \text{if } d \text{ is odd,} \\ (1 - 1/(2|AB|))/4 & \text{if } d \text{ is even.} \end{cases}$$

A_F

- If d is odd then

$$A_{F_{a,b,d}} = \frac{1}{d|ab|^{1/d}} \left(\frac{2\Gamma(1-2/d)\Gamma(1/d)}{\Gamma(1-1/d)} + \frac{\Gamma^2(1/d)}{\Gamma(2/d)} \right).$$

- If d is even

$$A_{F_{a,b,d}} = \frac{2}{d|ab|^{1/d}} \frac{\Gamma^2(1/d)}{\Gamma(2/d)} \qquad \text{if } ab > 0,$$

$$A_{F_{a,b,d}} = \frac{4}{d|ab|^{1/d}} \frac{\Gamma(1-2/d)\Gamma(1/d)}{\Gamma(1-1/d)} \qquad \text{if } ab < 0.$$

Family of binary binomial forms $aX^d + bY^d$

For each $d \geq 3$, let $\mathcal{E}_d$ be a finite set of $(a,b) \in \mathbb{Z}^2$ with $ab \neq 0$ and $\mathcal{F}_d$ the set of binary binomial forms $aX^d + bY^d$ with $(a,b) \in \mathcal{E}_d$.
For $m \in \mathbb{Z}$, let

$$\mathcal{G}_{\geq d}(m) = \Big\{(d', a, b, x, y) \ \mid \ m = ax^{d'} + by^{d'} \ \text{ with } \\ d' \geq d,\ (a,b) \in \mathcal{E}_{d'},\ (x,y) \in \mathbb{Z}^2 \ \text{ and } \ \max\{|x|,|y|\} \geq 2\Big\}.$$

For $d \geq 3$, let

$$\mathcal{R}_{\geq d} = \{m \in \mathbb{Z} \ \mid \ \mathcal{G}_{\geq d}(m) \neq \emptyset\}$$

and for $N \geq 1$, let $\mathcal{R}_{\geq d}(N) = \mathcal{R}_{\geq d} \cap [-N, N]$. So $\#\mathcal{R}_{\geq d}(N)$ is the number of $m \leq N$ which are represented by one of the forms $aX^{d'} + bY^{d'}$ with $d' \geq d$ and $(a,b) \in \mathcal{E}_{d'}$.

Isomorphisms between two binary binomial forms

Let $(a,b) \in \mathbb{Z}^2$ and $(a,b) \in \mathbb{Z}^2$ satisfy $ab \neq 0$ and $a'b' \neq 0$ and let $d \geq 2$. If the two conditions

(C1) : For every $(a,b) \neq (a',b') \in \mathcal{E}_d$, at least one of ratios a/a' and b/b' is not the d–th power of a rational number,

(C2) : For every $(a,b) \neq (a',b') \in \mathcal{E}_d$, at least one of ratios a/b' and b/a' is not the d–th power of a rational number

are satisfied, then the two forms $aX^d + bY^d$ and $a'X^d + b'Y^d$ are not isomorphic (and conversely).

Family of positive definite binary binomial forms

Assume $a > 0$, $b > 0$ for all $(a, b) \in \mathcal{E}_d$ for all $d \geq 4$ and $\mathcal{E}_d = \emptyset$ for odd d. Assume further

$$\frac{1}{d} \log(\sharp\mathcal{E}_d + 1) \to 0 \quad \text{as} \quad d \to \infty.$$

Then

(a) For all $m \in \mathbb{Z} \setminus \{0, 1\}$ and all $d \geq 4$, the set $\mathcal{G}_{\geq d}(m)$ is finite. Furthermore, for all $d \geq 4$ and all $\epsilon > 0$, we have, as $|m| \to \infty$,

$$\sharp\mathcal{G}_{\geq d}(m) = O\left(|m|^{(1/d)+\epsilon}\right).$$

(b) Let $d \geq 4$ be an integer such that the above conditions *(C1)* and *(C2)* hold. We have, as $N \to \infty$,

$$\sharp\mathcal{R}_{\geq d}(N) = \left(\sum_{(a,b) \in \mathcal{E}_d} C_{a,b,d} \right) N^{2/d} + O\Big(N^{\beta}\Big).$$

Family of binary binomial forms (general case)

Let $\epsilon > 0$. There exists an constant $\eta > 0$ depending only on ϵ with the following property. Assume that there exists $d_0 > 0$ such that, for all $d \geq d_0$, we have the inequality

$$\max_{(a,b)\in\mathcal{E}_d} \{|a|, |b|\} \leq \exp(\eta d/\log d).$$

Then

(a) For all $m \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and all $d \geq 3$, the set $\mathcal{G}_{\geq d}(m)$ is finite. Furthermore, for all $d \geq 3$, we have, as $|m| \to \infty$,

$$\sharp\mathcal{G}_{\geq d}(m) = O\left(|m|^{(1/d)+\epsilon}\right).$$

(b) Let $d \geq 3$ be an integer such that the above conditions *(C1)* and *(C2)* hold. We have, as $N \to \infty$,

$$\sharp\mathcal{R}_{\geq d}(N) = \left(\sum_{(a,b)\in\mathcal{E}_d} C_{a,b,d}\right) N^{2/d} + O\Big(N^{\beta}\Big).$$

Asymptotic estimate

Let $d_0 \geq 3$ be an integer. Let λ and μ be two real numbers such that $\lambda > 2$ and

$$0 < \mu < 2^{-81} 3^{-15} \frac{\lambda - 2}{\lambda}.$$

Assume

$$\max\{|a|, |b|\} \leq \exp(\mu d / \log d)$$

for all $(a, b) \in \mathcal{E}_d$ and $d \geq d_0$. Then

(a) For every $m \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and every $d \geq 3$, the set $\mathcal{G}_{\geq d}(m)$ is finite. Furthermore, for every $\epsilon > 0$ and as $|m| \to \infty$, we have

$$\sharp \mathcal{G}_{\geq d}(m) = O\left(|m|^{\epsilon + \lambda/(2d)}\right).$$

(b) For every $d \geq 3$, there exists $N_0 > 0$ such that, for every $N \geq N_0$, we have

$$\sharp \mathcal{R}_{\geq d}(N) \leq N^{\lambda/d}.$$

Diophantine tool : lower bound for a linear form in logarithms

Let d, a, b, x and y be rational integers. Let

$$\mathcal{A} := \max\{|a|, |b|\}, \quad X := \max\{|x|, |y|\}.$$

Assume $d \geq 2$, $\mathcal{A} \geq 2$, $X \geq 2$ and $ax^d + by^d \neq 0$. Then we have the lower bound

$$|ax^d + by^d| \geq \max\{|ax^d|, |by^d|\} \exp\{-2^{79} 3^{15} (\log d)(\log X)(\log \mathcal{A})\}.$$

$\mathcal{R}_{\geq d, X_0}(N)$

Let X_0 be an integer ≥ 2. we introduce the following subset of $\mathcal{R}_{\geq d}$:

$$\mathcal{R}_{\geq d, X_0} = \Big\{ m \in \mathbb{Z} \ \mid \ \text{there exists } (d', a, b, x, y) \text{ such that } m = ax^{d'} + by^{d'} \text{ with } d' \geq d,\ (a,b) \in \mathcal{E}_{d'},\ (x,y) \in \mathbb{Z}^2 \text{ and } \max\{|x|, |y|\} \geq X_0 \Big\},$$

such that $\mathcal{R}_{\geq d} = \mathcal{R}_{\geq d, 2}$. For N a positive integer we also denote

$$\mathcal{R}_{\geq d, X_0}(N) = \mathcal{R}_{\geq d, X_0} \cap [-N, N].$$

Conditional asymptotic uniform estimate

Let $\epsilon > 0$. Assume either

• a Conjecture on lower bounds for linear forms in logarithms (S. Lang, Elliptic Curves Diophantine Analysis) is true for this ϵ

or

• the abc Conjecture is true for this ϵ.

Let $\lambda > 2$. Let d_0 be a sufficiently large integer (depending on λ and ϵ) and let $X_0 \geq 2$. Assume

$$\max\{|a|, |b|\} \leq X_0^{d/d_0}$$

for all $(a, b) \in \mathcal{E}_d$ and $d \geq d_0$.

Then for every $d \geq d_0$, we have

$$\sharp\mathcal{R}_{\geq d, X_0}(N) \leq N^{\lambda/d}.$$

Regular family

Let $\mathcal{F}$ be an infinite set of binary forms with discriminants different from zero and with degrees ≥ 3. We assume that for each $d \geq 3$, the subset $\mathcal{F}_d$ of $\mathcal{F}$ of forms with degree d is finite. We will say this set $\mathcal{F}$ is *regular* if there exists a positive integer A satisfying the following two conditions
(i) Two forms of the family $\mathcal{F}$ are $\mathrm{GL}(2, \mathbb{Q})$–isomorphic if and only if they are equal,
(ii) For all $\epsilon > 0$, there exist two positive integers $N_0 = N_0(\epsilon)$ and $d_0 = d_0(\epsilon)$ such that, for all $N \geq N_0$, the number of integers m in the interval $[-N, N]$ for which there exists $d \in \mathbb{Z}$, $(x, y) \in \mathbb{Z}^2$ and $F \in \mathcal{F}_d$ satisfying

$$d \geq d_0, \quad \max\{|x|, |y|\} \geq A \quad \text{and} \quad F(x, y) = m$$

is bounded by N^{ϵ}.

Main result for a regular family

Let $\mathcal{F}$ be a regular family of binary forms. Then for every $d \geq 3$, the quantity

$$\mathcal{R}_{\geq d}(\mathcal{F}, N, A) := \sharp \Big\{ m : 0 \leq |m| \leq N, \text{ there is } F \in \mathcal{F} \text{ with } \deg F \geq d \text{ and } (x,y) \in \mathbb{Z}^2 \text{ with } \max\{|x|,|y|\} \geq A, \text{ such that } F(x,y) = m \Big\}$$

satisfies

$$\mathcal{R}_{\geq d}(\mathcal{F}, N, A) = \left(\sum_{F \in \mathcal{F}_d} A_F W_F \right) \cdot N^{2/d} + O\big(N^{\beta}\big),$$

uniformly as $N \to \infty$.

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