Updated: *June 16, 2010*

Diophantine approximation, irrationality and transcendence

Michel Waldschmidt

Course N°5, *May 3, 2010*

These are informal notes of my course given in April – June 2010 at IMPA (*Instituto Nacional de Matematica Pura e Aplicada*), Rio de Janeiro, Brazil.

6 Continued fractions

We first consider generalized continued fractions of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots}}},$$

which we denote by[5]

$$a_0 + \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \frac{b_3|}{\ddots}\cdot$$

Next we restrict to the special case where $b_1 = b_2 = \cdots = 1$, which yield the simple continued fractions

$$a_0 + \frac{1\,|}{|a_1} + \frac{1\,|}{|a_2} + \cdots = [a_0, a_1, a_2, \ldots],$$

already considered in section § 1.1.

[5] Another notation for $a_0 + \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \cdots + \frac{b_n|}{a_n}$ introduced by Th. Muir and used by Perron in [7] Chap. 1 is

$$K\begin{pmatrix} b_1, \ldots, b_n \\ a_0, a_1, \ldots, a_n \end{pmatrix}$$

6.1 Generalized continued fractions

To start with, $a_0, \dots, a_n, \dots$ and $b_1, \dots, b_n, \dots$ will be independent variables. Later, we shall specialize to positive integers (apart from a_0 which may be negative).

Consider the three rational fractions

$$a_0, \qquad a_0 + \frac{b_1}{a_1} \quad \text{and} \quad a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2}} \cdot$$

We write them as

$$\frac{A_0}{B_0}, \qquad \frac{A_1}{B_1} \quad \text{and} \quad \frac{A_2}{B_2}$$

with

$$\begin{aligned} &A_0 = a_0, \quad A_1 = a_0a_1 + b_1, \quad A_2 = a_0a_1a_2 + a_0b_2 + a_2b_1, \\ &B_0 = 1, \quad\;\; B_1 = a_1, \qquad\qquad B_2 = a_1a_2 + b_2. \end{aligned}$$

Observe that

$$A_2 = a_2A_1 + b_2A_0, \quad B_2 = a_2B_1 + b_2B_0.$$

Write these relations as

$$\begin{pmatrix} A_2 & A_1 \\ B_2 & B_1 \end{pmatrix} = \begin{pmatrix} A_1 & A_0 \\ B_1 & B_0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ b_2 & 0 \end{pmatrix}.$$

Define inductively two sequences of polynomials with positive rational coefficients A_n and B_n for $n \geq 3$ by

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix}. \tag{50}$$

This means

$$A_n = a_nA_{n-1} + b_nA_{n-2}, \qquad B_n = a_nB_{n-1} + b_nB_{n-2}.$$

This recurrence relation holds for $n \geq 2$. It will also hold for $n = 1$ if we set $A_{-1} = 1$ and $B_{-1} = 0$:

$$\begin{pmatrix} A_1 & A_0 \\ B_1 & B_0 \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix}$$

and it will hold also for $n = 0$ if we set $b_0 = 1$, $A_{-2} = 0$ and $B_{-2} = 1$:

$$\begin{pmatrix} A_0 & A_{-1} \\ B_0 & B_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ b_0 & 0 \end{pmatrix}.$$

Obviously, an equivalent definition is

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ b_0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ b_{n-1} & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix}. \tag{51}$$

These relations (51) hold for $n \geq -1$, with the empty product (for $n = -1$) being the identity matrix, as always.

Hence $A_n \in \mathbf{Z}[a_0, \ldots, a_n, b_1, \ldots, b_n]$ is a polynomial in $2n + 1$ variables, while $B_n \in \mathbf{Z}[a_1 \ldots, a_n, b_2, \ldots, b_n]$ is a polynomial in $2n - 1$ variables.

Exercise 6. *Check, for $n \geq -1$,*

$$B_n(a_1, \ldots, a_n, b_2, \ldots, b_n) = A_{n-1}(a_1, \ldots, a_n, b_2, \ldots, b_n).$$

Lemma 52. *For $n \geq 0$,*

$$a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_n|}{|a_n} = \frac{A_n}{B_n}\cdot$$

Proof. By induction. We have checked the result for $n = 0$, $n = 1$ and $n = 2$. Assume the formula holds with $n - 1$ where $n \geq 3$. We write

$$a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|a_{n-1}} + \frac{b_n|}{|a_n} = a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|x}$$

with

$$x = a_{n-1} + \frac{b_n}{a_n}\cdot$$

We have, by induction hypothesis and by the definition (50),

$$a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|a_{n-1}} = \frac{A_{n-1}}{B_{n-1}} = \frac{a_{n-1}A_{n-2} + b_{n-1}A_{n-3}}{a_{n-1}B_{n-2} + b_{n-1}B_{n-3}}\cdot$$

Since A_{n-2}, A_{n-3}, B_{n-2} and B_{n-3} do not depend on the variable a_{n-1}, we deduce

$$a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|x} = \frac{xA_{n-2} + b_{n-1}A_{n-3}}{xB_{n-2} + b_{n-1}B_{n-3}}\cdot$$

The product of the numerator by a_n is

$$\begin{aligned}(a_n a_{n-1} + b_n)A_{n-2} + a_n b_{n-1} A_{n-3} &= a_n(a_{n-1}A_{n-2} + b_{n-1}A_{n-3}) + b_n A_{n-2} \\ &= a_n A_{n-1} + b_n A_{n-2} = A_n\end{aligned}$$

and similarly, the product of the denominator by a_n is

$$\begin{aligned}(a_na_{n-1}+b_n)B_{n-2}+a_nb_{n-1}B_{n-3} &= a_n(a_{n-1}B_{n-2}+b_{n-1}B_{n-3})+b_nB_{n-2}\\ &= a_nB_{n-1}+b_nB_{n-2}=B_n.\end{aligned}$$

□

From (51), taking the determinant, we deduce, for $n\ge -1$,

$$A_nB_{n-1}-A_{n-1}B_n=(-1)^{n+1}b_0\cdots b_n. \tag{53}$$

which can be written, for $n\ge 1$,

$$\frac{A_n}{B_n}-\frac{A_{n-1}}{B_{n-1}}=\frac{(-1)^{n+1}b_0\cdots b_n}{B_{n-1}B_n}\cdot \tag{54}$$

Adding the telescoping sum, we get, for $n\ge 0$,

$$\frac{A_n}{B_n}=A_0+\sum_{k=1}^{n}\frac{(-1)^{k+1}b_0\cdots b_k}{B_{k-1}B_k}\cdot \tag{55}$$

We now substitute for $a_0,a_1,\dots$ and $b_1,b_2,\dots$ rational integers, all of which are ≥ 1, apart from a_0 which may be ≤ 0. We denote by p_n (resp. q_n) the value of A_n (resp. B_n) for these special values. Hence p_n and q_n are rational integers, with $q_n>0$ for $n\ge 0$. A consequence of Lemma 52 is

$$\frac{p_n}{q_n}=a_0+\frac{b_1|}{|a_1}+\cdots+\frac{b_n|}{|a_n}\quad\text{for}\quad n\ge 0.$$

We deduce from (50),

$$p_n=a_np_{n-1}+b_np_{n-2},\qquad q_n=a_nq_{n-1}+b_nq_{n-2}\quad\text{for}\quad n\ge 0,$$

and from (53),

$$p_nq_{n-1}-p_{n-1}q_n=(-1)^{n+1}b_0\cdots b_n\quad\text{for}\quad n\ge -1,$$

which can be written, for $n\ge 1$,

$$\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=\frac{(-1)^{n+1}b_0\cdots b_n}{q_{n-1}q_n}\cdot \tag{56}$$

Adding the telescoping sum (or using (55)), we get the alternating sum

$$\frac{p_n}{q_n}=a_0+\sum_{k=1}^{n}\frac{(-1)^{k+1}b_0\cdots b_k}{q_{k-1}q_k}\cdot \tag{57}$$

Recall that for real numbers a, b, c, d, with b and d positive, we have

$$\frac{a}{b} < \frac{c}{d} \implies \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}. \tag{58}$$

Since a_n and b_n are positive for $n \geq 0$, we deduce that for $n \geq 2$, the rational number

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + b_n p_{n-2}}{a_n q_{n-1} + b_n q_{n-2}}$$

lies between p_{n-1}/q_{n-1} and p_{n-2}/q_{n-2}. Therefore we have

$$\frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \frac{p_{2n}}{q_{2n}} < \cdots < \frac{p_{2m+1}}{q_{2m+1}} < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}. \tag{59}$$

From (56), we deduce, for $n \geq 3$, $q_{n-1} > q_{n-2}$, hence $q_n > (a_n + b_n) q_{n-2}$.

The previous discussion was valid without any restriction, now we assume $a_n \geq b_n$ for all sufficiently large n, say $n \geq n_0$. Then for $n > n_0$, using $q_n > 2b_n q_{n-2}$, we get

$$\left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right| = \frac{b_0 \cdots b_n}{q_{n-1} q_n} < \frac{b_n \cdots b_0}{2^{n-n_0} b_n b_{n-1} \cdots b_{n_0+1} q_{n_0} q_{n_0-1}} = \frac{b_{n_0} \cdots b_0}{2^{n-n_0} q_{n_0} q_{n_0-1}}$$

and the right hand side tends to 0 as n tends to infinity. Hence the sequence $(p_n/q_n)_{n \geq 0}$ has a limit, which we denote by

$$x = a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|a_{n-1}} + \frac{b_n|}{|a_n} + \cdots$$

From (57), it follows that x is also given by an alternating series

$$x = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} b_0 \cdots b_k}{q_{k-1} q_k}.$$

We now prove that x is irrational. Define, for $n \geq 0$,

$$x_n = a_n + \frac{b_{n+1}|}{|a_{n+1}} + \cdots$$

so that $x = x_0$ and, for all $n \geq 0$,

$$x_n = a_n + \frac{b_{n+1}}{x_{n+1}}, \qquad x_{n+1} = \frac{b_{n+1}}{x_n - a_n}$$

and $a_n < x_n < a_n + 1$. Hence for $n \geq 0$, x_n is rational if and only if x_{n+1} is rational, and therefore, if x is rational, then all x_n for $n \geq 0$ are also rational. Assume x is rational. Consider the rational numbers x_n with $n \geq n_0$ and select a value of n for which the denominator v of x_n is minimal, say $x_n = u/v$. From

$$x_{n+1} = \frac{b_{n+1}}{x_n - a_n} = \frac{b_{n+1}v}{u - a_n v} \quad \text{with} \quad 0 < u - a_n v < v,$$

it follows that x_{n+1} has a denominator strictly less than v, which is a contradiction. Hence x is irrational.

Conversely, given an irrational number x and a sequence $b_1, b_2, \ldots$ of positive integers, there is a unique integer a_0 and a unique sequence $a_1, \ldots, a_n, \ldots$ of positive integers satisfying $a_n \geq b_n$ for all $n \geq 1$, such that

$$x = a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|a_{n-1}} + \frac{b_n|}{|a_n} + \cdots$$

Indeed, the unique solution is given inductively as follows: $a_0 = \lfloor x \rfloor$, $x_1 = b_1/\{x\}$, and once $a_0, \ldots, a_{n-1}$ and $x_1, \ldots, x_n$ are known, then a_n and x_{n+1} are given by

$$a_n = \lfloor x_n \rfloor, \quad x_{n+1} = b_{n+1}/\{x_n\},$$

so that for $n \geq 1$ we have $0 < x_n - a_n < 1$ and

$$x = a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|a_{n-1}} + \frac{b_n|}{|x_n}.$$

Here is what we have proved.

Proposition 60. *Given a rational integer a_0 and two sequences $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ of positive rational integers with $a_n \geq b_n$ for all sufficiently large n, the infinite continued fraction*

$$a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|a_{n-1}} + \frac{b_n|}{|a_n} + \cdots$$

exists and is an irrational number.
Conversely, given an irrational number x and a sequence $b_1, b_2, \ldots$ of positive integers, there is a unique $a_0 \in \mathbf{Z}$ and a unique sequence $a_1, \ldots, a_n, \ldots$ of positive integers satisfying $a_n \geq b_n$ for all $n \geq 1$ such that

$$x = a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|a_{n-1}} + \frac{b_n|}{|a_n} + \cdots$$

These results are useful for proving the irrationality of π and e^r when r is a non–zero rational number, following the proof by Lambert. See for instance Chapter 7 (Lambert's Irrationality Proofs) of David Angell's course on Irrationality and Transcendence([6]) at the University of New South Wales:

`http://www.maths.unsw.edu.au/ angell/5535/`

The following example is related with Lambert's proof [20]:

$$\tanh z = \frac{z|}{|1} + \frac{z^2|}{|\,3} + \frac{z^2|}{|\,5} + \cdots + \frac{z^2\quad|}{|2n+1} + \cdots$$

Here, z is a complex number and the right hand side is a complex valued function. Here are other examples (see Sloane's Encyclopaedia of Integer Sequences([7]))

$$\frac{1}{\sqrt{e}-1} = 1 + \frac{2|}{|3} + \frac{4|}{|5} + \frac{6|}{|7} + \frac{8|}{|9} + \cdots = 1.541\,494\,082\ldots \qquad \text{(A113011)}$$

$$\frac{1}{e-1} = \frac{1|}{|1} + \frac{2|}{|2} + \frac{3|}{|3} + \frac{4|}{|4} + \cdots = 0.581\,976\,706\ldots \qquad \text{(A073333)}$$

Remark. *A variant of the algorithm of simple continued fractions is the following. Given two sequences $(a_n)_{n\ge 0}$ and $(b_n)_{n\ge 0}$ of elements in a field K and an element x in K, one defines a sequence (possibly finite) $(x_n)_{n\ge 1}$ of elements in K as follows. If $x = a_0$, the sequence is empty. Otherwise x_1 is defined by $x = a_0 + (b_1/x_1)$. Inductively, once $x_1, \ldots, x_n$ are defined, there are two cases:*

- *If $x_n = a_n$, the algorithm stops.*
- *Otherwise, x_{n+1} is defined by*

$$x_{n+1} = \frac{b_{n+1}}{x_n - a_n}, \qquad \textit{so that} \quad x_n = a_n + \frac{b_{n+1}}{x_{n+1}}\cdot$$

If the algorithm does not stop, then for any $n \ge 1$, one has

$$x = a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|a_{n-1}} + \frac{b_n|}{|x_n}\cdot$$

In the special case where $a_0 = a_1 = \cdots = b_1 = b_2 = \cdots = 1$, the set of x such that the algorithm stops after finitely many steps is the set $(F_{n+1}/F_n)_{n\ge 1}$ of

[6] I found this reference from the website of John Cosgrave `http://staff.spd.dcu.ie/johnbcos/transcendental_numbers.htm`.

[7] `http://www.research.att.com/~njas/sequences/`

quotients of consecutive Fibonacci numbers. In this special case, the limit of

$$a_0 + \frac{b_1|}{|a_1} + \cdots + \frac{b_{n-1}|}{|a_{n-1}} + \frac{b_n|}{|a_n}$$

is the Golden ratio, which is independent of x, of course!

6.2 Simple continued fractions

We restrict now the discussion of § 6.1 to the case where $b_1 = b_2 = \cdots = b_n = \cdots = 1$. We keep the notations A_n and B_n which are now polynomials in $\mathbf{Z}[a_0, a_1, \dots, a_n]$ and $\mathbf{Z}[a_1, \dots, a_n]$ respectively, and when we specialize to integers $a_0, a_1, \dots, a_n \dots$ with $a_n \geq 1$ for $n \geq 1$ we use the notations p_n and q_n for the values of A_n and B_n.

The recurrence relations (50) are now, for $n \geq 0$,

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}, \tag{61}$$

while (51) becomes, for $n \geq -1$,

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{62}$$

From Lemma 52 one deduces, for $n \geq 0$,

$$[a_0, \dots, a_n] = \frac{A_n}{B_n}.$$

Taking the determinant in (62), we deduce the following special case of (53)

$$A_n B_{n-1} - A_{n-1} B_n = (-1)^{n+1}. \tag{63}$$

The specialization of these relations to integral values of $a_0, a_1, a_2 \dots$ yields

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{for } n \geq 0, \tag{64}$$

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{for } n \geq -1, \tag{65}$$

$$[a_0, \dots, a_n] = \frac{p_n}{q_n} \qquad \text{for } n \geq 0 \tag{66}$$

and

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1} \qquad \text{for } n \geq -1. \tag{67}$$

From (67), it follows that for $n \geq 0$, the fraction p_n/q_n is in lowest terms: $\gcd(p_n, q_n) = 1$.

Transposing (65) yields, for $n \geq -1$,

$$\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}$$

from which we deduce, for $n \geq 1$,

$$[a_n, \ldots, a_0] = \frac{p_n}{p_{n-1}} \quad \text{and} \quad [a_n, \ldots, a_1] = \frac{q_n}{q_{n-1}}$$

Lemma 68. *For $n \geq 0$,*

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$

Proof. We multiply both sides of (64) on the left by the inverse of the matrix

$$\begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \qquad \text{which is} \quad (-1)^n \begin{pmatrix} q_{n-2} & -p_{n-2} \\ -q_{n-1} & p_{n-1} \end{pmatrix}.$$

We get

$$(-1)^n \begin{pmatrix} p_n q_{n-2} - p_{n-2} q_n & p_{n-1} q_{n-2} - p_{n-2} q_{n-1} \\ -p_n q_{n-1} + p_{n-1} q_n & 0 \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

□

6.2.1 Finite simple continued fraction of a rational number

Let u_0 and u_1 be two integers with u_1 positive. The first step in Euclid's algorithm to find the gcd of u_0 and u_1 consists in dividing u_0 by u_1:

$$u_0 = a_0 u_1 + u_2$$

with $a_0 \in \mathbf{Z}$ and $0 \leq u_2 < u_1$. This means

$$\frac{u_0}{u_1} = a_0 + \frac{u_2}{u_1},$$

which amonts to dividing the rational number $x_0 = u_0/u_1$ by 1 with quotient a_0 and remainder $u_2/u_1 < 1$. This algorithms continues with

$$u_m = a_m u_{m+1} + u_{m+2},$$

where a_m is the integral part of $x_m = u_m/u_{m+1}$ and $0 \leq u_{m+2} < u_{m+1}$, until some $u_{\ell+2}$ is 0, in which case the algorithms stops with

$$u_\ell = a_\ell u_{\ell+1}.$$

Since the gcd of u_m and u_{m+1} is the same as the gcd of u_{m+1} and u_{m+2}, it follows that the gcd of u_0 and u_1 is $u_{\ell+1}$. This is how one gets the regular continued fraction expansion $x_0 = [a_0, a_1, \ldots, a_\ell]$, where $\ell = 0$ in case x_0 is a rational integer, while $a_\ell \geq 2$ if x_0 is a rational number which is not an integer.

Exercise 7. *Compare with the geometrical construction of the continued fraction given in § 1.1.*
Give a variant of this geometrical construction where rectangles are replaced by segments.

Repeating what was already said in § 1.2, we can state

Proposition 69. *Any finite regular continued fraction*

$$[a_0, a_1, \ldots, a_n],$$

where $a_0, a_1, \ldots, a_n$ *are rational numbers with* $a_i \geq 2$ *for* $1 \leq i \leq n$ *and* $n \geq 0$*, represents a rational number. Conversely, any rational number* x *has two representations as a continued fraction, the first one, given by Euclid's algorithm, is*

$$x = [a_0, a_1, \ldots, a_n]$$

and the second one is

$$x = [a_0, a_1, \ldots, a_{n-1}, a_n - 1, 1].$$

If $x \in \mathbf{Z}$, then $n = 0$ and the two simple continued fractions representations of x are $[x]$ and $[x - 1, 1]$, while if x is not an integer, then $n \geq 1$ and $a_n \geq 2$.

We shall use later (in the proof of Lemma 81 in § 6.3.7) the fact that any rational number has one simple continued fraction expansion with an odd number of terms and one with an even number of terms.

6.2.2 Infinite simple continued fraction of an irrational number

Given a rational integer a_0 and an infinite sequence of positive integers $a_1, a_2, \ldots$, the continued fraction

$$[a_0, a_1, \ldots, a_n, \ldots]$$

represents an irrational number. Conversely, given an irrational number x, there is a unique representation of x as an infinite simple continued fraction

$$x = [a_0, a_1, \ldots, a_n, \ldots]$$

Definitions The numbers a_n are the *partial quotients*, the rational numbers

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

are the *convergents* (in French *réduites*), and the numbers

$$x_n = [a_n, a_{n+1}, \ldots]$$

are the *complete quotients.*

From these definitions we deduce, for $n \geq 0$,

$$x = [a_0, a_1, \ldots, a_n, x_{n+1}] = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}. \tag{70}$$

Lemma 71. *For $n \geq 0$,*

$$q_n x - p_n = \frac{(-1)^n}{x_{n+1}q_n + q_{n-1}}.$$

Proof. From (70) one deduces

$$x - \frac{p_n}{q_n} = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{(x_{n+1}q_n + q_{n-1})q_n}.$$

□

Corollary 72. *For $n \geq 0$,*

$$\frac{1}{q_{n+1} + q_n} < |q_n x - p_n| < \frac{1}{q_{n+1}}.$$

Proof. Since a_{n+1} is the integral part of x_{n+1}, we have

$$a_{n+1} < x_{n+1} < a_{n+1} + 1.$$

Using the recurrence relation $q_{n+1} = a_{n+1}q_n + q_{n-1}$, we deduce

$$q_{n+1} < x_{n+1}q_n + q_{n-1} < a_{n+1}q_n + q_{n-1} + q_n = q_{n+1} + q_n.$$

□

In particular, since $x_{n+1} > a_{n+1}$ and $q_{n-1} > 0$, one deduces from Lemma 71

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}. \tag{73}$$

Therefore any convergent p/q of x satisfies $|x - p/q| < 1/q^2$ (compare with (i) $\Rightarrow$ (v) in Proposition 4). Moreover, if a_{n+1} is large, then the approximation p_n/q_n is sharp. Hence, large partial quotients yield good rational approximations by truncating the continued fraction expansion just before the given partial quotient.