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Transcendental numbers and functions of several variables

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Abstract

Schwarz' Lemma provides an upper bound for the maximum modulus of a complex analytic functions in a disc. When the function has $many$ zeroes, then its maximum modulus is $small$. The proof is easy for functions of a single variable. In the higher dimensional case several statements are available, but the theory is not yet complete.

We survey known results and conjectures together with their connections with transcendental number theory.

Part I: Transcendence Results

Hermite-Lindemann

Gel'fond-Schneider

Schneider-Lang Criterion in one variable

Sketch of proof of the criterion

Part II: Schwarz' Lemma

A single variable

Cartesian products

Schneider-Lang Criterion in several variables (Cartesian products)

Schneider's result on Euler Beta integrals

Conjecture of Nagata - solution by Bombieri

Schwarz' Lemma in several variables: degrees of hypersurfaces

Part III: Back to transcendence

Interpolation determinants (M. Laurent)

Independence of logarithms of algebraic numbers

Schanuel's Conjecture

Transcendence Results

Theorem (Hermite, 1873). *The number e is transcendental.*

Means: for any $P \in \mathbf{Z}[X]$, $P \neq 0$, we have $P(e) \neq 0$.

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Theorem (Lindemann, 1882). *The number π is transcendental.*

Theorem (Hermite-Lindemann). *Let α be a non zero algebraic number. Then one at least of the two numbers α and e^α is transcendental.*

Examples: The numbers e, π, $\log 2$, $e^{\sqrt{2}}$ are transcendental.
Proof: *Select:* $\alpha = 1$, $i\pi$, $\log 2$, $\sqrt{2}$ *respectively.*

Hilbert's seventh problem (1900)

If α *and* β *are algebraic numbers with* $\alpha \neq 0$, $\alpha \neq 1$, *and* β *irrational, then the number* α^β *is transcendental.*

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Theorem (Gel'fond-Schneider, 1934). *Let* ℓ *and* β *be two complex numbers with* $\ell \neq 0$ *and* $\beta \notin \mathbf{Q}$. *Then one at least of the three numbers* e^ℓ, β *and* $e^{\ell\beta}$ *is transcendental.*

Write α for e^ℓ: then one at least of the three numbers α, β and α^β is transcendental.

Hilbert's seventh problem (1900)

If α and β are algebraic numbers with $\alpha \neq 0$, $\alpha \neq 1$, and β irrational, then the number α^β is transcendental.

Theorem (Gel'fond-Schneider, 1934). *Let ℓ and β be two complex numbers with $\ell \neq 0$ and $\beta \notin \mathbf{Q}$. Then one at least of the three numbers e^ℓ, β and $e^{\ell\beta}$ is transcendental.*

Examples: the numbers $2^{\sqrt{2}}$, e^π, $\log 2/\log 3$, are transcendental.
Select: $(\ell, \beta) = (\log 2, \sqrt{2})$, $(i\pi, i)$, $(\log 3, \log 2/\log 3)$, *respectively.*

Order of a complex function.

An entire function f in $\mathbf{C}$ is of order $\leq \varrho$ if there are positive real numbers c and R_0 such that $|f|_R \leq \exp\{cR^\varrho\}$ for any $R \geq R_0$.

Here,

$$|f|_R = \sup_{|z|=R} |f(z)|.$$

Examples.

For any $\epsilon > 0$, the function z is of order $\leq \epsilon$.
For any $\ell \in \mathbf{C}$, the function $e^{\ell z}$ is of order ≤ 1.

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A meromorphic function f in $\mathbf{C}$ is of order $\leq \varrho$ if there are two entire functions g and h of order $\leq \varrho$ with $h \neq 0$ such that $f = g/h$.

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Lemma. *If g and h are entire functions of order $\leq \varrho$ and if $f = g/h$ is also an entire function, then f is of order $\leq \varrho$.*

A meromorphic function f in $\mathbf{C}$ is of order $\leq \varrho$ if there are two entire functions g and h of order $\leq \varrho$ with $h \neq 0$ such that $f = g/h$.

Recall: A *number field* is a finite extension of $\mathbf{Q}$, i.e. a subfield of $\mathbf{C}$ which, as a $\mathbf{Q}$-vector space, has finite dimension.

Examples: $\mathbf{Q}(\sqrt{2})$, or $\mathbf{Q}(e^{2i\pi/n})$, but not $\mathbf{Q}(e)$ neither $\mathbf{Q}(\pi)$

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Also: Two functions f_1, f_2 are *algebraically independent* if $P(f_1, f_2) \neq 0$ whenever P is a non zero polynomial.

Examples: z and e^z, or e^z and $e^{z\sqrt{2}}$, but not e^{2z} and e^{3z}.

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Schneider-Lang Criterion in one variable. *Let K be a number field. Let $f_1, \ldots, f_d$ be meromorphic functions in $\mathbf{C}$ with $d \geq 2$. Assume f_1, f_2 are algebraically independent and of order $\leq \varrho$. Assume*

$$\frac{d}{dz} f_j \in K[f_1, \ldots, f_d] \quad \textit{for} \quad j = 1, \ldots, d.$$

Then the set S of $w \in \mathbf{C}$ which are not pole of any f_j and such that $f_j(w) \in K$ for $j = 1, \ldots, d$ is finite.

Hermite-Lindemann. *Assume* $\alpha \neq 0$ *and* e^α *are algebraic.*
Take

$$K = \mathbf{Q}(\alpha, e^\alpha),$$
$$d = 2,\ f_1(z) = z,\ f_2(z) = e^z,$$
$$S = \{\alpha, 2\alpha, \ldots\}.$$

Gel'fond-Schneider. *Let* $\ell \in \mathbf{C}$, $\ell \neq 0$. *Assume the three numbers* $\alpha = e^\ell$, β *and* $\alpha^\beta = e^{\ell\beta}$ *are algebraic.*
Take

$$K = \mathbf{Q}(\alpha, \beta, \alpha^\beta),$$
$$d = 2,\ f_1(z) = e^z,\ f_2(z) = e^{\beta z},$$
$$S = \{\ell, 2\ell, \ldots\}.$$

Sketch of proof of Schneider-Lang's Criterion

The proof produces an upper bound for the number of elements in S, namely

$$\mathrm{Card}(S) \le 2\varrho[K:\mathbf{Q}].$$

For simplicity assume f_1 and f_2 are entire in $\mathbf{C}$.

Let S' be a finite subset of S with more than $2\varrho[K:\mathbf{Q}]$ elements. Recall that for each $w \in S'$ we have $f_j(w) \in K$ for $j = 1, \ldots, d$. Hence

$$f_j^{(t)}(w) \in K \quad \text{for} \quad w \in S',\ t \ge 0 \ \text{ and } \ j = 1, 2.$$

Our aim is to contradict the algebraic independence of the two functions f_1 and f_2, hence to show that there is a non zero polynomial $P \in K[X_1, X_2]$ such that the function $F = P(f_1, f_2)$ is the zero function.

The first step is to show the existence of $P \neq 0$ for which the associated entire function $F = P(f_1, f_2)$ has many zeroes:

$$F^{(t)}(w) = 0 \quad \text{for} \quad t = 0, 1, \ldots, T \quad \text{and} \quad w \in S'.$$

Here T is an auxiliary parameter which is a sufficiently large positive integer.

The existence of P, together with upper bounds for the degrees of P, follow from linear algebra: one needs to solve a linear system of homogeneous equations.

Using a Lemma of Thue and Siegel (which relies on Dirichlet's box principle) one gets an upper bound for the coefficients of such a P.

The second step is an extrapolation: one proves, for each $T' \geq T$,

$$(\star)_{T'} \quad F^{(t)}(w) = 0 \quad \text{for} \quad t = 0, 1, \ldots, T' \quad \text{and} \quad w \in S'.$$

Clearly this property for any $T' \geq T$ implies $F = 0$.

Assume $(\star)_{T'}$ holds for some $T' \geq T$. Let $w \in S'$. Define $\gamma = F^{(T'+1)}(w)$. We want to prove $\gamma = 0$.

Let r be a positive real number $> \max\{|w| \; ; \; w \in S'\}$.

From the assumption $(\star)_{T'}$ together with Schwarz Lemma we deduce an upper bound for $|F|_r$, hence, by Cauchy's inequalities, an upper bound for $|\gamma|$. Using $\gamma \in K$, an arithmetic argument (Liouville's inequality) yields $\gamma = 0$.

Schwarz Lemma in one variable

Lemma. *Let F be analytic in a disc $|z| \leq R$ of $\mathbf{C}$. Let r be a real number in the range $0 < r < R$. Denote by $n(F, r)$ the number of zeroes (counting multiplicities) of F in the disc $|z| \leq r$. Then*

$$|F|_r \leq |F|_R \left(\frac{R - r}{2r} \right)^{-n(F,r)}.$$

Remark. When $R > 3r$ we have $R - r > 2r$ hence this lemma refines the maximum modulus principle $|F|_r \leq |F|_R$.

Proof of Schwarz Lemma in one variable

Let $\zeta_1, \ldots, \zeta_N$ be the zeroes (counting multiplicities) of F in the disc $|z| \le r$, with $N = n(F, r)$. The function

$$G(z) = F(z) \prod_{h=1}^{N} (z - \zeta_h)^{-1}$$

is analytic in the disc $|z| \le R$, hence by maximum modulus principle satisfies $|G|_r \le |G|_R$. The conclusion follows from

$$|F|_r \le |G|_r (2r)^N \quad \text{and} \quad |G|_R \le |F|_R (R - r)^{-N}.$$

Blaschke Products

Replacing $z - \zeta$ by

$$\frac{R(z-\zeta)}{R^2 - z\zeta}$$

enables one to replace

$$\frac{R-r}{2r} \quad \text{by} \quad \frac{R^2+r^2}{2rR}.$$

Schwarz Lemma in several variables
Cartesian products

Lemma. *Let F be analytic in a polydisc $|z_i| \leq R$ of $\mathbf{C}^n$. Let r be a real number in the range $0 < r < R$. Let $S_1, \ldots, S_n$ be finite subsets of the polydisc $|z_i| \leq r$, each having at least s elements. Let t be a positive integer. Assume F thas a zero of multiplicity $\geq t$ at each point of $S_1 \times \cdots \times S_n$. Then*

$$|F|_r \leq |F|_R \left(\frac{R-r}{2r}\right)^{-ts}.$$

Sketch of Proof of Schwarz Lemma in several variables

Iterate the relation

$$\frac{1}{t-z} = \frac{1}{t-\zeta} + \frac{z-\zeta}{t-\zeta} \cdot \frac{1}{t-z}$$

and use integral formulae (interpolation formulae).

Sketch of Proof of Schwarz Lemma in several variables

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Remark. In case of a single variable, this argument yields only

$$|F|_r \le |F|_R \left(\frac{R-r}{2r} \right)^{-n(F,r)} \frac{R}{R-r}.$$

One eliminates the extra term by a homogeneity argument (*Landau's trick*), applying the estimate for F^k with $k \to \infty$.

Schneider-Lang Criterion in several variables (Cartesian products). *Let* K *be a number field. Let* $f_1, \ldots, f_d$ *be meromorphic functions in* $\mathbf{C}^n$ *with* $d > n$. *Assume* $f_1, \ldots, f_{n+1}$ *are algebraically independent and of order* $\leq \varrho$. *Assume*

$$\frac{\partial}{\partial z_i} f_j \in K[f_1, \ldots, f_d] \quad \textit{for } j = 1, \ldots, d \textit{ and } i = 1, \ldots, n.$$

Let $(y_1, \ldots, y_n)$ *be a basis of* $\mathbf{C}^n$ *over* $\mathbf{C}$. *Then the numbers*

$$f_j(s_1 y_1 + \cdots + s_n y_n), \quad (1 \leq j \leq d, \ (s_1, \ldots, s_n) \in \mathbf{Z}^n)$$

do not all belong to K.

Schneider's result on Euler Beta integrals

Theorem (Th. Schneider, 1940). *Let a and b be two rational numbers such that none of the three numbers a, b, $a+b$ is an integer. Then the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt$$

is transcendental.

The proof rests on Schneider-Lang Criterion for Cartesian products, involving the periods of the Jacobian of the Fermat curve.

Further results on abelian integrals:

Th. Schneider, S. Lang, A. Baker, J. Coates,
D.W. Masser, G. Wüstholz,. . .

Transcendence of values of Euler Gamma function

Lindemann $\Rightarrow \Gamma(1/2) = \sqrt{\pi}$ is transcendental.

Theorem (G.V. Chudnovsky, 1976). $\Gamma(1/4)$ and $\Gamma(1/3)$ are transcendental.

Theorem (G.V. Chudnovsky, 1976): More precisely,

$\Gamma(1/4)$ and π are algebraically independent.

$\Gamma(1/3)$ and π are algebraically independent.

Means: if $P \in \mathbf{Q}[X,Y]$ is a non zero polynomial, then

$$P(\Gamma(1/4),\pi) \neq 0 \quad \text{and} \quad P(\Gamma(1/3),\pi) \neq 0.$$

For $\Gamma(1/5)$, only a weaker result is known:

Two at least of the three numbers

$$\Gamma(1/5),\ \Gamma(2/5),\ \pi$$

are algebraically independent.

Reason:

The Fermat curves $X^4+Y^4=Z^4$ and $X^3+Y^3=Z^3$ have genus 1, while $X^5+Y^5=Z^5$ has genus 2.

A stronger result is known for $\Gamma(1/3)$ and $\Gamma(1/4)$:

Theorem (Nesterenko, 1996). The three numbers

$$\Gamma(1/4),\ \pi,\ e^{\pi}$$

are algebraically independent. Also the three numbers

$$\Gamma(1/3),\ \pi,\ e^{\pi\sqrt{3}}$$

are algebraically independent.

Open Problem: Prove that three at least of the four numbers

$$\Gamma(1/5),\ \Gamma(2/5),\ \pi,\ e^{\pi\sqrt{5}}$$

are algebraically independent.

Conjecture of Nagata - solution by Bombieri

Theorem (E. Bombieri, 1970). *Let* K *be a number field. Let* $f_1, \ldots, f_d$ *be meromorphic functions in* $\mathbf{C}^n$ *with* $d > n$. *Assume* $f_1, \ldots, f_{n+1}$ *are algebraically independent and of order* $\le \varrho$. *Assume*

$$\frac{\partial}{\partial z_i} f_j \in K[f_1, \ldots, f_d] \quad \textit{for } j = 1, \ldots, d \textit{ and } i = 1, \ldots, n.$$

Then the set S *of* $w \in \mathbf{C}^n$ *where all* f_j *are regular and such that* $f_j(w) \in K$ *for* $j = 1, \ldots, d$ *is contained in an algebraic hypersurface.*

Schwarz' Lemma in several variables (continued)

Let S be a finite subset of $\mathbf{C}^n$.

Denote by $\omega_1(S)$ the smallest degree of a hypersurface in $\mathbf{C}^n$ containing S.

More generally, if t is a positive integer, denote by $\omega_t(S)$ the smallest degree of a non zero polynomial $P \in \mathbf{C}[z_1, \ldots, z_n]$ satisfying

$$\frac{\partial^{\tau_1}}{\partial z_1^{\tau_1}} \cdots \frac{\partial^{\tau_n}}{\partial z_n^{\tau_n}} P(\zeta) = 0$$

for all $\zeta \in S$ and $(\tau_1, \ldots, \tau_n) \in \mathbf{N}^n$ with $\tau_1 + \cdots + \tau_n < t$.

In case $n = 1$, $\omega_t(S) = t\mathrm{Card}(S)$.

In general,

$$\frac{1}{n}\omega_1(S) \leq \frac{1}{t}\omega_t(S) \leq \omega_1(S).$$

Lemma. *Let S be a finite subset of $\mathbf{C}^n$ and t a positive integer. There exists a positive number $r_0 = r_0(S,t)$ such that, for $R > r > r_0$, for any analytic function F in $|z| \leq R$ vanishing with multiplicity $\geq t$ at each point of S,*

$$|F|_r \leq |F|_R \left(\frac{R}{e^n r}\right)^{-\omega_t(S)}.$$

Tools for the proof of Schwarz' Lemma in several variables involving degrees of hypersurfaces:

- Lelong mass of zeroes (Bombieri-Lang)
- L^2-estimates (Hörmander-Bombieri).

Zero estimates: lower bounds for $\omega_t(S)$:

Hermite, Schneider, Gel'fond, Mahler, Tijdeman,
Masser, Brownawell, Wüstholz, Nesterenko, Philippon,. . .

Open problem: upper bound for $r_0(S,t)$:

Work of J-C. Moreau.

Special case: Fix $y_1, \ldots, y_\ell$ in $\mathbf{C}^n$. For $N \geq 1$ define

$$Y(N) = \{b_1 y_1 + \cdots + b_\ell y_\ell \; ; \; b_j \in \mathbf{Z}, \; |b_j| \leq N\}.$$

Conjecture. *There exists a constant c which depends on $y_1, \ldots, y_\ell$ and n but not on N such that, for any $N \geq 1$,*

$$r_0\big(Y(N), 1\big) \leq cN.$$

Interpolation determinants (M. Laurent)

Lemma (M. Laurent). *Let* $\varphi_1, \ldots, \varphi_L$ *be analytic functions in* $|z| \leq R$ *and let* $\zeta_1, \ldots, \zeta_L$ *be complex numbers in* $|z| \leq r$ *with* $r \leq R$. *Define*

$$\Delta = \det\bigl(\varphi_\lambda(\zeta_\mu)\bigr)_{1\leq\lambda,\mu\leq L}.$$

Then

$$|\Delta| \leq \left(\frac{R}{r}\right)^{-L(L-1)/2} L! \prod_{\lambda=1}^{L} |\varphi_\lambda|_R.$$

Proof. *The function*

$$\Psi(z) = \det\bigl(\varphi_\lambda(z\zeta_\mu)\bigr)_{1\leq\lambda,\mu\leq L}$$

has a zero at $z = 0$ *of order* $\geq L(L-1)/2$.

Generalizations: multiplicities, several variables,. . .

Baker's theorem on linear independence of logarithms of algebraic numbers

Theorem (A. Baker, 1968). *Let* $\ell_1, \ldots, \ell_n$ *be complex numbers which are linearly independent over* $\mathbf{Q}$*. Assume the numbers* $\alpha_i = e^{\ell_i}$ *for* $1 \le i \le n$ *are all algebraic. Then the numbers* $1, \ell_1, \ldots, \ell_n$ *are linearly independent over the field of algebraic numbers.*

Hence there is no non trivial linear relation

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n = 0$$

when the α_i and β_j are all algebraic.

First sketch of proof of Baker's Theorem. Assume

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n$$

Functions: $z_0,\ e^{z_1},\ \ldots,\ e^{z_{n-1}},\ e^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$

Points: $\mathbf{Z}(1, \log \alpha_1, \ldots, \log \alpha_{n-1}) \in \mathbf{C}^n$

Derivatives: $\partial / \partial z_i$, $(0 \le i \le n-1)$.

$n+1$ functions, n variables, 1 point, n derivatives

Another sketch of proof of Baker's Theorem

Assume again

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n$$

Functions: $z_0, \; z_1, \; \ldots, z_{n-1},$

$$e^{z_0} \alpha_1^{z_1} \cdots \alpha_{n-1}^{z_{n-1}} = \exp\{z_0 + z_1 \log \alpha_1 + \cdots + z_{n-1} \log \alpha_{n-1}\}$$

Points: $\{0\} \times \mathbf{Z}^{n-1} + \mathbf{Z}(\beta_0, , \ldots, \beta_{n-1}) \in \mathbf{C}^n$

Derivative: $\partial / \partial z_0$.

$n+1$ functions, n variables, n points, 1 derivative

Algebraic independence of logarithms of algebraic numbers

Conjecture. *Let* $\ell_1, \ldots, \ell_n$ *be complex numbers which are linearly independent over* $\mathbf{Q}$. *Assume the numbers* $\alpha_i = e^{\ell_i}$ *for* $1 \leq i \leq n$ *are all algebraic. Then the numbers* $\ell_1, \ldots, \ell_n$ *are algebraicallly independent.*

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Open problem: *Show that the field*

$$\mathbf{Q}\bigl(\{\ell\,;\; e^{\ell} \in \overline{\mathbf{Q}}^{\times}\}\bigr) = \mathbf{Q}\bigl(\{\log\alpha\,;\; \alpha \in \overline{\mathbf{Q}}^{\times}\}\bigr)$$

has transcendence degree ≥ 2.

Algebraic independence of logarithms of algebraic numbers

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Homogeneous conjecture: under the same hypotheses, *no non zero* homogeneous *polynomial* $P \in \mathbf{Z}[z_1, \ldots, z_n]$ *vanishes at* $(\ell_1, \ldots, \ell_n)$.

Structural rank of a matrix.

Let $M = \bigl(\ell_{ij}\bigr)_{\substack{1\le i\le d\\ 1\le j\le m}}$ be a matrix. Let $(\lambda_1,\ldots,\lambda_s)$ be a basis of the $\mathbf{Q}$-vector space spanned by the numbers ℓ_{ij}. Write

$$M = M_1\lambda_1+\cdots+M_s\lambda_s$$

where the matrices $M_1,\ldots,M_s$ have rational entries. The *structural rank* $\mathrm{r}_{\mathrm{str}}(M)$ of M is the rank of the matrix

$$M_1z_1+\cdots+M_sz_s\in\mathbf{Q}(z_1,\ldots,z_s).$$

According to the homogeneous conjecture of algebraic independence of logarithms, if the entries of M are logarithms of algebraic numbers:

$$e^{\ell_{ij}} \in \overline{\mathbf{Q}},$$

then the rank of the matrix M is equal to $\mathrm{r}_{\mathrm{str}}(M)$.

Conversely, according to Damien Roy, if the rank of any square matrix whose entries are logarithms of algebraic numbers is equal to $\mathrm{r}_{\mathrm{str}}(M)$, then the homogeneous conjecture of algebraic independence of logarithms is true.

Lemma (D. Roy). *For any $P \in \mathbf{Q}[z_1, \ldots, z_s]$ there is a square matrix with entries in the $\mathbf{Q}$-vector space spanned by $\mathbf{Q} + \mathbf{Q}z_1 + \cdots + \mathbf{Q}z_n$ whose determinant is P.*

Theorem. *The rank* r *of a matrix* M *whose entries are logarithms of algebraic numbers satisfies*

$$r \geq \frac{1}{2}\mathrm{r}_{\mathrm{str}}.$$

Schanuel's Conjecture *Let* $x_1, \ldots, x_n$ *be* $\mathbf{Q}$*-linearly independent complex numbers. Then the transcendence degree of the field*

$$\mathbf{Q}(x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n})$$

is at least n.

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