Updated: *June 21, 2010*

Diophantine approximation, irrationality and transcendence

Michel Waldschmidt

Course N°19, *June 21, 2010*

These are informal notes of my course given in April – June 2010 at IMPA (*Instituto Nacional de Matematica Pura e Aplicada*), Rio de Janeiro, Brazil.

10.7 Elliptic functions

10.7.1 Introduction to elliptic functions

Among many references for this section are the books by Chandrasekharan [4], Chap. 1–6; by S. Lang [16], Chap. 1–6 and [14], § 1–6; by Alain Robert, [20], Chap I; by J. Silverman [23, 24], and by M. Hindry and J. Silverman [9].

The text below is taken from [29] § 2 and § 3.

An elliptic curve may be defined as

- $y^2 = C(x)$ for a squarefree cubic polynomial $C(x)$,
- a connected compact Lie group of dimension 1,
- a complex torus $\mathbf{C}/\Omega$ where Ω is a lattice in $\mathbf{C}$,
- a Riemann surface of genus 1,
- a non-singular cubic in $\mathbf{P}_2(\mathbf{C})$ (together with a point at infinity),
- an algebraic group of dimension 1, with underlying projective algebraic variety.

We shall use the Weierstraß form

$$E = \left\{(t:x:y)\ ;\ y^2t = 4x^3 - g_2xt^2 - g_3t^3\right\} \subset \mathbf{P}_2.$$

Here g_2 and g_3 are complex numbers, with the only assumption $g_2^3 \neq 27g_3^2$, which means that the discriminant of the polynomial $4X^3 - g_2X - g_3$ does not vanish.

An analytic parametrization of the complex points $E(\mathbf{C})$ of E is given by means of *the Weierstraß elliptic function* $\wp$*,* which satisfies the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3. \tag{158}$$

It has a double pole at the origin with principal part $1/z^2$ and also satisfies an addition formula

$$\wp(z_1+z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\cdot\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2. \tag{159}$$

The exponential map of the Lie group $E(\mathbf{C})$ is

$$\begin{array}{rccc} \exp_E : & \mathbf{C} & \to & E(\mathbf{C}) \\ & z & \mapsto & \bigl(1:\wp(z):\wp'(z)\bigr). \end{array}$$

The kernel of this map is a *lattice* in $\mathbf{C}$ (that is a discrete rank 2 subgroup),

$$\Omega = \ker \exp_E = \{\omega \in \mathbf{C}\ ;\ \wp(z+\omega) = \wp(z)\} = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2.$$

Hence $\exp_E$ induces an isomorphism between the quotient additive group $\mathbf{C}/\Omega$ and $E(\mathbf{C})$ with the law given by (159). The elements of Ω are the *periods* of $\wp$. A pair (ω_1, ω_2) of fundamental periods is given by (cf. [30] § 20.32 Example 1)

$$\omega_i = 2\int_{e_i}^{\infty} \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}, \qquad (i = 1, 2),$$

where

$$4x^3 - g_2x - g_3 = 4(x-e_1)(x-e_2)(x-e_3).$$

Indeed, since $\wp'$ is periodic and odd, it vanishes at $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$, hence the values of $\wp$ at these points are the three distinct complex numbers e_1, e_2 and e_3 (recall that the discriminant of $4x^3 - g_2x - g_3$ is not 0).

Conversely, given a lattice Ω, there is a unique Weierstraß elliptic function $\wp_\Omega$ whose period lattice is Ω (see § 10.7.5). We denote its invariants in the differential equation (158) by $g_2(\Omega)$ and $g_3(\Omega)$.

We shall be interested mainly (but not only) with elliptic curves which are defined over the field of algebraic numbers: they have a Weierstraß equation with algebraic g_2 and g_3. However we shall also use the Weierstraß elliptic function associated with the lattice $\lambda\Omega$ where $\lambda \in \mathbf{C}^\times$ may be transcendental; the relations are

$$\wp_{\lambda\Omega}(\lambda z) = \lambda^{-2}\wp_\Omega(z), \qquad g_2(\lambda\Omega) = \lambda^{-4}g_2(\Omega), \qquad g_3(\lambda\Omega) = \lambda^{-6}g_3(\Omega). \tag{160}$$

The lattice $\Omega = \mathbf{Z} + \mathbf{Z}\tau$, where τ is a complex number with positive imaginary part, satisfies

$$g_2(\mathbf{Z} + \mathbf{Z}\tau) = 60G_2(\tau) \quad \text{and} \quad g_3(\mathbf{Z} + \mathbf{Z}\tau) = 140G_3(\tau),$$

where, for $G_k(\tau)$ (with $k \geq 2$) are the Eisenstein series (see, for instance, [22] Chap. VII, § 2.3, [11] Chap. III § 2 or [23] Chap. VI § 3— the normalization in [31] p. 240 is different):

$$G_k(\tau) = \sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} (m + n\tau)^{-2k}. \tag{161}$$

10.7.2 Morphisms between elliptic curves. The modular invariant

If Ω and Ω' are two lattices in $\mathbf{C}$ and if $f : \mathbf{C}/\Omega \to \mathbf{C}/\Omega'$ is an analytic homomorphism, then the map $\mathbf{C} \to \mathbf{C}/\Omega \to \mathbf{C}/\Omega'$ factors through a homothecy $\mathbf{C} \to \mathbf{C}$ given by some $\lambda \in \mathbf{C}$ such that $\lambda\Omega \subset \Omega'$:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\lambda} & \mathbf{C} \\ \downarrow & & \downarrow \\ \mathbf{C}/\Omega & \xrightarrow[f]{} & \mathbf{C}/\Omega' \end{array}$$

If $f \neq 0$, then $\lambda \in \mathbf{C}^\times$ and f is surjective.

Conversely, if there exists $\lambda \in \mathbf{C}$ such that $\lambda\Omega \subset \Omega'$, then $f_\lambda(x + \Omega) = \lambda x + \Omega'$ defines an analytic surjective homomorphism $f_\lambda : \mathbf{C}/\Omega \to \mathbf{C}/\Omega'$. In this case $\lambda\Omega$ is a subgroup of finite index in Ω', hence the kernel of f_λ is finite and there exists $\mu \in \mathbf{C}^\times$ with $\mu\Omega' \subset \Omega$: the two elliptic curves $\mathbf{C}/\Omega$ and $\mathbf{C}/\Omega'$ are *isogeneous*.

If Ω and Ω^* are two lattices, $\wp$ and $\wp^*$ the associated Weierstraß elliptic functions and g_2, g_3 the invariants of $\wp$, the following statements are equivalent:
(i) There is a 2×2 matrix with rational coefficients which maps a basis of Ω to a basis of Ω^*.

(ii) There exists $\lambda \in \mathbf{Q}^\times$ such that $\lambda\Omega \subset \Omega^*$.
(iii) There exists $\lambda \in \mathbf{Z} \setminus \{0\}$ such that $\lambda\Omega \subset \Omega^*$.
(iv) The two functions $\wp$ and $\wp^*$ are algebraically dependent over the field $\mathbf{Q}(g_2, g_3)$.
(v) The two functions $\wp$ and $\wp^*$ are algebraically dependent over $\mathbf{C}$.

The map f_λ is an isomorphism if and only if $\lambda\Omega = \Omega'$.

The number
$$j = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}$$
is the *modular invariant* of the elliptic curve E. Two elliptic curves over $\mathbf{C}$ are isomorphic if and only if they have the same modular invariant.

Set $\tau = \omega_2/\omega_1$, $q = e^{2\pi i\tau}$ and $J(e^{2\pi i\tau}) = j(\tau)$. Then
$$\begin{aligned} J(q) &= q^{-1}\left(1 + 240\sum_{m=1}^{\infty} m^3 \frac{q^m}{1-q^m}\right)^3 \prod_{n=1}^{\infty}(1-q^n)^{-24} \\ &= \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + \cdots \end{aligned}$$
— see [19] § 4.12 or [22] Chap. VII § 3.3 and § 4.

10.7.3 Endomorphisms of an elliptic curve; complex multiplications

Let Ω be a lattice in $\mathbf{C}$. The set of analytic endomorphisms of $\mathbf{C}/\Omega$ is the subring
$$\mathrm{End}(\mathbf{C}/\Omega) = \{f_\lambda \;;\; \lambda \in \mathbf{C} \;\text{ with }\; \lambda\Omega \subset \Omega\}$$
of $\mathbf{C}$. We also call it the ring of endomorphisms of the associated elliptic curve, or of the corresponding Weierstraß $\wp$ function and we identify it with the subring
$$\{\lambda \in \mathbf{C} \;;\; \lambda\Omega \subset \Omega\}$$
of $\mathbf{C}$. The *field of endomorphisms* is the quotient field $\mathrm{End}(\mathbf{C}/\Omega) \otimes_{\mathbf{Z}} \mathbf{Q}$ of this ring.

If $\lambda \in \mathbf{C}$ satisfies $\lambda\Omega \subset \Omega$, then λ is either a rational integer or else an algebraic integer in an imaginary quadratic field. For such a λ, $\wp_\Omega(\lambda z)$ is a rational function of $\wp_\Omega(z)$; the degree of the numerator is λ^2 if $\lambda \in \mathbf{Z}$ and $N(\lambda)$ otherwise (here, N is the norm of the imaginary quadratic field); the degree of the denominator is $\lambda^2 - 1$ if $\lambda \in \mathbf{Z}$ and $N(\lambda) - 1$ otherwise.

Let E be the elliptic curve attached to the Weierstraß $\wp$ function. The ring of endomorphisms $\mathrm{End}(E)$ of E is either $\mathbf{Z}$ or else an order in an imaginary quadratic field k. The latter case arises if and only if the quotient

$\tau = \omega_2/\omega_1$ of a pair of fundamental periods is a quadratic number; in this case the field of endomorphisms of E is $k = \mathbf{Q}(\tau)$ and the curve E has *complex multiplications* – this is the so-called *CM case*. This means also that the two functions $\wp(z)$ and $\wp(\tau z)$ are algebraically dependent. In this case, the value $j(\tau)$ of the modular invariant j is an algebraic integer whose degree is the class number of the quadratic field $k = \mathbf{Q}(\tau)$.

Remark. *From Gel'fond–Schneider Theorem (§ 10.1) one deduces the transcendence of the number*

$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.999\ 999\ 999\ 999\ 250\,072\,59\ldots$$

If we set

$$\tau = \frac{1+i\sqrt{163}}{2}, \quad q = e^{2\pi i\tau} = -e^{-\pi\sqrt{163}},$$

then the class number of the imaginary quadratic field $\mathbf{Q}(\tau)$ is 1, we have $j(\tau) = -(640\ 320)^3$ and

$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$

Also ([6] § 2.4)

$$\left(e^{\pi\sqrt{163}} - 744\right)^{1/3} = 640\,319.999\,999\,999\,999\,999\,999\,999\,390\,31\ldots$$

Let $\wp$ be a Weierstraß elliptic function with field of endomorphisms k. Hence $k = \mathbf{Q}$ if the associated elliptic curve has no complex multiplication, while in the other case k is an imaginary quadratic field, namely $k = \mathbf{Q}(\tau)$, where τ is the quotient of two linearly independent periods of $\wp$. Let $u_1, \ldots, u_d$ be non-zero complex numbers. Then the functions $\wp(u_1 z), \ldots, \wp(u_d z)$ are algebraically independent (over $\mathbf{C}$ or over $\mathbf{Q}(g_2, g_3)$, this is equivalent) if and only if the numbers $u_1, \ldots, u_d$ are linearly independent over k. This generalizes the fact that $\wp(z)$ and $\wp(\tau z)$ are algebraically dependent if and only if the elliptic curve has complex multiplications. Much more general and deeper results of algebraic independence of functions (exponential and elliptic functions, zeta functions...) were proved by W.D. Brownawell and K.K. Kubota [3].

If $\wp$ is a Weierstraß elliptic function with algebraic invariants g_2 and g_3, if E is the associated elliptic curve and if k denotes its field of endomorphisms, then the set

$$\mathcal{L}_E = \Omega \cup \left\{ u \in \mathbf{C} \setminus \Omega \ ;\ \wp(u) \in \overline{\mathbf{Q}} \right\}$$

is a k-vector subspace of $\mathbf{C}$: this is the set of *elliptic logarithms of algebraic points on* E. It plays a role with respect to E similar to the role of $\mathcal{L}$ for the multiplicative group $\mathbf{G}_m$.

Let $k = \mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic field with class number $h(-d) = h$. There are h non-isomorphic elliptic curves $E_1, \ldots, E_h$ with ring of endomorphisms the ring of integers of k. The numbers $j(E_i)$ are conjugate algebraic integers of degree h; each of them generates the Hilbert class field H of k (maximal unramified abelian extension of k). The Galois group of H/k is isomorphic to the ideal class group of k.

Since the group of roots of units of an imaginary quadratic field is $\{-1, +1\}$ except for $\mathbf{Q}(i)$ and $\mathbf{Q}(\varrho)$, where $\varrho = e^{2\pi i/3}$, it follows that there are exactly two elliptic curves over $\mathbf{Q}$ (up to isomorphism) having an automorphism group bigger than $\{-1, +1\}$. They correspond to Weierstraß elliptic functions $\wp$ for which there exists a complex number $\lambda \neq \pm 1$ with $\lambda^2 \wp(\lambda z) = \wp(z)$.

The first one has $g_3 = 0$ and $j = 1728$. An explicit value for a pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4xt^2$$

follows from computations by Legendre using Gauss's lemniscate function ([30] § 22.8) and yields (see [1], as well as Appendix 1 of [28])

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = \frac{1}{2} B(1/4, 1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}} \quad \text{and} \quad \omega_2 = i\omega_1. \tag{162}$$

The lattice $\mathbf{Z}[i]$ has $g_2 = 4\omega_1^4$, thus

$$\sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} (m + ni)^{-4} = \frac{\Gamma(1/4)^8}{2^6 \cdot 3 \cdot 5 \cdot \pi^2} \cdot$$

The second one has $g_2 = 0$ and $j = 0$. Again from computations by Legendre ([30] § 22.81 II) one deduces that a pair of fundamental periods of the elliptic curve

$$y^2 t = 4x^3 - 4t^3$$

is (see once more [1] and Appendice 1 of [28])

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - 1}} = \frac{1}{3} B(1/6, 1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} \quad \text{and} \quad \omega_2 = \varrho\omega_1. \tag{163}$$

The lattice $\mathbf{Z}[\varrho]$ has $g_3 = 4\omega_1^6$, thus

$$\sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} (m+n\varrho)^{-6} = \frac{\Gamma(1/3)^{18}}{2^8\cdot 5\cdot 7\cdot \pi^6}\cdot$$

These two examples involve special values of Euler's Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}t^z\cdot\frac{dt}{t} = e^{-\gamma z}z^{-1}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}, \tag{164}$$

where

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577\,215\,664\,901\,532\,860\,606\,512\,09\ldots$$

is Euler's constant (§ 12.1 in [30]), while Euler's Beta function is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

More generally, the formula of Chowla and Selberg (1966) [5] (see also [2, 7, 8, 10, 12, 26] for related results) expresses periods of elliptic curves with complex multiplications as products of Gamma values: *if k is an imaginary quadratic field and $\mathcal{O}$ an order in k, if E is an elliptic curve with complex multiplications by $\mathcal{O}$, then the corresponding lattice Ω determines a vector space $\Omega\otimes_{\mathbf{Z}}\mathbf{Q}$ which is invariant under the action of k and thus has the form $k\cdot\omega$ for some $\omega\in\mathbf{C}^\times$ defined up to elements in $k^\times$. In particular, if $\mathcal{O}$ is the ring of integers $\mathbf{Z}_k$ of k, then*

$$\omega = \alpha\sqrt{\pi}\prod_{\substack{0<a<d\\(a,d)=1}}\Gamma(a/d)^{w\epsilon(a)/4h},$$

where α is a non-zero algebraic number, w is the number of roots of unity in k, h is the class number of k, ϵ is the Dirichlet character modulo the discriminant d of k.

10.7.4 Standard relations among Gamma values

Euler's Gamma function satisfies the following relations ([30] Chap. XII):
(Translation)

$$\Gamma(z+1) = z\Gamma(z);$$

(Reflection)

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)};$$

(Multiplication) For any positive integer n,

$$\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right)=(2\pi)^{(n-1)/2}n^{-nz+(1/2)}\Gamma(nz).$$

D. Rohrlich conjectured that any multiplicative relation among Gamma values is a consequence of these standard relations, while S. Lang was more optimistic (see [15], [17] I Chap. 2 Appendix p. 66 and [2] Chap. 24):

Conjecture 165 (D. Rohrlich)**.** *Any multiplicative relation*

$$\pi^{b/2}\prod_{a\in\mathbf{Q}}\Gamma(a)^{m_a}\in\overline{\mathbf{Q}}$$

with b and m_a in $\mathbf{Z}$ is a consequence of the standard relations.

Conjecture 166 (S. Lang)**.** *Any algebraic dependence relation with algebraic coefficients among the numbers $(2\pi)^{-1/2}\Gamma(a)$ with $a\in\mathbf{Q}$ is in the ideal generated by the standard relations.*

10.7.5 Quasi-periods of elliptic curves and elliptic integrals of the second kind

Let $\Omega=\mathbf{Z}\omega_1+\mathbf{Z}\omega_2$ be a lattice in $\mathbf{C}$. The *Weierstraß canonical product* attached to this lattice is the entire function σ_Ω defined by ([30] § 20.42)

$$\sigma_\Omega(z)=z\prod_{\omega\in\Omega\setminus\{0\}}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{z^2}{2\omega^2}}.$$

It has a simple zero at any point of Ω.

Hence the Weierstraß sigma function plays, for the lattice Ω, the role which is played by the function

$$z\prod_{n\geq 1}\left(1-\frac{z}{n}\right)e^{z/n}=-e^{\gamma z}\Gamma(-z)^{-1}$$

for the set of positive integers $\mathbf{N}\setminus\{0\}=\{1,2,\dots\}$ (see the infinite product (164) for Euler's Gamma function), and also by the function

$$\pi^{-1}\sin(\pi z)=z\prod_{n\in\mathbf{Z}\setminus\{0\}}\left(1-\frac{z}{n}\right)e^{z/n}$$

for the set $\mathbf{Z}$ of rational integers ([4] Chap. IV § 2).

The Weierstraß sigma function σ associated with a lattice in $\mathbf{C}$ is an entire function of *order* 2:

$$\limsup_{r\to\infty} \frac{1}{\log r} \cdot \log\log \sup_{|z|=r} |\sigma(z)| = 2;$$

the product $\sigma^2\wp$ is also an entire function of order 2 (this can be checked by using infinite products, but it is easier to use the quasi-periodicity of σ, see formula (167) below).

The logarithmic derivative of the sigma function is *the Weierstraß zeta function* $\zeta = \sigma'/\sigma$ whose Laurent expansion at the origin is

$$\zeta(z) = \frac{1}{z} - \sum_{k\ge 2} s_k z^{2k-1},$$

where, for $k \in \mathbf{Z}$, $k \ge 2$,

$$s_k = s_k(\Omega) = \sum_{\substack{\omega\in\Omega\\ \omega\neq 0}} \omega^{-2k} = \omega_1^{-2k} G_k(\tau)$$

The derivative of ζ is $-\wp$. From

$$\wp'' = 6\wp^2 - (g_2/2)$$

one deduces that $s_k(\Omega)$ is a homogenous polynomial in $\mathbf{Q}[g_2, g_3]$ of weight $2k$ for the graduation of $\mathbf{Q}[g_2, g_3]$ determined by assigning to g_2 the degree 4 and to g_3 the degree 6.

As a side remark, we notice that for any $u \in \mathbf{C} \setminus \Omega$ we have

$$\mathbf{Q}(g_2, g_3) \subset \mathbf{Q}\big(\wp(u), \wp'(u), \wp''(u)\big).$$

Since its derivative is periodic, the function ζ is *quasi-periodic*: for each $\omega \in \Omega$ there is a complex number $\eta = \eta(\omega)$ such that

$$\zeta(z+\omega) = \zeta(z) + \eta.$$

These numbers η are the *quasi-periods* of the elliptic curve. If (ω_1, ω_2) is a pair of fundamental periods and if we set $\eta_1 = \eta(\omega_1)$ and $\eta_2 = \eta(\omega_2)$, then, for $(a, b) \in \mathbf{Z}^2$,

$$\eta(a\omega_1 + b\omega_2) = a\eta_1 + b\eta_2.$$

Coming back to the sigma function, one deduces that

$$\sigma(z+\omega_i) = -\sigma(z)\exp\Big(\eta_i\big(z+(\omega_i/2)\big)\Big) \qquad (i=1,2). \tag{167}$$

The zeta function also satisfies an addition formula:

$$\zeta(z_1+z_2) = \zeta(z_1)+\zeta(z_2)+\frac{1}{2}\cdot\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\cdot$$

The Legendre relation relating the periods and the quasi-periods

$$\omega_2\eta_1-\omega_1\eta_2 = 2\pi i,$$

when ω_2/ω_1 has positive imaginary part, can be obtained by integrating $\zeta(z)$ along the boundary of a fundamental parallelogram.

In the case of complex multiplication, if τ is the quotient of a pair of fundamental periods of $\wp$, then the function $\zeta(\tau z)$ is algebraic over the field $\mathbf{Q}\big(g_2, g_3, z, \wp(z), \zeta(z)\big)$.

Examples For the curve $y^2t = 4x^3-4xt^2$ the quasi-periods attached to the pair of fundamental periods (162) are

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \qquad \eta_2 = -i\eta_1; \tag{168}$$

it follows that the fields $\mathbf{Q}(\omega_1,\omega_2,\eta_1,\eta_2)$ and $\mathbf{Q}\big(\pi,\Gamma(1/4)\big)$ have the same algebraic closure over $\mathbf{Q}$, hence the same transcendence degree. For the curve $y^2t = 4x^3-4t^3$ with periods (163), they are

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \qquad \eta_2 = \varrho^2\eta_1. \tag{169}$$

In this case the fields $\mathbf{Q}(\omega_1,\omega_2,\eta_1,\eta_2)$ and $\mathbf{Q}\big(\pi,\Gamma(1/3)\big)$ have the same algebraic closure over $\mathbf{Q}$, hence the same transcendence degree.

10.7.6 Elliptic integrals

Let

$$\mathcal{E} = \{(t:x:y)\in\mathbf{P}_2; y^2t = 4x^3-g_2xt^2-g_3t^3\}$$

be an elliptic curve. The field of rational (meromorphic) functions on $\mathcal{E}$ over $\mathbf{C}$ is $\mathbf{C}(\mathcal{E}) = \mathbf{C}(\wp,\wp') = \mathbf{C}(x,y)$ where x and y are related by the cubic equation $y^2 = 4x^3-g_2x-g_3$. Under the isomorphism $\mathbf{C}/\Omega\rightarrow\mathcal{E}(\mathbf{C})$ given by $(1 : \wp : \wp')$, the differential form dz is mapped to dx/y. The holomorphic differential forms on $\mathbf{C}/\Omega$ are λdz with $\lambda \in \mathbf{C}$.

The differential form $d\zeta = \zeta'/\zeta$ is mapped to $-xdx/y$. The differential forms of second kind on $\mathcal{E}(\mathbf{C})$ are $adz + bd\zeta + d\chi$, where a and b are complex numbers and $\chi \in \mathbf{C}(x, y)$ is a meromorphic function on $\mathcal{E}$.

Assume that the elliptic curve $\mathcal{E}$ is defined over $\overline{\mathbf{Q}}$: the invariants g_2 and g_3 are algebraic. We shall be interested with differential forms which are defined over $\overline{\mathbf{Q}}$. Those of second kind are $adz + bd\zeta + d\chi$, where a and b are algebraic numbers and $\chi \in \overline{\mathbf{Q}}(x, y)$.

An elliptic integral is an integral

$$\int R(x, y)dx$$

where R is a rational function of x and y, while y^2 is a polynomial in x of degree 3 or 4 without multiple roots, with the proviso that the integral cannot be integrated by means of elementary functions. One may transform this integral as follows: one reduces it to an integral of $dx/\sqrt{P(x)}$ where P is a polynomial of 3rd or 4th degree; in case P has degree 4 one replaces it with a degree 3 polynomial by sending one root to infinity; finally one reduces it to a Weierstraß equation by means of a birational transformation. The value of the integral is not modified.

For transcendence purposes, if the initial differential form is defined over $\overline{\mathbf{Q}}$, then all these transformations involve only algebraic numbers.

10.7.7 Transcendence results of numbers related with elliptic functions

The main references for this section are [13, 21, 27, 29].

The first transcendence result on periods of elliptic functions was proved by C.L. Siegel as early as 1932.

Theorem 170 (Siegel, 1932)**.** *Let $\wp$ be a Weierstraß elliptic function with period lattice $\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$. Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then at least one of the two numbers ω_1, ω_2 is transcendental.*

In the case of complex multiplication, it follows from Theorem 170 that *any non-zero period of $\wp$ is transcendental.*

From formulae (162) and (163) it follows as a consequence of Siegel's 1932 result that both numbers $\Gamma(1/4)^4/\pi$ and $\Gamma(1/3)^3/\pi$ are transcendental.

Other consequences of Siegel's result concern the transcendence of the length of an arc of an ellipse [21]

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}\;dx$$

for algebraic a and b, as well as the transcendence of an arc of the lemniscate $(x^2+y^2)^2=2a^2(x^2-y^2)$ with a algebraic.

A further example of application of Siegel's Theorem is the transcendence of values of hypergeometric series related with elliptic integrals

$$\begin{aligned} K(z) &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}} \\ &= \frac{\pi}{2}\cdot {}_2F_1\left(1/2,\ 1/2\ ;\ 1\ \middle|\ z^2\right), \end{aligned}$$

where ${}_2F_1$ denotes Gauss hypergeometric series

$${}_2F_1\left(a,\ b\ ;\ c\ \middle|\ z\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}$$

with $(a)_n=a(a+1)\cdots(a+n-1)$.

Further results on this topic were obtained by Th. Schneider in 1934 and in a joint work by K. Mahler and J. Popken in 1935 using Siegel's method. These results were superseded by Th. Schneider's work in 1936 where he proved a number of definitive results on the subject, including:

Theorem 171 (Schneider, 1936)**.** *Assume that the invariants g_2 and g_3 of $\wp$ are algebraic. Then for any non-zero period ω of $\wp$, the numbers ω and $\eta(\omega)$ are transcendental.*

It follows from Theorem 171 that any non-zero period of an elliptic integral of the first or second kind is transcendental:

Corollary 172. *Let $\mathcal{E}$ be an elliptic curve over $\overline{\mathbf{Q}}$, p_1 and p_2 two algebraic points on $\mathcal{E}(\overline{\mathbf{Q}})$, w a differential form of first or second kind on $\mathcal{E}$ which is defined over $\overline{\mathbf{Q}}$, holomorphic at p_1 and p_2 and which is not the differential of a rational function. Let γ be a path on $\mathcal{E}$ from p_1 to p_2. In case $p_1=p_2$ one assumes that γ is not homologous to 0. Then the number*

$$\int_\gamma w$$

is transcendental.

Examples: Using Corollary 172 and formulae (168) and (169), one deduces that the numbers
$$\Gamma(1/4)^4/\pi^3 \quad \text{and} \quad \Gamma(1/3)^3/\pi^2$$
are transcendental.

The main results of Schneider's 1936 paper are as follows (see [21]):

Theorem 173 (Schneider, 1936)**. 1.** *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Let β be a non-zero algebraic number. Then β is not a pole of $\wp$ and $\wp(\beta)$ is transcendental.*
More generally, if a and b are two algebraic numbers with $(a,b) \neq (0,0)$, then for any $u \in \mathbf{C} \setminus \Omega$ at least one of the two numbers $\wp(u)$, $au + b\zeta(u)$ is transcendental.
2. *Let $\wp$ and $\wp^*$ be two algebraically independent elliptic functions with algebraic invariants g_2, g_3, g_2^*, g_3^*. If $t \in \mathbf{C}$ is not a pole of $\wp$ or of $\wp^*$, then at least one of the two numbers $\wp(t)$ and $\wp^*(t)$ is transcendental.*
3. *Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3. Then for any $t \in \mathbf{C} \setminus \Omega$, at least one of the two numbers $\wp(t)$, e^t is transcendental.*

It follows from Theorem 173.2 that the quotient of an elliptic integral of the first kind (between algebraic points) by a non-zero period is either in the field of endomorphisms (hence a rational number, or a quadratic number in the field of complex multiplications), or a transcendental number.

Here is another important consequence of Theorem 173.2.

Corollary 174 (Schneider, 1936)**.** *Let $\tau \in \mathcal{H}$ be a complex number in the upper half plane $\Im m(\tau) > 0$ such that $j(\tau)$ is algebraic. Then τ is algebraic if and only if τ is imaginary quadratic.*

In this connection we quote Schneider's second problem in [21], which is still open (see papers by Wakabayashi

Conjecture 175 (Schneiders' second problem)**.** *Prove Corollary 174 without using elliptic functions.*

Sketch of proof of Corollary 174 as a consequence of part 2 of Theorem 173.

Assume that both $\tau \in \mathcal{H}$ and $j(\tau)$ are algebraic. There exists an elliptic function with algebraic invariants g_2, g_3 and periods ω_1, ω_2 such that
$$\tau = \frac{\omega_2}{\omega_1} \quad \text{and} \quad j(\tau) = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}.$$

Set $\wp^*(z) = \tau^2\wp(\tau z)$. Then $\wp^*$ is a Weierstraß function with algebraic invariants g_2^*, g_3^*. For $u = \omega_1/2$ the two numbers $\wp(u)$ and $\wp^*(u)$ are algebraic. Hence the two functions $\wp(z)$ and $\wp^*(z)$ are algebraically dependent. It follows that the corresponding elliptic curve has non-trivial endomorphisms, therefore τ is quadratic. $\square$

A quantitative refinement of Schneider's Theorem on the transcendence of $j(\tau)$ given by A. Faisant and G. Philibert in 1984 became useful 10 years later in connection with Nesterenko's result. (see § 12).

We will not review the results related with abelian integrals, but only quote the first result on this topic, which involves the Jacobian of a Fermat curve: in 1941 Schneider proved that *for a and b in $\mathbf{Q}$ with a, b and $a+b$ not in $\mathbf{Z}$, the number*

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental. We notice that in his 1932 paper, C.L. Siegel had already announced partial results on the values of the Euler Gamma function.

Schneider's above mentioned results deal with elliptic (and abelian) integrals of the first or second kind. His method can be extended to deal with elliptic (and abelian) integrals of the third kind (this is Schneider's third problem in [21]).

As pointed out by J-P. Serre in 1979, it follows from the quasi-periodicity of the Weierstraß sigma function (167) that the function

$$F_u(z) = \frac{\sigma(z+u)}{\sigma(z)\sigma(u)} e^{-z\zeta(u)}$$

satisfies

$$F_u(z+\omega_i) = F_u(z) e^{\eta_i u - \omega_i \zeta(u)}.$$

Theorem 176. *Let u_1 and u_2 be two non-zero complex numbers. Assume that g_2, g_3, $\wp(u_1)$, $\wp(u_2)$, β are algebraic and $\mathbf{Z}u_1 \cap \Omega = \{0\}$. Then the number*

$$\frac{\sigma(u_1+u_2)}{\sigma(u_1)\sigma(u_2)} e^{\big(\beta - \zeta(u_1)\big)u_2}$$

is transcendental.

From the next corollary, one can deduce that non-zero periods of elliptic integrals of the third kind are transcendental.

Corollary 177. *For any non-zero period ω and for any $u \in \mathbf{C} \setminus \Omega$ the number $e^{\omega\zeta(u) - \eta u + \beta\omega}$ is transcendental.*

References

[1] M. Abramowitz & I. A. Stegun – *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, A Wiley-Interscience Publication, New York: John Wiley & Sons, Inc; Washington, D.C, 1984, Reprint of the 1972 ed.

[2] Y. André – *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses, vol. 17, Société Mathématique de France, Paris, 2004.

[3] W. D. Brownawell & K. K. Kubota – "The algebraic independence of Weierstrass functions and some related numbers", *Acta Arith.* **33** (1977), no. 2, p. 111–149.

[4] K. Chandrasekharan, *Elliptic functions*, vol. 281 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1985.

[5] S. Chowla & A. Selberg – "On Epstein's zeta-function", *J. reine angew. Math.* **227** (1967), p. 86–110.

[6] H. Cohen – "Elliptic curves", in *From Number Theory to Physics (Les Houches, 1989)*, Springer, Berlin, 1992, p. 212–237.

[7] B. H. Gross – "On the periods of abelian integrals and a formula of Chowla and Selberg", *Invent. Math.* **45** (1978), no. 2, p. 193–211, With an appendix by David E. Rohrlich.

[8] — , "On an identity of Chowla and Selberg", *J. Number Theory* **11** (1979), no. 3 S. Chowla Anniversary Issue, p. 344–348.

[9] M. Hindry & J. H. Silverman – *Diophantine geometry*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000, An introduction.

[10] N. Koblitz – "Gamma function identities and elliptic differentials on Fermat curves", *Duke Math. J.* **45** (1978), no. 1, p. 87–99.

[11] — , *Introduction to elliptic curves and modular forms*, second ed., Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York, 1993.

[12] N. Koblitz & D. Rohrlich – "Simple factors in the Jacobian of a Fermat curve", *Canad. J. Math.* **30** (1978), no. 6, p. 1183–1205.

[13] S. Lang, *Introduction to transcendental numbers*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[14] — , *Elliptic curves: Diophantine analysis*, vol. **231** of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1978.

[15] — , "Relations de distributions et exemples classiques", in *Séminaire Delange-Pisot-Poitou, 19e année: 1977/78, Théorie des nombres, Fasc. 2*, Secrétariat Math., Paris, 1978, p. Exp. No. 40, 6 (= [18] p. 59–65).

[16] — , *Elliptic functions*, second ed., Graduate Texts in Mathematics, vol. 112, Springer-Verlag, New York, 1987, With an appendix by J. Tate.

[17] — , *Cyclotomic fields I and II*, second ed., Graduate Texts in Mathematics, vol. 121, Springer-Verlag, New York, 1990, With an appendix by Karl Rubin.

[18] — , *Collected papers. Vol. III*, Springer-Verlag, New York, 2000, 1978–1990.

[19] Y. Manin – "Cyclotomic fields and modular curves.", *Uspekhi Mat. Nauk* **26** (1971), no. 6, p. 7–71, Engl. Transl. Russ. Math. Surv. **26** (1971), No 6, 7-78.

[20] A. Robert, *Elliptic curves*, Lecture Notes in Mathematics, Vol. **326**, Springer-Verlag, Berlin, 1973. Notes from postgraduate lectures given in Lausanne 1971/72.

[21] T. Schneider, *Einführung in die transzendenten Zahlen.* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. *Introduction aux nombres transcendants.* Traduit de l'allemand par P. Eymard. Gauthier-Villars, Paris 1959.

[22] J.-P. Serre – *Cours d'arithmétique*, Collection SUP: "Le Mathématicien", vol. 2, Presses Universitaires de France, Paris, 1970, reprinted 1977. Engl. transl.: *A course in arithmetic*, Graduate Texts in Mathematics, Vol. 7. Springer-Verlag, New York, 1978.

[23] J. H. Silverman – *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1986.

[24] — , *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994.

[25] M. Waldschmidt, *Nombres transcendants*, Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 402.
`http://www.springerlink.com/content/110312/`

[26] — , "Diophantine properties of the periods of the Fermat curve.", in *Number theory related to Fermat's last theorem*, Proc. Conf., Prog. Math. 26, 79-88 , 1982.

[27] — , *Nombres transcendants et groupes algébriques*, Astérisque, (1987), p. 218. With appendices by Daniel Bertrand and Jean-Pierre Serre.

[28] — , "Transcendance et indépendance algébrique de valeurs de fonctions modulaires", in *Number theory (Ottawa, ON, 1996)*, CRM Proc. Lecture Notes, vol. 19, Amer. Math. Soc., Providence, RI, 1999, p. 353–375.

[29] — , *Elliptic functions and transcendence*, in Surveys in number theory, vol. 17 of Dev. Math., Springer, New York, 2008, pp. 143–188.

[30] E. Whittaker & G. Watson – *A course of modern analysis. An introduction to the general theory on infinite processes and of analytic functions; with an account of the principal transcendental functions. 4th ed., reprinted*, Cambridge: At the University Press. 608 p. , 1962.

[31] D. Zagier – "Introduction to modular forms", in *From Number Theory to Physics (Les Houches, 1989)*, Springer, Berlin, 1992, p. 238–291.