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Diophantine approximation, irrationality and transcendence

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These are informal notes of my course given in April – June 2010 at IMPA (*Instituto Nacional de Matematica Pura e Aplicada*), Rio de Janeiro, Brazil.

We complete the proof of the transcendence of e, following Hermite.

We shall substitute 1 to x in the relations

$$Q(x)e^{\mu x} = P_\mu(x) + R_\mu(x)$$

and deduce simultaneous rational approximations $(p_1/q, p_2/q, \dots, p_m/q)$ to the numbers $e, e^2, \dots, e^m$. In order to use Proposition 14, we need to have independent such approximations. This is a subtle point which Hermite did not find easy to overcome, according to his owns comments: we quote from p. 77 of [3]

Mais une autre voie conduira à une démonstration plus rigoureuse

The following approach is due to K. Mahler, we can view it as an extension of the simple non-vanishing argument used in § 8.1.3 for the irrationality of π.

We fix integers $n_0, \dots, n_1$, all ≥ 1. We set $N = n_0 + \cdots + n_m$. For $j = 0, 1, \dots, m$ we denote by $Q_j, P_{j1}, \dots, P_{jm}$ the Hermite-Padé polynomials attached to the parameters

$$n_0 - \delta_{j0}, n_1 - \delta_{j1}, \dots, n_m - \delta_{jm},$$

where δ_{ji} is Kronecker's symbol

$$\delta_{ji} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

These parameters are said to be *contiguous* to $n_0, n_1, \ldots, n_m$. They are the rows of the matrix

$$\begin{pmatrix} n_0 - 1 & n_1 & n_2 & \cdots & n_m \\ n_0 & n_1 - 1 & n_2 & \cdots & n_m \\ \vdots & \vdots & \ddots & \vdots & \\ n_0 & n_1 & n_2 & \cdots & n_m - 1 \end{pmatrix}.$$

We are going to use the previous results, but one should notice that the sum of the parameters on each row is now $N' = N - 1$, not N as before.

Proposition 148. *There exists a non-zero constant* c *such that the determinant*

$$\Delta = \begin{vmatrix} Q_0 & P_{10} & \cdots & P_{m0} \\ \vdots & \vdots & \ddots & \vdots \\ Q_m & P_{1m} & \cdots & P_{mm} \end{vmatrix}$$

is the monomial cx^{mN}.

Proof. The matrix of degrees of the entries in the determinant defining Δ is

$$\begin{pmatrix} N - n_0 & N - n_1 - 1 & \cdots & N - n_m - 1 \\ N - n_0 - 1 & N - n_1 & \cdots & N - n_m - 1 \\ \vdots & \vdots & \ddots & \vdots \\ N - n_0 - 1 & N - n_1 - 1 & \cdots & N - n_m \end{pmatrix}.$$

Therefore Δ is a polynomial of exact degree $N - n_0 + N - n_1 + \cdots + N - n_m = mN$, the leading coefficient arising from the diagonal. This leading coefficient is $c = c_0 c_1 \cdots c_m$, where c_0 is the leading coefficient of Q_0 and c_μ is the leading coefficient of $P_{\mu\mu}$, $1 \le \mu \le m$.

It remains to check that Δ has a multiplicity at least mN at the origin. Linear combinations of the columns yield

$$\Delta(x) = \begin{vmatrix} Q_0(x) & P_{10}(x) - e^x Q_0(x) & \cdots & P_{m0}(x) - e^{mx} Q_0(x) \\ \vdots & \vdots & \ddots & \vdots \\ Q_m(x) & P_{1m}(x) - e^x Q_m(x) & \cdots & P_{mm}(x) - e^{mx} Q_m(x) \end{vmatrix}.$$

Each $P_{\mu j}(x) - e^{\mu x} Q_j(x)$, $1 \le \mu \le m$, $0 \le j \le m$, has multiplicity at least N at the origin, because for each contiguous triple $(1 \le j \le m)$ we have

$$\sum_{i=0}^{m} (n_i - \delta_{ji}) = n_0 + n_1 + \cdots + n_m - 1 = N - 1.$$

Looking at the multiplicity at the origin, we can write

$$\Delta(x) = \begin{vmatrix} Q_0(x) & \mathcal{O}(x^N) & \cdots & \mathcal{O}(x^N) \\ \vdots & \vdots & \ddots & \vdots \\ Q_m(x) & \mathcal{O}(x^N) & \cdots & \mathcal{O}(x^N) \end{vmatrix}.$$

This completes the proof of Proposition 148. □

Now we fix a sufficiently large integer n and we use the previous results for $n_0 = n_1 = \cdots = n_m = n$ with $N = (m+1)n$. We define, for $0 \le j \le m$, the integers $q_j, p_{1j}, \ldots, p_{nj}$ by

$$(n-1)!q_j = Q_j(1),\ (n-1)!p_{\mu j} = P_{\mu j}(1), \quad (1 \le \mu \le m).$$

Proposition 149. *There exists a constant $\kappa > 0$ independent on n such that*

$$\max_{1\le\mu\le m}\max_{0\le j\le m} |q_i e^{\mu} - p_{\mu j}| \le \frac{\kappa^n}{n!}.$$

Further, the determinant

$$\begin{vmatrix} q_0 & p_{10} & \cdots & p_{m0} \\ \vdots & \vdots & \ddots & \vdots \\ q_m & p_{1m} & \cdots & p_{mm} \end{vmatrix}$$

is not zero.

Proof. Recall Hermite's formulae in Proposition 146:

$$Q_j(x)e^{\mu x} - P_{\mu j}(X) = x^{mn}e^{\mu x}\int_0^{\mu} e^{-xt} f_j(t)dt, \quad (1 \le \mu \le m,\ 0 \le j \le m),$$

where

$$\begin{aligned} f_j(t) &= (t-j)^{-1}\big(t(t-1)\cdots(t-m)\big)^n \\ &= (t-j)^{n-1}\prod_{\substack{1\le i\le m\\ i\ne j}} (t-i)^n. \end{aligned}$$

We substitute 1 to x and we divide by $(n-1)!$:

$$q_j e^{\mu} - p_{\mu j} = \frac{1}{(n-1)!}\big(Q_j(1)e^{\mu} - P_{\mu j}(1)\big) = \frac{e^{\mu}}{(n-1)!}\int_0^{\mu} e^{-t} f_j(t)dt.$$

Now the integral is bounded from above by

$$\int_0^\mu e^{-t}|f_j(t)|dt \le m \sup_{0\le t\le m} |f_j(t)| \le m^{1+(m+1)n}.$$

Finally the determinant in the statement of Proposition 149 is

$$\frac{\Delta(1)}{(n-1)!^{m+1}},$$

where Δ is the determinant of Proposition 148. Hence it does not vanish since $\Delta(1) \neq 0$.

□

Since $\kappa^n/n!$ tends to 0 as n tends to infinity, we may apply the criterion for linear independence Proposition 14. Therefore the numbers $1, e, e^2, \dots, e^m$ are linearly independent, and since this is true for all integers m, Hermite's Theorem on the transcendence of e follows.

Exercise 8. Using Hermite's method as explained in § 8.3, prove that for any non-zero $r \in \mathbf{Q}(i)$, the number e^r is transcendental.

Exercise 9. Let m be a positive integer and $\epsilon > 0$ a real number. Show that there exists $q_0 > 0$ such that, for any tuple $(q, p_1, \dots, p_m)$ of rational integers with $q > q_0$,

$$\max_{1\le\mu\le m} \left| e^\mu - \frac{p_\mu}{q} \right| \ge \frac{1}{q^{1+(1/m)+\epsilon}}.$$

Check that it is not possible to replace the exponent $1 + (1/m)$ by a smaller number.

`Hint.` Consider Hermite's proof of the transcendence of e (§ 8.3.2), especially Proposition 149. First check (for instance, using Cauchy's formulae)

$$\max_{0\le j\le m} \frac{1}{k!}|D^k f_j(\mu)| \le c_1^n,$$

where c_1 is a positive real number which does not depend on n. Next, check that the numbers p_j and $q_{\mu j}$ satisfy

$$\max\{q_j, |p_{\mu j}|\} \le (n!)^m c_2^m$$

for $1 \le \mu \le m$ and $0 \le j \le n$, where again $c_2 > 0$ does not depend on n. Then repeat the proof of Hermite in § 8.3 with n satisfying

$$(n!)^m c_3^{-2mn} \le q < \big((n+1)!\big)^m c_3^{-2m(n+1)},$$

where $c_3 > 0$ is a suitable constant independent on n. One does not need to compute c_1, c_2 and c_3 in terms of m, one only needs to show their existence so that the proof yields the desired estimate.

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9 Interpolation

9.1 Weierstraß question

Weierstraß (see [3]) initiated the question of investigating the set of algebraic numbers where a given transcendental entire function f takes algebraic values.

Denote by $\overline{\mathbf{Q}}$ the *field of algebraic numbers* (algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$). For an entire function f, we define the *exceptional set* S_f of f as the set of algebraic numbers α such that $f(\alpha)$ is algebraic:

$$S_f := \{\alpha \in \overline{\mathbf{Q}}\,;\; f(\alpha) \in \overline{\mathbf{Q}}\}.$$

For instance, the Hermite–Lindemann's Theorem on the transcendence of $\log \alpha$ and e^{β} for α and β algebraic numbers is the fact that the exceptional set of the function e^z is $\{0\}$. Also, the exceptional set of $e^z + e^{1+z}$ is empty, by the Theorem of Lindemann–Weierstrass. The exceptional set of functions like 2^z or $e^{i\pi z}$ is $\mathbf{Q}$, as shown by the Theorem of Gel'fond and Schneider.

The exceptional set of a polynomial is $\overline{\mathbf{Q}}$ if the polynomial has algebraic coefficients, otherwise it is finite. Also, any finite set of algebraic numbers is the exceptional set of some entire function: for $s \geq 1$ the set $\{\alpha_1, \dots, \alpha_s\}$ is the exceptional set of the polynomial $\pi(z-\alpha_1)\cdots(z-\alpha_s) \in \mathbf{C}[z]$ and also of the transcendental entire function $(z-\alpha_2)\cdots(z-\alpha_s)e^{z-\alpha_1}$. Assuming Schanuel's conjecture, further explicit examples of exceptional sets for entire functions can be produced, for instance $\mathbf{Z}_{\geq 0}$ or $\mathbf{Z}$.

The study of exceptional sets started in 1886 with a letter of Weierstrass to Strauss. This study was later developed by Strauss, Stäckel, Faber – see [3]. Further results are due to van der Poorten, Gramain, Surroca and others (see [1, 5]).

Among the results which were obtained, a typical one is the following: *if A is a countable subset of $\mathbf{C}$ and if E is a dense subset of $\mathbf{C}$, there exist transcendental entire functions f mapping A into E.*

Also, van der Poorten noticed in [4] that there are transcendental entire functions f such that $D^k f(\alpha) \in \mathbf{Q}(\alpha)$ for all $k \geq 0$ and all algebraic α.

The question of possible sets S_f has been solved in [2]: *any set of algebraic numbers is the exceptional set of some transcendental entire function.* Also multiplicities can be included, as follows: define the *exceptional set with multiplicity* of a transcendental entire function f as the subset of $(\alpha, t) \in \overline{\mathbf{Q}} \times \mathbf{Z}_{\geq 0}$ such that $f^{(t)}(\alpha) \in \overline{\mathbf{Q}}$. Here, $f^{(t)}$ stands for the t-th derivative of f, which we denote also by $D^t f$.

Then *any subset of* $\overline{\mathbf{Q}} \times \mathbf{Z}_{\geq 0}$ *is the exceptional set with multiplicities of some transcendental entire function* f. More generally, the main result of [2] is the following:

> *Let* A *be a countable subset of* $\mathbf{C}$. *For each pair* (α, s) *with* $\alpha \in A$, *and* $s \in \mathbf{Z}_{\geq 0}$, *let* $E_{\alpha,s}$ *be a dense subset of* $\mathbf{C}$. *Then there exists a transcendental entire function* f *such that*
>
> $$\left(\frac{d}{dz}\right)^s f(\alpha) \in E_{\alpha,s} \qquad (150)$$
>
> *for all* $(\alpha, s) \in A \times \mathbf{Z}_{\geq 0}$.

One may replace $\mathbf{C}$ by $\mathbf{R}$: it means that one may take for the sets $E_{\alpha,s}$ dense subsets of $\mathbf{R}$, provided that one requires A to be a countable subset of $\mathbf{R}$.

The proof is a construction of an interpolation series on a sequence where each w occurs infinitely often. The coefficients of the interpolation series are selected recursively to be sufficiently small (and nonzero), so that the sum f of the series is a transcendental entire function.

This process yields uncountably many such functions. Further, one may also require that they are algebraically independent over $\mathbf{C}(z)$ together with their derivatives. Furthermore, at the same time, one may request further restrictions on each of these functions f. For instance, given any transcendental function g with $g(0) \neq 0$, one may require $|f|_R \leq |g|_R$ for all $R \geq 0$.

As a very special case of 150 (selecting A to be the set $\overline{\mathbf{Q}}$ of algebraic numbers and each $E_{\alpha,s}$ to be either $\overline{\mathbf{Q}}$ or its complement in $\mathbf{C}$), one deduces the existence of uncountably many algebraic independent transcendental entire functions f such that any Taylor coefficient at any algebraic point α takes a prescribed value, either algebraic or transcendental.

Exercise 10. . Check that a consequence of the main result (150) of [2] is the following.
Let A *be a countable subset of* $\mathbf{C}$. *For any non negative integer* s *and any* $\alpha \in A$, *let* $E_{\alpha s}$ *be a dense subset in* $\mathbf{C}$. *Let* g *be a transcendental entire function with* $g(0) \neq 0$. *Then there exists a set* $\{f_i \; i \in I\}$ *of entire functions, with* I *a set having the power of continuum, with the following properties.*

- *For any* $i \in I$, *any* $\alpha \in A$ *and any integer* $s \geq 0$, $f_i^{(s)}(\alpha) \in E_{\alpha s}$.
- *For any* $i \in I$ *and any real number* $r \geq 0$, $|f_i|_r \leq |g|_r$.
- *The functions* $f_i^{(s)}$, *(*$i \in I$, $s \geq 0$*) are algebraically independent over* $\mathbf{C}(z)$.

`Hint.` Use (150) with A replaced by $A \cup \{z_1, z_2\}$, where z_1, z_2 are two algebraically independent complex numbers which do not belong to A. For $s \geq 0$, set $E_{z_1,s} = \overline{\mathbf{Q}}$. If there is a non–trivial relation of algebraic dependence among some of the functions $f_i^{(s)}$, then there is such a relation with coefficients in $\overline{\mathbf{Q}}(z_1)$. Next select a set of numbers $x_{i,s}$, $i \in I$, $s \geq 0$, having the power of continuum, which are algebraically independent over $\mathbf{Q}(z_1, z_2)$ – it is easy to give explicit examples with Liouville numbers. To produce f_i, set $E_{z_2,s} = \overline{\mathbf{Q}} x_{i,s} \setminus \{0\}$.

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