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On transcendental entire functions with infinitely many derivatives taking integer values at several points

by

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Let $s_0, s_1, \dots, s_{m-1}$ be complex numbers and $r_0, \dots, r_{m-1}$ rational integers in the range $0 \le r_j \le m-1$. Our first goal is to prove that if an entire function f of sufficiently small exponential type satisfies $f^{(mn+r_j)}(s_j) \in \mathbb{Z}$ for $0 \le j \le m-1$ and all sufficiently large n, then f is a polynomial. Under suitable assumptions on $s_0, s_1, \dots, s_{m-1}$ and $r_0, \dots, r_{m-1}$, we introduce interpolation polynomials Λ_{nj}, $(n \ge 0,\ 0 \le j \le m-1)$ satisfying

$$\Lambda_{nj}^{(mk+r_\ell)}(s_\ell) = \delta_{j\ell}\delta_{nk}, \quad \text{for} \quad n,k \ge 0 \quad \text{and} \quad 0 \le j,\ell \le m-1$$

and we show that any entire function f of sufficiently small exponential type has a convergent expansion

$$f(z) = \sum_{n\ge 0} \sum_{j=0}^{m-1} f^{(mn+r_j)}(s_j)\Lambda_{nj}(z).$$

The case $r_j = j$ for $0 \le j \le m-1$ involves successive derivatives $f^{(n)}(w_n)$ of f evaluated at points of a periodic sequence $\mathbf{w} = (w_n)_{n\ge 0}$ of complex numbers, where $w_{mh+j} = s_j$ $(h \ge 0,\ 0 \le j \le m)$. More generally, given a bounded (not necessarily periodic) sequence $\mathbf{w} = (w_n)_{n\ge 0}$ of complex numbers, we consider similar interpolation formulae

$$f(z) = \sum_{n\ge 0} f^{(n)}(w_n)\Omega_{\mathbf{w},n}(z)$$

involving polynomials $\Omega_{\mathbf{w},n}(z)$ which were introduced by W. Gontcharoff in 1930. Under suitable assumptions, we show that the hypothesis $f^{(n)}(w_n) \in \mathbb{Z}$ for all sufficiently large n implies that f is a polynomial.

1 Introduction

Given a finite set of points S in the complex plane and an infinite subset $\mathscr{S}$ of $S \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers, we ask for a lower bound on the order of growth of a transcendental entire function f such that $f^{(n)}(s) \in \mathbb{Z}$ for all $(s,n) \in \mathscr{S}$. In [Waldschmidt 2019], we discussed the case $S = \{s_0, s_1\}$ using interpolation

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polynomials of Lidstone, Whittaker and Gontcharoff, together with results of Schoenberg and Macintyre.

Here we introduce generalizations of these interpolation polynomials to several points and we deduce lower bounds for the growth of transcendental entire functions with corresponding integral values of their derivatives. We first consider periodic sequences: given complex numbers $s_0, s_1, \dots, s_{m-1}$ and rational integers $r_0, \dots, r_{m-1}$ in the range $0 \le r_j \le m-1$, we set

$$\mathscr{S} = \{(s_j, mn + r_j) \mid n \ge 0,\ 0 \le j \le m-1\};$$

under suitable assumptions, we give a lower bound for the growth order of a transcendental entire function f satisfying $f^{(mn+r_j)}(s_j) \in \mathbb{Z}$ for $0 \le j \le m-1$ and all sufficiently large n (Theorem 2). That some assumption is necessary is obvious from the example $m = 2$, $s_0 = s_1 = r_0 = r_1 = 0$: given any transcendental entire function g, say of order 0, the function $f(z) = zg(z^2)$ is a transcendental entire function of the same order satisfying $f^{(2n)}(s_0) = 0$ for all $n \ge 0$.

Next, we consider a sequence $(w_n)_{n\ge 0}$ of elements in S and we prove that an entire function of sufficiently small exponential type satisfying $f^{(n)}(w_n) \in \mathbb{Z}$ for all sufficiently large n is a polynomial (Theorem 5(a)).

In Section 4, we show how to interpolate entire functions of sufficiently small exponential type with respect to periodic subsets of $\{s_0, s_1, \dots, s_{m-1}\} \times \mathbb{N}$. Our approach requires that some determinant $D(s_0, s_1, \dots, s_{m-1})$ (depending also on $r_0, \dots, r_{m-1}$) does not vanish; this assumption cannot be omitted (it could be weakened, but we do not address this issue here).

In Section 5, we introduce interpolation polynomials attached to a sequence of elements belonging to $\{s_0, s_1, \dots, s_{m-1}\}$. We deduce that if f is an entire function f of sufficiently small exponential type such that, for all sufficiently large n, one at least of the $2^m - 1$ nonempty products of elements $f^{(n)}(s_0), f^{(n)}(s_1), \dots, f^{(n)}(s_{m-1})$ is in $\mathbb{Z}$, then f is a polynomial (Theorem 5(b)).

2 Notation and auxiliary results

We denote by δ_{ij} the Kronecker symbol:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j, \end{cases}$$

and by $f^{(n)}$ the n–th derivative $(\mathrm{d}^n/\mathrm{d}z^n)f$ of an analytic function $f(z)$.

The order of an entire function f is

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log\log|f|_r}{\log r} \quad \text{where} \quad |f|_r = \sup_{|z|=r} |f(z)|,$$

and the exponential type is

$$\tau(f) = \limsup_{r\to\infty} \frac{\log|f|_r}{r}\cdot$$

For each $z_0 \in \mathbb{C}$, we have

$$\limsup_{n\to\infty} |f^{(n)}(z_0)|^{1/n} = \tau(f). \tag{2.1}$$

Cauchy's inequalities

$$\frac{|f^{(n)}(z_0)|}{n!} r^n \le |f|_{r+|z_0|}, \tag{2.2}$$

are valid for any entire function f and all $z_0 \in \mathbb{C}$, $n \ge 0$ and $r > 0$. We will also use Stirling's Formula: for all $N \ge 1$, we have

$$N^N \mathrm{e}^{-N}\sqrt{2\pi N} < N! < N^N \mathrm{e}^{-N}\sqrt{2\pi N}\mathrm{e}^{1/(12N)}. \tag{2.3}$$

For the arithmetical applications, our main assumption on the growth of our functions f is

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\max\{|s_0|,|s_1|,\dots,|s_{m-1}|\}}. \tag{2.4}$$

This condition arises from the following auxiliary result, based on Cauchy's upper bound for the derivatives and Stirling approximation formula for $n!$ [Waldschmidt 2019, Proposition 12]:

Proposition 1. *Let f be an entire function and let $A \ge 0$. Assume*

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{\mathrm{e}^{-A}}{\sqrt{2\pi}}. \tag{2.5}$$

Then there exists $n_0 > 0$ such that, for $n \ge n_0$ and for all $z \in \mathbb{C}$ in the disc $|z| \le A$, we have

$$|f^{(n)}(z)| < 1.$$

3 Integer values of derivatives of entire functions

3.1 Periodic sequences

Let $s_0, s_1, \dots, s_{m-1}$ be complex numbers, not necessarily distinct. We write $\mathbf{s}$ for the tuple $(s_0, s_1, \dots, s_{m-1})$. Let ζ be a primitive m-th root of unity and let $r_0, \dots, r_{m-1}$ be m integers satisfying $0 \le r_j \le j$ $(0 \le j \le m-1)$. Our main assumption is that the determinant

$$\mathrm{D}(\mathbf{s}) = \det\left(\frac{k!}{(k-r_j)!}s_j^{k-r_j}\right)_{0\le j,k\le m-1}$$

does not vanish. Here, $a!/(a-b)!$ is understood to be 0 for $a < b$. This assumption means that the linear map

$$\begin{array}{ccc} \mathbb{C}[z]_{\le m-1} & \longrightarrow & \mathbb{C}^m \\ L(z) & \longmapsto & \left(L^{(r_j)}(s_j)\right)_{0\le j\le m-1} \end{array} \tag{3.1}$$

is an isomorphism of $\mathbb{C}$–vector spaces, $\mathbb{C}[z]_{\le m-1}$ being the space of polynomials of degree $\le m-1$.

For $t \in \mathbb{C}$, consider the $m \times m$ matrix

$$M(t) = \left(\zeta^{kr_\ell} \mathrm{e}^{\zeta^k t s_\ell}\right)_{0 \le k,\ell \le m-1}$$

and its determinant $\Delta(t)$:

$$\Delta(t) = \det \begin{pmatrix} \mathrm{e}^{ts_0} & \mathrm{e}^{ts_1} & \cdots & \mathrm{e}^{ts_{m-1}} \\ \zeta^{r_0}\mathrm{e}^{\zeta t s_0} & \zeta^{r_1}\mathrm{e}^{\zeta t s_1} & \cdots & \zeta^{r_{m-1}}\mathrm{e}^{\zeta t s_{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta^{(m-1)r_0}\mathrm{e}^{\zeta^{m-1} t s_0} & \zeta^{(m-1)r_1}\mathrm{e}^{\zeta^{m-1} t s_1} & \cdots & \zeta^{(m-1)r_{m-1}}\mathrm{e}^{\zeta^{m-1} t s_{m-1}} \end{pmatrix}.$$

We will show (Lemma 9) that the exponential polynomial $\Delta(t)$ is not the zero function.

Theorem 2. *Assume* $\mathrm{D}(\mathbf{s}) \ne 0$. *Let* $\tau > 0$ *be such that* $\Delta(t)$ *does not vanish for* $0 < |t| < \tau$. *Let* f *be an entire function of exponential type* $< \tau$ *which satisfies* (2.4) *and also, for each* n *sufficiently large,*

$$f^{(mn+r_j)}(s_j) \in \mathbb{Z} \text{ for } j = 0, \dots, m-1.$$

Then f *is a polynomial.*

In the case $m = 1$, we can take $\tau = 1$ and the assumption that the exponential type is < 1 can be replaced by the weaker condition (2.5) with $A = 0$, according to a classical result of Pólya on Hurwitz functions; see [Waldschmidt 2019, §2].

Let us give two further examples. Proofs will be given in Section 4.1.

Our first example is with $r_0 = r_1 = \cdots = r_{m-1} = 0$. In this case, the assumption $\mathrm{D}(\mathbf{s}) \ne 0$ is satisfied if and only if $s_0, s_1, \dots, s_{m-1}$ are pairwise distinct (Section 4.1 Example 1).

Corollary 3. *Assume that* $s_0, s_1, \dots, s_{m-1}$ *are pairwise distinct. An entire function of sufficiently small exponential type, satisfying*

$$f^{(mn)}(s_j) \in \mathbb{Z}$$

for $j = 0, \dots, m-1$ *and for all sufficiently large* n*, is a polynomial.*

For $m = 2$ (Lidstone interpolation), with $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$, Corollary 3 follows also from [Waldschmidt 2019, Corollary 2], where the assumption on the exponential type $\tau(f)$ of f is

$$\tau(f) < \min\{1, \pi/|s_0 - s_1|\},$$

and this assumption is best possible. Indeed,

- The function

$$f(z) = \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)}$$

has exponential type 1 and satisfies $f^{(2n)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

- The function

$$f(z) = \sin\left(\pi \frac{z - s_0}{s_1 - s_0}\right)$$

has exponential type $\pi/|s_1 - s_0|$ and satisfies $f^{(2n)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

Our second example is $r_j = j$ for $j = 0, 1, \dots, m-1$. The assumption $\mathrm{D}(\mathbf{s}) \neq 0$ is always satisfied (Section 4.1 Example 2).

Corollary 4. *An entire function of sufficiently small exponential type satisfying*

$$f^{(mn+j)}(s_j) \in \mathbb{Z}$$

for $j = 0, \dots, m-1$ and for all sufficiently large n is a polynomial.

In the case $m = 2$ (Whittaker interpolation), with $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$, Corollary 4 also follows from [Waldschmidt 2019, Corollary 6] (after permutation of s_0 and s_1), where the assumption is

$$\tau(f) < \min\left\{1, \frac{\pi}{2|s_0 - s_1|}\right\},$$

and this assumption is best possible. Indeed,

- The function

$$f(z) = \frac{\sinh(z - s_0)}{\cosh(s_1 - s_0)}$$

has exponential type 1 and satisfies $f^{(2n)}(s_0) = 0$ and $f^{(2n+1)}(s_1) = 1$ for all $n \geq 0$.

- The function

$$f(z) = \cos\left(\frac{\pi}{2} \cdot \frac{z - s_1}{s_1 - s_0}\right)$$

has exponential type $\pi/(2|s_1 - s_0|)$ and satisfies $f^{(2n)}(s_0) = f^{(2n+1)}(s_1) = 0$ for all $n \geq 0$.

3.2 Sequence of derivatives at finitely many points

The next result deals with a situation more general than Corollary 4.

Theorem 5. *Let $A > 0$, let f be an entire function satisfying (2.5) and let the exponential type $\tau(f)$ of f satisfy*

$$\tau(f) < \frac{\log 2}{A}\cdot$$

(a) Assume that for all sufficiently large integers n, there exists $w_n \in \mathbb{C}$ with $|w_n| < A$ such that $f^{(n)}(w_n) \in \mathbb{Z}$. Then f is a polynomial.
(b) Let $s_0, s_1, \ldots, s_{m-1}$ be m complex numbers, not necessarily distinct, satisfying

$$\max_{0\leq j\leq m-1} |s_j| < A.$$

Assume that, for all sufficiently large n, there exists a nonempty subset I_n of $\{0, 1, \ldots, m-1\}$ such that the product

$$\prod_{j\in I_n} f^{(n)}(s_j)$$

is in $\mathbb{Z}$. Then f is a polynomial.

The case $m = 2$ in part (b) of Theorem 5 is [Waldschmidt 2019, Theorem 8].

3.3 Content

In Section 4 we deal with periodic subsets of $S \times \mathbb{N}$: we generalize the construction of Lidstone polynomials to several points and we prove Theorem 2 and Corollaries 3 and 4. In Section 5, we introduce and study interpolation polynomials associated with a sequence of elements in S and we prove Theorem 5.

4 Periodic case

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and $r_0, \ldots, r_{m-1}$ rational integers satisfying $0 \leq r_0 \leq r_1 \leq \cdots \leq r_{m-1} \leq m-1$.

4.1 The determinant $\mathrm{D}(\mathbf{z})$ – proofs of Corollaries 3 and 4

Let $z_0, z_1, \ldots, z_{m-1}$ be independent variables. Write $\mathbf{z}$ for $(z_0, z_1, \ldots, z_{m-1})$. Let K be the field $\mathbb{Q}(z_0, z_1, \ldots, z_{m-1})$ and $\mathrm{D}(\mathbf{z})$ be the determinant

$$\det\left(\frac{k!}{(k-r_j)!} z_j^{k-r_j}\right)_{0\leq j,k\leq m-1} \in \mathbb{Q}[\mathbf{z}] \subset K.$$

Recall $a!/(a-b)! = 0$ for $a < b$.

For $j = 0, 1, \ldots, m-1$, the row vector

$$\begin{aligned} v_j &= \left(\frac{k!}{(k-r_j)!} z_j^{k-r_j}\right)_{k=0,1,\ldots,m-1} \\ &= \left(0, 0, \ldots, 0, r_j!, \frac{(r_j+1)!}{1!} z_j, \frac{(r_j+2)!}{2!} z_j^2, \ldots, \frac{(m-1)!}{(m-1-r_j)!} z_j^{m-1-r_j}\right) \end{aligned}$$

belongs to $\{0\}^{r_j} \times K^{m-r_j}$. If $r_j > j$ for some $j \in \{0, 1, \dots, m-1\}$, then the $m-j$ vectors $v_j, v_{j+1}, \dots, v_{m-1}$ all belong to the subspace $\{0\}^{j+1} \times K^{m-j-1}$ of K^m, the dimension of which is $m-j-1$; hence the determinant $\mathrm{D}(\mathbf{z})$ vanishes.

Assume $r_j \le j$ for $0 \le j \le m-1$. For the degree given by the lexicographic order, the leading term of the polynomial $\mathrm{D}(\mathbf{z})$ is the product of the elements on the diagonal. The degree in z_j of $\mathrm{D}(\mathbf{z})$ is $\le m-1-r_j$. For $k = 0, 1, \dots, m-1$, define

$$\mathcal{E}(k) = \{(i,j) \ \mid \ 0 \le i < j \le m-1, \ r_i = r_j\}.$$

In the ring $\mathbb{Q}[z_0, z_1, \dots, z_{m-1}]$, the polynomial $\mathrm{D}(\mathbf{z})$ is divisible by

$$\prod_{(i,j)\in\mathcal{E}(k)} (z_j - z_i).$$

If there is no extra nonconstant factor, the only zeros of $\mathrm{D}(\mathbf{z})$ are given by $z_i = z_j$ with $r_i = r_j$ and $i < j$. But extra factors may occur.

Examples

(1) [Poritsky 1932], quoted by [Macintyre 1954, §3] and [Buck 1955]:

$$r_0 = r_1 = \cdots = r_{m-1} = 0.$$

The Vandermonde determinant

$$D(\mathbf{s}) = \det\left(s_j^k\right)_{0\le j,k\le m-1} = \det\begin{pmatrix} 1 & s_0 & s_0^2 & \cdots & s_0^{m-1} \\ 1 & s_1 & s_1^2 & \cdots & s_1^{m-1} \\ 1 & s_2 & s_2^2 & \cdots & s_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_{m-1} & s_{m-1}^2 & \cdots & s_{m-1}^{m-1} \end{pmatrix} = \prod_{0\le j<\ell\le m-1} (s_\ell - s_j)$$

does not vanish if and only if $s_0, s_1, \dots, s_{m-1}$ are pairwise distinct.

(2) [Gontcharoff 1930], quoted by [Macintyre 1954, §4] and [Buck 1955]:

$$r_j = j \quad \text{for} \quad j = 0, 1, \dots, m-1.$$

Then

$$D(\mathbf{s}) = \det\left(\frac{k!}{(k-j)!} s_j^{k-j}\right)_{0\le j,k\le m-1}$$

$$= \det\begin{pmatrix} 1 & s_0 & s_0^2 & s_0^3 & \cdots & s_0^{m-2} & s_0^{m-1} \\ 0 & 1 & 2s_1 & 3s_1^2 & \cdots & (m-2)s_1^{m-3} & (m-1)s_1^{m-2} \\ 0 & 0 & 2 & 6s_2 & \cdots & (m-2)(m-3)s_3^{m-4} & (m-1)(m-2)s_2^{m-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (m-2)! & (m-1)!s_{m-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix} = \prod_{j=0}^{m-1} j!$$

does not vanish.
(3) Take $m=3$, $r_0=r_1=0$, $r_2=1$. Then

$$\mathrm{D}(z_0,z_1,z_2)=\begin{vmatrix}1 & z_0 & z_0^2\\ 1 & z_1 & z_1^2\\ 0 & 1 & 2z_2\end{vmatrix}=(z_1-z_0)(2z_2-z_1-z_0).$$

A polynomial of degree 2 vanishing at s and $-s$ with $s\neq 0$ has a zero derivative at the origin. For the study of entire functions f satisfying

$$f^{(3n)}(s_0)\in\mathbb{Z},\ f^{(3n)}(s_1)\in\mathbb{Z},\ f^{(3n+1)}(s_2)\in\mathbb{Z}\quad\text{for}\quad n\geq 0,$$

the assumption $\mathrm{D}(\mathbf{s})\neq 0$ amounts to $2s_2\neq s_1+s_0$.

4.2 Interpolation polynomials

The following interpolation polynomials generalize the sequences of polynomials introduced by Lidstone, Whittaker, Poritsky, Gontcharoff and others.

Proposition 6. *Assume* $\mathrm{D}(\mathbf{s})\neq 0$. *Then there exists a unique family of polynomials* $(\Lambda_{nj}(z))_{n\geq 0,0\leq j\leq m-1}$ *satisfying*

$$\Lambda_{nj}^{(mk+r_\ell)}(s_\ell)=\delta_{j\ell}\delta_{nk},\quad \textit{for}\quad n,k\geq 0\quad \textit{and}\quad 0\leq j,\ell\leq m-1. \tag{4.1}$$

For $n\geq 0$ *and* $0\leq j\leq m-1$ *the polynomial* Λ_{nj} *has degree* $\leq mn+m-1$.

This result plays a main role in our paper; we give two proofs of it.

First proof of Proposition 6. . Assuming $\mathrm{D}(\mathbf{s})\neq 0$, we prove by induction on n that the linear map

$$\begin{array}{rccc}\psi_n: & \mathbb{C}[z]_{\leq m(n+1)-1} & \longrightarrow & \mathbb{C}^{m(n+1)}\\ & L(z) & \longmapsto & \bigl(L^{(mk+r_\ell)}(s_\ell)\bigr)_{0\leq\ell\leq m-1,0\leq k\leq n}\end{array}$$

is an isomorphism of $\mathbb{C}$–vector spaces. For $n=0$ this is the assumption (3.1). Assume ψ_{n-1} is injective for some $n\geq 1$. Let $L\in\ker\psi_n$. Then $L^{(m)}\in\ker\psi_{n-1}$, hence $L^{(m)}=0$, which means that L has degree $<m$. From (3.1) we conclude $L=0$.

The fact that ψ_n is injective for all n implies that if a polynomial $f\in\mathbb{C}[z]$ satisfies $f^{(mk+r_\ell)}(s_\ell)=0$ for all $k\geq 0$ and all ℓ with $0\leq\ell\leq m-1$, then $f=0$. This shows the unicity of the solution Λ_{nj} of (4.1).

Since ψ_n is injective, it is an isomorphism, and hence surjective: for $0\leq j\leq n-1$ there exists a unique polynomial $\Lambda_{nj}\in\mathbb{C}[z]_{\leq m(n+1)-1}$ such that $\Lambda_{nj}^{(mk+r_\ell)}(s_\ell)=\delta_{j\ell}\delta_{nk}$ for $0\leq j,\ell\leq m-1$. These conditions show that the set of polynomials Λ_{kj} for $0\leq k\leq n$, $0\leq j\leq m-1$, is a basis of $\mathbb{C}[z]_{\leq m(n+1)-1}$: any polynomial $f\in\mathbb{C}[z]$ of degree $\leq m(n+1)-1$ can be written in a unique way

$$f(z)=\sum_{j=0}^{m-1}\sum_{k=0}^{n}a_{kj}\Lambda_{kj}(z),$$

and therefore the coefficients are given by $a_{kj}=f^{(mk+r_j)}(s_j)$. □

Second proof of Proposition 6. . The conditions (4.1) mean that any polynomial $f \in \mathbb{C}[z]$ has an expansion

$$f(z) = \sum_{j=0}^{m-1} \sum_{n\geq 0} f^{(mn+r_j)}(s_j)\Lambda_{nj}(z), \tag{4.2}$$

where only finitely many terms on the right hand side are nonzero.

Assuming $\mathrm{D}(\mathbf{s}) \neq 0$, we first prove the unicity of such an expansion by induction on the degree of f. The assumption $\mathrm{D}(\mathbf{s}) \neq 0$ shows that there is no nonzero polynomial of degree $< m$ satisfying $f^{(mn+r_j)}(s_j) = 0$ for all (n, j) with $0 \leq n, j \leq m-1$. Now if f is a polynomial satisfying $f^{(mn+r_j)}(s_j) = 0$ for all (n, j) with $n \geq 0$ and $0 \leq j \leq m-1$, then $f^{(m)}$ satisfies the same conditions and has a degree less than the degree of f. By the induction hypothesis we deduce $f^{(m)} = 0$, which means that f has degree $< m$, hence $f = 0$. This proves the unicity.

For the existence, let us show that, under the assumption $\mathrm{D}(\mathbf{s}) \neq 0$, the recurrence relations

$$\Lambda_{nj}^{(m)} = \Lambda_{n-1,j}, \quad \Lambda_{nj}^{(r_\ell)}(s_\ell) = 0 \text{ for } n \geq 1, \quad \Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \text{ for } 0 \leq j, \ell \leq m-1$$

have a unique solution given by polynomials $\Lambda_{nj}(z)$, $(n \geq 0,\ j = 0, \dots, m-1)$, where Λ_{nj} has degree $\leq mn + m - 1$. Clearly, these polynomials will satisfy (4.1).

From the assumption $\mathrm{D}(\mathbf{s}) \neq 0$ we deduce that, for $0 \leq j \leq m-1$, there is a unique polynomial Λ_{0j} of degree $< m$ satisfying

$$\Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \text{ for } 0 \leq \ell \leq m-1.$$

By induction, given $n \geq 1$ and $j \in \{0, 1, \dots, m-1\}$, once we know $\Lambda_{n-1,j}(z)$, we choose a solution L of the differential equation $L^{(m)} = \Lambda_{n-1,j}$; using again the assumption $\mathrm{D}(\mathbf{s}) \neq 0$, we deduce that there is a unique polynomial $\widetilde{L}$ of degree $< m$ satisfying $\widetilde{L}^{(r_\ell)}(s_\ell) = L^{(r_\ell)}(s_\ell)$ for $0 \leq \ell \leq m-1$; then the solution is given by $\Lambda_{nj} = L - \widetilde{L}$. □

Remark. The following converse of Proposition 6 is plain: if there exists a unique tuple

$$\big(\Lambda_{00}(z), \Lambda_{01}(z), \dots, \Lambda_{0,m-1}(z)\big)$$

of polynomials of degree $\leq m-1$ satisfying

$$\Lambda_{0j}^{(r_\ell)}(s_\ell) = \delta_{j\ell} \text{ for } 0 \leq j, \ell \leq m-1,$$

then $\mathrm{D}(\mathbf{s}) \neq 0$.

Examples

Special cases of Proposition 6 have already been introduced in the literature.

(1) Lidstone polynomials with $\{0, 1\}$ [Waldschmidt 2019, §3.1]:

$$m = 2,\ s_0 = 0,\ s_1 = 1,\ r_0 = r_1 = 0,\ \Lambda_{n0}(z) = \Lambda_n(1-z),\ \Lambda_{n1}(z) = \Lambda_n(z).$$

(2) Lidstone polynomials with $\{s_0, s_1\}$ and $s_0 \neq s_1$; with the notation of [Waldschmidt 2019, §3.2]:

$$m = 2,\ r_0 = r_1 = 0,\ \Lambda_{n0}(z) = -\widetilde{\Lambda}_n(z - s_1),\ \Lambda_{n1}(z) = \widetilde{\Lambda}_n(z - s_0).$$

(3) Whittaker polynomials with $\{0, 1\}$; with the notation of [Waldschmidt 2019, §5.1]:

$$m = 2,\ s_0 = 1,\ s_1 = 0,\ r_0 = 0,\ r_1 = 1,\ \Lambda_{n0}(z) = M_n(z),\ \Lambda_{n1}(z) = M'_{n+1}(z - 1).$$

(4) Whittaker polynomials with $\{s_0, s_1\}$; with the notation of [Waldschmidt 2019, §5.2] (beware that this reference deals with the even derivatives at s_0 and the odd derivatives at s_1, while here we impose $r_0 \le r_1$):

$$m = 2,\ r_0 = 0,\ r_1 = 1,\ \Lambda_{n0}(z) = \widetilde{M}_n(z - s_1),\ \Lambda_{n1}(z) = \widetilde{M}'_{n+1}(z - s_0).$$

(5) [Poritsky 1932], quoted by [Macintyre 1954, §3], [Buck 1955] (see also [Gel'fond 1971, Chap. 3, §4.3]): assuming $s_0, s_1, \dots, s_{m-1}$ are pairwise distinct,

$$r_0 = r_1 = \cdots = r_{m-1} = 0.$$

(6) [Gontcharoff 1930], quoted by [Macintyre 1954, §4], [Buck 1955] (see also [Gel'fond 1971, Chap. 3, §4.2]):

$$r_j = j \quad \text{for} \quad j = 0, 1, \dots, m - 1.$$

4.3 Exponential sums, following D. Roy

This section is due to D. Roy (private communication).

Given complex numbers $a_0, a_1, \cdots$ and non negative real numbers $c_0, c_1, \dots,$ we write

$$\sum_{i \ge 0} a_i z^i \preceq_z \sum_{i \ge 0} c_i z^i$$

if $|a_i| \le c_i$ for all $i \ge 0$. In the same way, given two power series $\sum_{i \ge 0, j \ge 0} a_{ij} t^i z^j$ and $\sum_{i \ge 0, j \ge 0} c_{ij} t^i z^j$ with $a_{ij} \in \mathbb{C}$ and $c_{ij} \in \mathbb{R}_{\ge 0}$, we write

$$\sum_{i \ge 0} \sum_{j \ge 0} a_{ij} t^i z^j \preceq_{t,z} \sum_{i \ge 0} \sum_{j \ge 0} c_i t^i z^j$$

if $|a_{ij}| \le c_{ij}$ for all i, j.

We first give a quantitative version of Proposition 6.

Lemma 7. *There exists a constant $\Theta > 0$ such that*

$$\Lambda_{nj}(z) \preceq_z \sum_{i=0}^{m(n+1)-1} \frac{\Theta^{m(n+1)-i}}{i!} z^i$$

for all $n \ge 0$ and $j = 0, 1, \dots, m - 1$.

Proof. We proceed by induction. For $n = 0$ it suffices to choose $\Theta > 0$ sufficiently large so that

$$\Lambda_{0j}(z) \preceq_z \sum_{i=0}^{m-1} \frac{\Theta^{m-i}}{i!} z^i$$

for $j = 0, 1, \ldots, m-1$. Assume

$$\Lambda_{n-1,j}(z) \preceq_z \sum_{i=0}^{mn-1} \frac{\Theta^{mn-i}}{i!} z^i$$

for some integer $n \geq 1$ and for $j = 0, 1, \ldots, m-1$. Fix an integer j and let $L(z) \in \mathbb{C}[z]$ be the polynomial satisfying

$$L^{(m)}(z) = \Lambda_{n-1,j}(z) \quad \text{and} \quad L(0) = L'(0) = \cdots = L^{(m-1)}(0) = 0.$$

We have

$$L(z) \preceq_z \sum_{i=0}^{mn-1} \frac{\Theta^{mn-i}}{(i+m)!} z^{i+m} = \sum_{i=m}^{m(n+1)-1} \frac{\Theta^{m(n+1)-i}}{i!} z^i.$$

Set $A = \max\{1, |s_0|, \ldots, |s_{m-1}|\}$. For $\ell = 0, 1, \ldots, m-1$, we have

$$|L^{(r_\ell)}(s_\ell)| \leq \sum_{i=0}^{mn-1} \frac{\Theta^{mn-i} A^{i+m-r_\ell}}{(i+m-r_\ell)!} = \Theta^{mn} A^{m-r_\ell} \sum_{i=0}^{mn-1} \frac{(A/\Theta)^i}{(i+m-r_\ell)!} \leq \Theta^{mn} A^m \exp(A/\Theta).$$

From the isomorphism (3.1) it follows that there is a constant $B > 0$ such that, for any polynomial $\widetilde{L}(z) \in \mathbb{C}[z]_{\leq m-1}$,

$$\widetilde{L}(z) \preceq_z \left(\max_{0 \leq \ell \leq m-1} |\widetilde{L}^{(r_\ell)}(s_\ell)| \right) B \sum_{i=0}^{m-1} \frac{z^i}{i!}.$$

Choosing $\widetilde{L}(z)$ such that

$$\widetilde{L}^{(r_\ell)}(s_\ell) = L^{(r_\ell)}(s_\ell)$$

for $\ell = 0, 1, \ldots, m-1$ and assuming $\Theta \geq 1$ sufficiently large so that

$$\Theta \geq B A^m \exp(A/\Theta),$$

we get

$$\widetilde{L}(z) \preceq_z \Theta^{mn+1} \sum_{i=0}^{m-1} \frac{z^i}{i!} \preceq_z \sum_{i=0}^{m-1} \frac{\Theta^{m(n+1)-i}}{i!} z^i,$$

hence

$$\Lambda_{nj}(z) = L(z) - \widetilde{L}(z) \preceq_z \sum_{i=0}^{m(n+1)-1} \frac{\Theta^{m(n+1)-i}}{i!} z^i.$$

□

For $j = 0, 1, \dots, m-1$ and $z \in \mathbb{C}$, consider the power series $\varphi_j(t,z) \in \mathbb{C}[[t]]$ defined by

$$\varphi_j(t,z) = \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}(z). \tag{4.3}$$

From Lemma 7 it follows that we have

$$\varphi_j(t,z) \preceq_{t,z} \sum_{n\geq 0} \sum_{i=0}^{m(n+1)-1} \frac{\Theta^{m(n+1)-i}}{i!} t^{mn+r_j} z^i,$$

and therefore the function of two complex variables $(t,z) \mapsto \varphi_j(t,z)$ is analytic in the domain $|t| < 1/\Theta$, $z \in \mathbb{C}$.

Lemma 8. *For $|t| < 1/\Theta$ and $z \in \mathbb{C}$, we have*

$$\mathrm{e}^{tz} = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \varphi_j(t,z).$$

Proof. Define, for $|t| < 1/\Theta$ and $z \in \mathbb{C}$,

$$F(t,z) = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \varphi_j(t,z) - \mathrm{e}^{tz}.$$

We have

$$F(t,z) = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}(z) - \mathrm{e}^{tz} = \sum_{n\geq 0} a_n(z) t^n,$$

where $a_n(z) \in \mathbb{C}[z]_{\leq n+m-1}$ for all $n \geq 0$. We obtain, for all $k \geq 0$ and $\ell = 0, 1, \dots, m-1$,

$$\left(\frac{\partial}{\partial z}\right)^{mk+r_\ell} F(t,z)\bigg|_{z=s_\ell} = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}^{(mk+r_\ell)}(s_\ell) - t^{mk+r_\ell} \mathrm{e}^{ts_\ell} = 0,$$

hence

$$\sum_{n\geq 0} a_n^{(mk+r_\ell)}(s_\ell) t^n = 0$$

for $|t| < 1/\Theta$. Therefore $a_n^{(mk+r_\ell)}(s_\ell) = 0$ for all $k \geq 0$, $n \geq 0$ and $\ell = 0, 1, \dots, m-1$. We conclude $a_n(z) = 0$ for all $n \geq 0$, which proves Lemma 8. □

For $0 < |t| < 1/\Theta$ and $j = 0, 1, \dots, m-1$, we have

$$\left(\frac{\partial}{\partial z}\right)^{m} \varphi_j(t,z) = \sum_{n\geq 0} t^{mn+r_j} \Lambda_{nj}^{(m)}(z) = \sum_{n\geq 1} t^{mn+r_j} \Lambda_{n-1,j}(z) = t^m \varphi_j(t,z).$$

The functions $\varphi_0(t,z), \varphi_1(t,z), \dots, \varphi_{m-1}(t,z)$ are the solutions of the differential equation

$$f^{(m)}(z) = t^m f(z)$$

with the initial conditions

$$\left(\frac{\partial}{\partial z}\right)^{r_\ell} \varphi_j(t, s_\ell) = t^{r_\ell} \delta_{j\ell} \quad \text{for} \quad 0 \le j, \ell \le m-1. \tag{4.4}$$

Recall that ζ is a primitive m-th root of unity. The general solution of this differential equation is a linear combination of the functions $\mathrm{e}^{\zeta^k tz}$ $(k = 0, 1, \dots, m-1)$ with coefficients depending on t. Hence for $0 < |t| < 1/\Theta$ there exist complex numbers $c_{jk}(t)$ $(j, k = 0, 1 \dots, m-1)$ such that

$$\varphi_j(t, z) = \sum_{k=0}^{m-1} c_{jk}(t) \mathrm{e}^{\zeta^k tz}. \tag{4.5}$$

For $\ell = 0, 1, \dots, m-1$, this yields

$$\sum_{k=0}^{m-1} c_{jk}(t)(\zeta^k t)^\ell = \left(\frac{\partial}{\partial z}\right)^\ell \varphi_j(t, z)\bigg|_{z=0} = \sum_{n \ge 0} t^{mn+r_j} \Lambda_{nj}^{(\ell)}(0),$$

and thus we deduce that

$$t^\ell \sum_{k=0}^{m-1} c_{jk}(t) \zeta^{k\ell} \preceq_t \sum_{n \ge 0} \Theta^{m(n+1)-\ell} t^{mn+r_j}.$$

Since the matrix $(\zeta^{k\ell})_{0 \le k, \ell \le m-1}$ is invertible, this shows that the functions $c_{jk}(t)$ are meromorphic for $|t| < 1/\Theta$ with at most a pole at $t = 0$ of order $\le m-1$.

4.4 Analytic continuation of $\varphi_j(t, z)$

From (4.4) and (4.5) we deduce that for $j = 0, 1, \dots, m-1$ and $0 < |t| < 1/\Theta$, we have

$$\sum_{k=0}^{m-1} c_{jk}(t) \zeta^{kr_\ell} \mathrm{e}^{\zeta^k ts_\ell} = \delta_{j\ell} \quad (0 \le \ell \le m-1).$$

Hence for $|t| < 1/\Theta$ the matrix $\big(c_{jk}(t)\big)_{0 \le j,k \le m-1}$ is the inverse of the matrix $M(t)$. Recall (Section 3.1) that $\Delta(t)$ is the determinant of the matrix $M(t) = \left(\zeta^{kr_\ell} \mathrm{e}^{\zeta^k ts_\ell}\right)_{0 \le k, \ell \le m-1}$. We deduce:

Lemma 9. *The determinant $\Delta(t)$ does not vanish for $0 < |t| < 1/\Theta$.*

The determinant $\Delta(t)$ defines a nonzero entire function in $\mathbb{C}$. We extend the definition of $c_{jk}(t)$ to meromorphic functions in $\mathbb{C}$ by the condition that the matrix $\bigl(c_{jk}(t)\bigr)_{0\le j,k\le m-1}$ is the inverse of the matrix $M(t)$. From the assumption in Theorem 2 that $\Delta(t)$ does not vanish for $0<|t|<\tau$, we infer that $c_{jk}(t)$ is analytic in the domain $0<|t|<\tau$. By means of (4.5), this defines $\varphi_j(t,z)$ for all $z\in\mathbb{C}$ and for all t with $\Delta(t)\neq 0$. In particular the function of two variables $t\mapsto\varphi_j(t,z)$ is analytic in the domain $|t|<\tau$, $z\in\mathbb{C}$, and (4.3) is valid in this domain.

Lemma 10. *Let ϱ satisfy $0<\varrho<\tau$. For $z\in\mathbb{C}$ and $0\le j\le m-1$ we have*

$$|\Lambda_{nj}(z)|\le\varrho^{-mn-r_j}\sup_{|t|=\varrho}|\varphi_j(t,z)|.$$

Proof. The Taylor expansion at the origin of the meromorphic function $t\mapsto\varphi_j(t,z)$ is given by the formula (4.3), which is therefore valid for $|t|<\tau$. Hence

$$\Lambda_{nj}(z)=\frac{1}{2i\pi}\int_{|t|=\varrho}\varphi_j(t,z)t^{-mn-r_j-1}\mathrm{d}t.$$

Lemma 10 follows. □

Examples

(1) Lidstone [Waldschmidt 2019, §3.1]: $m=2$, $s_0=0$, $s_1=1$, $r_0=r_1=0$,

$$\varphi_0(t,z)=\frac{\sinh((1-z)t)}{\sinh(t)},\quad\varphi_1(t,z)=\frac{\sinh(tz)}{\sinh(t)}\cdot$$

(2) Whittaker [Waldschmidt 2019, §5.1]: $m=2$, $s_0=1$, $s_1=0$, $r_0=0$, $r_1=1$,

$$\varphi_0(t,z)=\frac{\cosh(tz)}{\cosh(t)},\quad\varphi_1(t,z)=\frac{\sinh((z-1)t)}{\cosh(t)}\cdot$$

(3) Poritsky interpolation – see [Macintyre 1954, §3]: $r_0=r_1=\cdots=r_{m-1}=0$. The condition $\mathrm{D}(\mathbf{s})=0$ means that $s_0,s_1,\dots,s_{m-1}$ are pairwise distinct. The coefficient of $t^{m(m-1)/2}$ in the Taylor expansion at the origin of $\Delta(t)$ is given by the following formula involving two Vandermonde determinants:

$$\frac{1}{1!2!\cdots(m-1)!}\det\begin{pmatrix}1&1&\cdots&1\\1&\zeta&\cdots&\zeta^{m-1}\\1&\zeta^2&\cdots&\zeta^{2(m-1)}\\\vdots&\vdots&\ddots&\vdots\\1&\zeta^{m-1}&\cdots&\zeta^{(m-1)^2}\end{pmatrix}\det\begin{pmatrix}1&1&\cdots&1\\s_0&s_1&\cdots&s_{m-1}\\s_0^2&s_1^2&\cdots&s_{m-1}^2\\\vdots&\vdots&\ddots&\vdots\\s_0^{m-1}&s_1^{m-1}&\cdots&s_{m-1}^{m-1}\end{pmatrix}.$$

Hence $\Delta(t)$ has a zero at the origin of multiplicity $m(m-1)/2$.

For $0\le j\le m-1$, the order of the zero at $t=0$ of $\Delta(t)\varphi_j(t,z)$ is at least $m(m-1)/2$.

(4) Gontcharoff interpolation – see [Macintyre 1954, §4]: $r_j = j$ for $j = 0, 1, \dots, m-1$. In this case $\Delta(0)$ is the Vandermonde determinant

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \zeta & \cdots & \zeta^{m-1} \\ 1 & \zeta^2 & \cdots & \zeta^{2(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{m-1} & \cdots & \zeta^{(m-1)^2} \end{pmatrix},$$

and hence is not zero.

4.5 Laplace transform

The main tool for the proof of Theorem 2 is the following result.

Proposition 11. *Assume* $\mathrm{D}(\mathbf{s}) \neq 0$ *and* $\Delta(t) \neq 0$ *for* $0 < |t| < \tau$. *Then any entire function* f *of exponential type* $< \tau$ *has an expansion of the form* (4.2), *where the series in the right hand side is absolutely and uniformly convergent for* z *on any compact space in* $\mathbb{C}$.

As a consequence:

Corollary 12. *Under the assumptions of Proposition 11, if an entire function* f *has exponential type* $< \tau$ *and satisfies*

$$f^{(mn+r_j)}(s_j) = 0 \textit{ for } j = 0, \dots, m-1 \textit{ and all sufficiently large } n,$$

then f *is a polynomial.*

The bound for the exponential type is sharp: if $\alpha \neq 0$ is a zero of Δ, then there exists a transcendental entire function of exponential type $|\alpha|$ satisfying the vanishing conditions of Corollary 12; for the proof, see Proposition 9(a) of [Waldschmidt 2019].

The strategy for the proof of Proposition 11 will be to check that for $|t| < \tau$, the function $f_t(z) = \mathrm{e}^{tz}$ admits the expansion (4.2), and then to deduce the general case by means of the Laplace transform, which is called the method of the kernel expansion in [Buck 1955], [Boas and Buck 1964, Chap. I §3], [Macintyre 1954, §1].

We have $f_t^{(m)} = t^m f_t$ and

$$f_t^{(r_j)}(s_j) = t^{r_j}\mathrm{e}^{ts_j}.$$

Proof of Proposition 11. Let

$$f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$$

be an entire function of exponential type $\tau(f)$. Using (2.1), we deduce that the Laplace transform of f,

$$F(t) = \sum_{n\geq 0} a_n t^{-n-1},$$

is analytic in the domain $|t| > \tau(f)$. From Cauchy's residue Theorem, it follows that for $\varrho > \tau(f)$ we have

$$f(z) = \frac{1}{2\pi i} \int_{|t|=\varrho} \mathrm{e}^{tz} F(t) \mathrm{d}t.$$

Hence

$$f^{(mn+r_j)}(z) = \frac{1}{2\pi i} \int_{|t|=\varrho} t^{mn+r_j} \mathrm{e}^{tz} F(t) \mathrm{d}t.$$

Assume $\tau(f) < \tau$. Let ϱ satisfy $\tau(f) < \varrho < \tau$. We deduce from (4.3) (which is valid for $|t| < \tau$) and Lemma 8 that, for $|t| = \varrho$, we have

$$\mathrm{e}^{tz} = \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} \varphi_j(t, z) = \sum_{n \geq 0} \sum_{j=0}^{m-1} \mathrm{e}^{ts_j} t^{mn+r_j} \Lambda_{nj}(z),$$

which is the expansion (4.2) for the function $f_t(z) = \mathrm{e}^{tz}$.

We now use Lemma 10 and permute the integral and the series to deduce

$$f(z) = \sum_{n \geq 0} \sum_{j=0}^{m-1} \left(\frac{1}{2\pi i} \int_{|t|=\varrho} t^{mn+r_j} \mathrm{e}^{ts_j} F(t) \mathrm{d}t \right) \Lambda_{nj}(z) = \sum_{n \geq 0} f^{(mn+r_j)}(s_j) \Lambda_{nj}(z).$$

Using again Lemma 10 together with (2.1), we check that the last series is absolutely and uniformly convergent for z on any compact space in $\mathbb{C}$. □

Proof of Theorem 2. Let f be an entire function satisfying the assumptions of Theorem 2. From the assumption (2.4) and Proposition 1, we deduce that for n sufficiently large, we have

$$f^{(mn+r_j)}(s_j) = 0 \text{ for } j = 0, \ldots, m-1.$$

Since the exponential type of f is $< \tau$, we deduce from Corollary 12 that f is a polynomial. □

5 Sequence of derivatives at several points

Given a sequence $\mathbf{w} = (w_n)_{n \geq 0}$ of complex numbers, we investigate the entire functions f such that the numbers $f^{(n)}(w_n)$ are in $\mathbb{Z}$. Under suitable assumptions, we reduce this question to the case where these numbers all vanish.

5.1 Abel–Gontcharoff interpolation

We start with any sequence $\mathbf{w} = (w_n)_{n \geq 0}$ of complex numbers. Following [Gontcharoff 1930] (see also [Evgrafov 1954], [Popov 2002]), we define a sequence of polynomials $(\Omega_{w_0, w_1, \ldots, w_{n-1}})_{n \geq 0}$ in $\mathbb{C}[z]$ as follows: we set $\Omega_\emptyset = 1$, $\Omega_{w_0}(z) = z - w_0$, and, for $n \geq 1$, we define $\Omega_{w_0, w_1, w_2, \ldots, w_n}(z)$ as the polynomial of degree $n+1$ which is the primitive of $\Omega_{w_1, w_2, \ldots, w_n}$ vanishing at w_0. For $n \geq 0$, we write $\Omega_{n;\mathbf{w}}$ for $\Omega_{w_0, w_1, \ldots, w_{n-1}}$, a polynomial of degree n which depends only on the first n terms of the sequence $\mathbf{w}$. The leading term of $\Omega_{n;\mathbf{w}}$ is $(1/n!)z^n$. An equivalent definition is

$$\Omega_{n;\mathbf{w}}^{(k)}(w_k) = \delta_{kn}$$

for $n \geq 0$ and $k \geq 0$. As a consequence, any polynomial P can be written as a finite sum

$$P(z) = \sum_{n\geq 0} P^{(n)}(w_n)\Omega_{n;\mathbf{w}}(z).$$

In particular, for $N \geq 0$ we have

$$\frac{z^N}{N!} = \sum_{n=0}^{N} \frac{1}{(N-n)!} w_n^{N-n}\Omega_{n;\mathbf{w}}(z).$$

This gives an inductive formula defining $\Omega_{N;\mathbf{w}}$: for $N \geq 0$,

$$\Omega_{N;\mathbf{w}}(z) = \frac{z^N}{N!} - \sum_{n=0}^{N-1} \frac{1}{(N-n)!} w_n^{N-n}\Omega_{n;\mathbf{w}}(z). \tag{5.1}$$

We also have

$$\Omega_{w_0,w_1,\ldots,w_n}(z) = \Omega_{0,w_1-w_0,w_2-w_0,\ldots,w_n-w_0}(z-w_0).$$

With $w_0 = 0$, the first polynomials are given by

$$\begin{aligned}
2!\Omega_{0,w_1}(z) &= (z-w_1)^2 - w_1^2,\\
3!\Omega_{0,w_1,w_2}(z) &= (z-w_2)^3 - 3(w_1-w_2)^2 z + w_2^3,\\
4!\Omega_{0,w_1,w_2,w_3}(z) &= (z-w_3)^4 - 6(w_2-w_3)^2(z-w_1)^2\\
&\quad - 4(w_1-w_3)^3 z + 6w_1^2(w_2-w_3)^2 - w_3^4.
\end{aligned}$$

From the definition we deduce the following formula, involving iterated integrals

$$\Omega_{w_0,w_1,\ldots,w_{n-1}}(z) = \int_{w_0}^{z} \mathrm{d}t_1 \int_{w_1}^{t_1} \mathrm{d}t_2 \cdots \int_{w_{n-1}}^{t_{n-1}} \mathrm{d}t_n.$$

These polynomials are also given by a determinant [Gontcharoff 1930, p. 7]

$$\Omega_{w_0,w_1,\ldots,w_{n-1}}(z) = (-1)^n \begin{vmatrix}
1 & \frac{z}{1!} & \frac{z^2}{2!} & \cdots & \frac{z^{n-1}}{(n-1)!} & \frac{z^n}{n!}\\
1 & \frac{w_0}{1!} & \frac{w_0^2}{2!} & \cdots & \frac{w_0^{n-1}}{(n-1)!} & \frac{w_0^n}{n!}\\
0 & 1 & \frac{w_1}{1!} & \cdots & \frac{w_1^{n-2}}{(n-2)!} & \frac{w_1^{n-1}}{(n-1)!}\\
0 & 0 & 1 & \cdots & \frac{w_2^{n-3}}{(n-3)!} & \frac{w_2^{n-2}}{(n-2)!}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 1 & \frac{w_{n-1}}{1!}
\end{vmatrix}.$$

With the sequence $\mathbf{w} = (1, 0, 1, 0, \dots, 0, 1, \dots)$, we recover the Whittaker polynomials [Waldschmidt 2019, §5]

$$\Omega_{2n;\mathbf{w}}(z) = M_n(z), \quad \Omega_{2n+1,\mathbf{w}}(z) = M'_{n+1}(z-1).$$

Another example, considered by N. Abel (see [Halphén 1882], [Gontcharoff 1930, p. 7], [Buck 1948, §7]), is the arithmetic progression $\mathbf{w} = (a + nt)_{n\geq 0}$ with a in $\mathbb{C}$ and t in $\mathbb{C} \setminus \{0\}$, where

$$\Omega_{n;\mathbf{w}}(z) = \frac{1}{n!}(z-a)(z-a-nt)^{n-1}$$

for $n \geq 1$, which satisfies

$$\Omega'_{n;\mathbf{w}}(z) = \Omega_{n-1;\mathbf{w}}(z-t).$$

Theorem III in [Gontcharoff 1930, p. 29] gives sufficient conditions on the sequence $(w_n)_{n\geq 0}$ so that an entire function f satisfying some growth condition has an expansion

$$f(z) = \sum_{n\geq 0} f^{(n)}(w_n)\Omega_{n;\mathbf{w}}(z).$$

In the case that we are going to consider where the sequence $(|w_n|)_{n\geq 0}$ is bounded, say $|w_n - w_0| \leq r$, the condition [Gontcharoff 1930, (31') p. 33] reduces to $\tau < 1/(\mathrm{e}r)$. See also [Whittaker 1934, §10] for an improvement in the case $m = 2$.

From now on we assume that the sequence $(|w_n|)_{n\geq 0}$ is bounded. Let $A > \sup_{n\geq 0} |w_n|$.

Proposition 13. *Let $\kappa > 1/\log 2$. For n sufficiently large, we have, for all $r \geq |A|$,*

$$|\Omega_{n;\mathbf{w}}|_r \leq (\kappa r)^n.$$

Proof. Let $c_0, c_1, c_2, \dots$ be the sequence of positive numbers defined by induction as follows: $c_0 = 1$ and, for $n \geq 1$,

$$c_n = \frac{1}{n!} + \frac{c_0}{n!} + \frac{c_1}{(n-1)!} + \cdots + \frac{c_{n-2}}{2!} + c_{n-1}.$$

From (5.1) we deduce by induction, for $|z| \leq r$ and all $n \geq 0$,

$$|\Omega_{n;\mathbf{w}}(z)| \leq c_n r^n.$$

Let κ_1 satisfy $1/\log 2 < \kappa_1 < \kappa$ and let $A > 0$ satisfy

$$A \geq \left(2 - \mathrm{e}^{\frac{1}{\kappa_1}}\right)^{-1} \max_{n\geq 0} \frac{1}{\kappa_1^n n!}.$$

One checks by induction $c_n \leq A\kappa_1^n$ for all $n \geq 0$ thanks to the upper bound

$$\frac{1}{n!} + A\kappa_1^n \left(\mathrm{e}^{\frac{1}{\kappa_1}} - 1\right) \leq A\kappa_1^n.$$

Therefore, for sufficiently large n, we have $c_n < \kappa^n$. □

In the case $m = 2$ and $w_n \in \{0, 1\}$ for all $n \geq 0$, a sharper estimate has been achieved in [Whittaker 1934, §10], namely

$$|\Omega_{n;\mathbf{w}}(z)| \leq \frac{1}{2}\mathrm{e}^2 \left(\frac{1}{2} + R\right)^n$$

for $|z - \frac{1}{2}| = R$. The proof relies on explicit formulae for the polynomials $\Omega_{n;\mathbf{w}}(z)$.

From Proposition 13 we deduce the following interpolation formula:

Proposition 14. *Let f be an entire function of exponential type $\tau(f)$ satisfying*

$$\tau(f) < \frac{\log 2}{A}.$$

Let r be a real number in the range

$$A \leq r < \frac{\log 2}{\tau(f)}.$$

Then

$$f(z) = \sum_{n \geq 0} f^{(n)}(w_n)\Omega_{n;\mathbf{w}}(z),$$

where the series on the right hand side is absolutely and uniformly convergent in the disk $|z| \leq r$.

Proof. Let κ and τ satisfy

$$\kappa > \frac{1}{\log 2}, \quad \tau(f) < \tau < \frac{1}{\kappa r}.$$

Write the Taylor expansion of f at the origin:

$$f(z) = \sum_{N \geq 0} a_N \frac{z^N}{N!}.$$

From (2.1) we deduce that there exists a constant $c > 0$ such that, for all $N \geq 0$, we have $|a_N| \leq c\tau^N$. For $|z| \leq r$, we have

$$|a_N| \sum_{n=0}^{N} \left| \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\mathbf{w}}(z) \right| \leq c\tau^N \sum_{n=0}^{N} \frac{A^{N-n}(\kappa r)^n}{(N-n)!} \leq c\mathrm{e}^{A/\kappa r}(\tau\kappa r)^N,$$

which is the general term of a convergent series, since $\tau\kappa r < 1$. Hence

$$\begin{aligned} f(z) &= \sum_{N \geq 0} a_N \sum_{n=0}^{N} \frac{1}{(N-n)!} w_n^{N-n} \Omega_{n;\mathbf{w}}(z) \\ &= \sum_{n \geq 0} \Omega_{n;\mathbf{w}}(z) \sum_{N \geq n} a_N \frac{1}{(N-n)!} w_n^{N-n} = \sum_{n \geq 0} \Omega_{n;\mathbf{w}}(z) f^{(n)}(w_n). \end{aligned}$$

□

Remark. Notice that here the expansions are valid in a bounded domain of $\mathbb{C}$, not in the entire complex plane as in Section 4.5 for instance.

Corollary 15. *If an entire function f of exponential type $\tau(f) < \log 2/A$ satisfies $f^{(n)}(w_n) = 0$ for all sufficiently large n, then f is a polynomial.*

Replacing z by Az, one may assume $A = 1$, and then Corollary 15 is [Whittaker 1964, Theorem 8], a special case of one of Takenaka's theorems.

In the special case where the set $\{w_0, w_1, w_2, \dots\}$ is finite, say $S = \{s_0, s_1, \dots, s_{m-1}\}$ with

$$\max\{|s_0|, |s_1|, \dots, |s_{m-1}|\} < A,$$

Corollary 15 reduces to the following statement:

Corollary 16. *If an entire function f of exponential type $\tau(f) < \log 2/A$ satisfies*

$$\prod_{j=0}^{m-1} f^{(n)}(s_j) = 0$$

for all sufficiently large n, then f is a polynomial.

5.2 Sequence of elements in S

Proof of Theorem 5. Denote by $\tau(f)$ the exponential type of f. Since f satisfies the hypothesis (2.5) of Proposition 1, for n sufficiently large we have $|f^{(n)}(z)| < 1$ for all $|z| < A$.

Under the assumption (a) of Theorem 5, for n sufficiently large we have $f^{(n)}(w_n) = 0$. Corollary 15 implies that f is a polynomial.

For each sufficiently large n, the product $\prod_{j\in I_n} f^{(n)}(s_j)$ is an integer of absolute value less than 1, and hence it vanishes. Part (b) of Theorem 5 follows from Corollary 16. □

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References

[Boas and Buck 1964] R. P. Boas, Jr. and R. C. Buck, *Polynomial expansions of analytic functions*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F. **19**, Academic Press, New York, 1964. MR Zbl

[Buck 1948] R. C. Buck, "Interpolation series", *Trans. Amer. Math. Soc.* **64** (1948), 283–298. MR Zbl

[Buck 1955] R. C. Buck, "On n-point expansions of entire functions", *Proc. Amer. Math. Soc.* **6** (1955), 793–796. MR Zbl

[Evgrafov 1954] M. A. Evgrafov, *Interpolyacionnaya zadača Abelya-Gončarova*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. MR Zbl

[Gel'fond 1971] A. O. Gel'fond, *Calculus of finite differences*, Hindustan Publishing Corp., Delhi, 1971. MR Zbl

[Gontcharoff 1930] W. Gontcharoff, "Recherches sur les dérivées successives des fonctions analytiques. Généralisation de la série d'Abel", *Ann. Sci. École Norm. Sup.* (3) **47** (1930), 1–78. MR Zbl

[Halphén 1882] G. Halphén, "Sur une série d'Abel", *Bull. Soc. Math. France* **10** (1882), 67–87. MR JFM

[Macintyre 1954] A. J. Macintyre, "Interpolation series for integral functions of exponential type", *Trans. Amer. Math. Soc.* **76** (1954), 1–13. MR Zbl

[Popov 2002] A. Y. Popov, "Bounds for the convergence and uniqueness of Abel-Goncharov interpolation problems", *Mat. Sb.* **193**:2, 97–128. In Russian; translated in Sb. Math. **193**:2 (2002), 247–277. MR Zbl

[Poritsky 1932] H. Poritsky, "On certain polynomial and other approximations to analytic functions", *Trans. Amer. Math. Soc.* **34**:2 (1932), 274–331. MR Zbl

[Waldschmidt 2019] M. Waldschmidt, "On transcendental entire functions with infinitely many derivatives taking integer values at two points", preprint, 2019. arXiv: 1912.00173

[Whittaker 1934] J. M. Whittaker, "On Lidstone's series and two-point expansions of analytic functions", *Proc. Lond. Math. Soc. (2)* **36** (1934), 451–469. MR Zbl

[Whittaker 1964] J. M. Whittaker, *Interpolatory function theory*, Cambridge Tracts in Mathematics and Mathematical Physics **33**, Stechert-Hafner, New York, 1964. MR Zbl

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