Interpolation of analytic functions
and arithmetic applications.

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Abstract

The very first interpolation formula for analytic functions is given by Taylor series. There are many other ways of interpolating analytic functions. Lagrange interpolation polynomials involve the values of the function at several points; some derivatives may be included. We discuss other types of interpolation formulae, starting with Lidstone interpolation of a function of exponential type $< \pi$ by its derivatives of even order at $0$ and $1$. A new result is a lower bound for the exponential type of a transcendental entire function having derivatives of even order at two points taking integer values.
The interpolation problem

An entire function is a holomorphic (＝analytic) map \( \mathbb{C} \to \mathbb{C} \). The graph \( \{(z, f(z)) \mid z \in \mathbb{C}\} \) has the power of continuum.

However, such a function is uniquely determined by a countable set; for instance by the sequence of coefficients of its Taylor series at a given point \( z_0 \):

\[
f(z) = \sum_{n \geq 0} f^{(n)}(z_0) \frac{(z - z_0)^n}{n!}.
\]

Notation:

\[
f^{(n)}(z) = \frac{d^n}{dz^n} f(z).
\]

There are other sequences of numbers which determine uniquely an entire function, at least if we restrict to some classes of entire functions.
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References on interpolation:


Interpolation data

Given complex numbers \( \{\sigma_i\}_{i \in I}, \{a_i\}_{i \in I} \) and nonnegative integers \( \{k_i\}_{i \in I} \), the interpolation problem is to decide whether there exists an analytic function \( f \) satisfying

\[
f^{(k_i)}(\sigma_i) = a_i \quad \text{for all } i \in I.
\]

We will consider this question for \( f \) analytic everywhere in \( \mathbb{C} \) (i.e. \( f \) an entire function) and \( I = \mathbb{N} \).

The unicity is given by the answer to the same question with \( a_i = 0 \) for all \( i \in I \).

Taylor series: \( \sigma_n = 0 \) and \( k_n = n \) for all \( n \geq 0 \). The solution, if it exists, is unique

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Calculus of finite differences

Another classical interpolation problem is given by the data $k_n = 0$ and $\sigma_n = n$ for all $n \geq 0$. Given complex numbers $a_n$, does there exist an entire function $f$ satisfying

$$f(n) = a_n \text{ for all } n \geq 0?$$

The answer depends on the growth of the sequence $(a_n)_{n \geq 0}$. The example of the function $\sin(\pi z)$ shows that the solution is not unique in general. However we recover unicity by adding a condition on the growth of the solution $f$.

For the existence, one uses interpolation formulae based on

$$f(z) = f(0) + zf_1(z), \quad f_1(z) = f_1(1) + (z - 1)f_2(z),$$
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Further interpolation problems

We are going to consider the following interpolation problems:

- **(Lidstone):**
  \[ f^{(2n)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0. \]

- **(Whittaker):**
  \[ f^{(2n+1)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0. \]

- **(Poritsky):** For \( m \geq 2 \) and \( \sigma_0, \ldots, \sigma_{m-1} \) in \( \mathbb{C} \),
  \[ f^{(mn)}(\sigma_j) = a_{nj} \text{ for } n \geq 0 \text{ and } j = 0, 1, \ldots, m - 1. \]

- **(Gontcharoff):** For \( (\sigma_n)_{n \geq 0} \) a sequence of complex numbers,
  \[ f^{(n)}(\sigma_n) = a_n \text{ for } n \geq 0. \]
Lidstone interpolation problem

The following interpolation problem was considered by G.J. Lidstone in 1930.

*Given two sequences of complex numbers \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\), does there exist an entire function \(f\) satisfying*

\[
f^{(2n)}(0) = a_n, \quad f^{(2n)}(1) = b_n \quad \text{for } n \geq 0 \quad ?
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*Is such a function \(f\) unique?*

The answer to unicity is plain: the function \(\sin(\pi z)\) satisfies these conditions with \(a_n = b_n = 0\), hence there is no unicity, unless we restrict the question to entire functions satisfying some extra condition. Such a condition is a bound on the growth of \(f\).

We start with unicity \((a_n = b_n = 0)\) and polynomials.
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Lemma. Let $f$ be a polynomial satisfying

$$f^{(2n)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

Then $f = 0$.

First proof.

By induction on the degree of the polynomial $f$.
If $f$ has degree $\leq 1$, say $f(z) = a_0 z + a_1$, the conditions $f(0) = f(1) = 0$ imply $a_0 = a_1 = 0$, hence $f = 0$.
If $f$ has degree $\leq n$ with $n \geq 2$ and satisfies the hypotheses, then $f''$ also satisfies the hypotheses and has degree $< n$, hence by induction $f'' = 0$ and therefore $f$ has degree $\leq 1$. The result follows.
Even derivatives at 0 and 1: first proof

Lemma. Let $f$ be a polynomial satisfying

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If \( f \) has degree \( \leq n \) with \( n \geq 2 \) and satisfies the hypotheses, then \( f'' \) also satisfies the hypotheses and has degree \( < n \), hence by induction \( f'' = 0 \) and therefore \( f \) has degree \( \leq 1 \).
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Second proof.

Let $f$ be a polynomial satisfying

$$f^{(2n)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$ 

The assumption $f^{(2n)}(0) = 0$ for all $n \geq 0$ means that $f$ is an odd function: $f(-z) = -f(z)$. The assumption $f^{(2n)}(1) = 0$ for all $n \geq 0$ means that $f(1 - z)$ is an odd function: $f(1 - z) = -f(1 + z)$. We deduce

$$f(z + 2) = f(1 + z + 1) = -f(1 - z - 1) = -f(-z) = f(z),$$

together with

hence the polynomial $f$ is periodic, and therefore it is a constant. Since $f(0) = 0$, we conclude $f = 0$. $\square$
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Even derivatives at 0 and 1: third proof

Third proof.

Assume
\[ f^{(2n)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0. \]

Write
\[ f(z) = a_1 z + a_3 z^3 + a_5 z^5 + a_7 z^7 + \cdots + a_{2n+1} z^{2n+1} + \cdots \]

(finite sum). We have \( f(1) = f''(1) = f^{(iv)}(1) = \cdots = 0 \):

\[
\begin{array}{ccccccccc}
  a_1 & +a_3 & +a_5 & +a_7 & + \cdots & +a_{2n+1} & + \cdots & = 0 \\
  6a_3 & +20a_5 & +42a_7 & + \cdots & +2n(2n+1)a_{2n+1} & + \cdots & = 0 \\
  120a_5 & +840a_7 & + \cdots & +\frac{(2n+1)!}{(2n-3)!}a_{2n+1} & + \cdots & = 0 \\
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\]

The matrix of this system is triangular with maximal rank. \( \square \)
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\begin{align*}
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120a_5 &+ 840a_7 &+ \cdots &+ \frac{(2n+1)!}{(2n-3)!} a_{2n+1} &+ \cdots = 0 \\
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Even derivatives at 0 and 1

The fact that this matrix has maximal rank means that a polynomial $f$ is uniquely determined by the numbers

$$f^{(2n)}(0) \text{ and } f^{(2n)}(1) \text{ for } n \geq 0.$$ 

Given numbers $a_n$ and $b_n$, all but finitely many of them are 0, there is a unique polynomial $f$ such that

$$f^{(2n)}(0) = a_n \text{ and } f^{(2n)}(1) = b_n \text{ for all } n \geq 0.$$ 

Involution : $z \mapsto 1 - z :$

$$0 \mapsto 1, \quad 1 \mapsto 0, \quad 1 - z \mapsto z.$$
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Given numbers $a_n$ and $b_n$, all but finitely many of them are 0, there is a unique polynomial $f$ such that

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Involution : $z \mapsto 1 - z$ :

$$0 \mapsto 1, \quad 1 \mapsto 0, \quad 1 - z \mapsto z.$$
Lidstone expansion of a polynomial

G. J. Lidstone (1930). There exists a unique sequence of polynomials \( \Lambda_0(z) \), \( \Lambda_1(z) \), \( \Lambda_2(z) \), \ldots such that any polynomial \( f \) can be written as a finite sum

\[
f(z) = \sum_{n \geq 0} f(2n)(0)\Lambda_n(1 - z) + \sum_{n \geq 0} f(2n)(1)\Lambda_n(z).
\]

This is equivalent to

\[\Lambda_n^{(2k)}(0) = 0 \text{ and } \Lambda_n^{(2k)}(1) = \delta_{nk} \text{ for } n \geq 0 \text{ and } k \geq 0.\]

(Kronecker symbol).

A basis of the \( \mathbb{Q} \)-space of polynomials in \( \mathbb{Q}[z] \) of degree \( \leq 2n + 1 \) is given by the \( 2n + 2 \) polynomials

\( \Lambda_0(z), \Lambda_1(z), \ldots, \Lambda_n(z), \Lambda_0(1 - z), \Lambda_1(1 - z), \ldots, \Lambda_n(1 - z) \).
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Analogy with Taylor series

Given a sequence \((a_n)_{n \geq 0}\) of complex numbers, the unique analytic solution (if it exists) \(f\) of the interpolation problem

\[ f^{(n)}(0) = a_n \text{ for all } n \geq 0 \]

is given by the Taylor expansion

\[ f(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}. \]

The polynomials \(z^n/n!\) satisfy

\[ \frac{d^k}{dz^k} \left( \frac{z^n}{n!} \right)_{z=0} = \delta_{nk} \text{ for } n \geq 0 \text{ and } k \geq 0. \]
**Lidstone polynomials**

\[ \Lambda_0(z) = z : \]

\[ \Lambda_0(0) = 0, \quad \Lambda_0(1) = 1, \quad \Lambda_0^{(2n)}(0) = 0 \text{ for } n \geq 1. \]

Induction: the sequence of Lidstone polynomials is determined by \( \Lambda_0(z) = z \) and

\[ \Lambda''_n = \Lambda_{n-1} \text{ for } n \geq 1 \]

with the initial conditions \( \Lambda_n(0) = \Lambda_n(1) = 0 \) for \( n \geq 1 \).

Let \( L_n(z) \) be any solution of

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\[ \Lambda_n(z) = -L_n(1)z + L_n(z). \]
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For \( n \geq 0 \), the polynomial \( \Lambda_n \) is odd, it has degree \( 2n + 1 \) and leading term \( \frac{1}{(2n+1)!} z^{2n+1} \).

For instance

\[ \Lambda_1(z) = \frac{1}{6} (z^3 - z) \]

and

\[ \Lambda_2(z) = \frac{1}{120} z^5 - \frac{1}{36} z^3 + \frac{7}{360} z = \frac{1}{360} z (z^2 - 1)(3z^2 - 7). \]
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The polynomial \( f(z) = z^{2n+1} \) satisfies

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 f^{(2k)}(0) = 0 \text{ for } k \geq 0, \quad f^{(2k)}(1) = \begin{cases} 
 \frac{(2n+1)!}{(2n-2k+1)!} & \text{for } 0 \leq k \leq n, \\
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\end{cases}
\]

One deduces

\[
z^{2n+1} = \sum_{k=0}^{n-1} \frac{(2n+1)!}{(2n-2k+1)!} \Lambda_k(z) + (2n+1)!\Lambda_n(z),
\]

which yields the induction formula

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Order and exponential type

Order of an entire function:

\[ \varrho(f) = \limsup_{r \to \infty} \frac{\log \log |f|_r}{\log r} \text{ where } |f|_r = \sup_{|z|=r} |f(z)|. \]

Exponential type of an entire function:

\[ \tau(f) = \limsup_{r \to \infty} \frac{\log |f|_r}{r}. \]

If the exponential type is finite, then \( f \) has order \( \leq 1 \). If \( f \) has order \( < 1 \), then the exponential type is 0.

For \( \tau \in \mathbb{C} \setminus \{0\} \), the function \( e^{\tau z} \) has order 1 and exponential type \( |\tau| \).
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For \( \tau \in \mathbb{C} \setminus \{0\} \), the function \( e^{\tau z} \) has order \( 1 \) and exponential type \( |\tau| \).
An alternative definition of the exponential type is the following: \( f \) is of exponential type \( \tau(f) \) if and only if, for all \( z_0 \in \mathbb{C} \),

\[
\limsup_{n \to \infty} \left| f^{(n)}(z_0) \right|^{1/n} = \tau(f).
\]

The equivalence between the two definitions follows from Cauchy’s inequalities and Stirling’s Formula.
Exponential type

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Theorem (H. Poritsky, 1932).

Let $f$ be an entire function of exponential type $< \pi$ satisfying $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all sufficiently large $n$. Then $f$ is a polynomial.

This is best possible: the entire function $\sin(\pi z)$ has exponential type $\pi$ and satisfies $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$. 
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Proof.
Let $\tilde{f} = f - P$, where $P$ is the polynomial satisfying

$$P^{(2n)}(0) = f^{(2n)}(0) \quad \text{and} \quad P^{(2n)}(1) = f^{(2n)}(1) \quad \text{for} \quad n \geq 0.$$ 

We have $\tilde{f}^{(2n)}(0) = \tilde{f}^{(2n)}(1) = 0$ for all $n \geq 0$. The functions $\tilde{f}(z)$ and $\tilde{f}(1 - z)$ are odd, hence $\tilde{f}(z)$ is periodic of period 2. Therefore there exists an entire function $g$ such that $\tilde{f}(z) = g(e^{i\pi z})$. Since $\tilde{f}(z)$ has exponential type $< \pi$, we deduce $\tilde{f} = 0$ and $f = P$. \qed
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Lidstone series: exponential type $< \pi$

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Theorem (H. Poritsky, 1932).

The expansion

\[ f(z) = \sum_{n=0}^{\infty} f^{(2n)}(0) \Lambda_n(1 - z) + \sum_{n=0}^{\infty} f^{(2n)}(1) \Lambda_n(z) \]

holds for any entire function \( f \) of exponential type \( < \pi \).

We will check this formula for \( f_t(z) = e^{tz} \) with \( |t| < \pi \), then deduce the general case.
Solution of the Lidstone interpolation problem

Consequence of Poritsky’s expansion formula:
Let \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) be two sequences of complex numbers satisfying

\[
\limsup_{n \to \infty} |a_n|^{1/n} < \pi \quad \text{and} \quad \limsup_{n \to \infty} |b_n|^{1/n} < \pi.
\]

Then the function

\[
f(z) = \sum_{n=0}^{\infty} a_n \Lambda_n (1 - z) + \sum_{n=0}^{\infty} b_n \Lambda_n (z)
\]

is the unique entire function of exponential type \(< \pi\) satisfying

\[
f^{(2n)}(0) = a_n \quad \text{and} \quad f^{(2n)}(1) = b_n \quad \text{for all} \quad n \geq 0.
\]
Special case: $e^{tz}$ for $|t| < \pi$

Consider Poritsky’s expansion formula

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(0) \Lambda_n(1 - z) + \sum_{n=0}^{\infty} f^{(2n)}(1) \Lambda_n(z)$$

for the function $f_t(z) = e^{tz}$ where $|t| < \pi$. Since $f_t^{(2n)}(0) = t^{2n}$ and $f_t^{(2n)}(1) = t^{2n}e^t$ it gives

$$e^{tz} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(1 - z) + e^t \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).$$

Replacing $t$ with $-t$ yields

$$e^{-tz} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(1 - z) + e^{-t} \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).$$

Hence

$$e^{tz} - e^{-tz} = (e^t - e^{-t}) \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).$$
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Generating series

Let $t \in \mathbb{C}$, $t \not\in i\pi\mathbb{Z}$. The entire function

$$f(z) = \frac{\sinh(tz)}{\sinh(t)} = \frac{e^{tz} - e^{-tz}}{e^t - e^{-t}}$$

satisfies

$$f'' = t^2 f, \quad f(0) = 0, \quad f(1) = 1,$$

hence $f^{(2n)}(0) = 0$ and $f^{(2n)}(1) = t^{2n}$ for all $n \geq 0$.

For $0 < |t| < \pi$ and $z \in \mathbb{C}$, we deduce

$$\frac{\sinh(tz)}{\sinh(t)} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).$$

Notice that

$$e^{tz} = \frac{\sinh((1 - z)t)}{\sinh(t)} + e^t \frac{\sinh(tz)}{\sinh(t)}.$$
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Special case: $e^{tz}$

From Poritsky's expansion of an entire function of exponential type $< \pi$ we deduced the formula

$$\frac{\sinh(tz)}{\sinh(t)} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).$$

Let us prove this formula directly. We will deduce

$$e^{tz} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(1-z) + e^t \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z)$$

for $|t| < \pi$. 
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Expansion of $F(z, t) = \frac{\sinh(tz)}{\sinh(t)}$

For $z \in \mathbb{C}$ and $|t| < \pi$ let

$$F(z, t) = \frac{\sinh(tz)}{\sinh(t)}$$

with $F(z, 0) = z$.

Fix $z \in \mathbb{C}$. The function $t \mapsto F(z, t)$ is analytic in the disc $|t| < \pi$ and is an even function: $F(z, -t) = F(z, t)$. Consider its Taylor series at the origin:

$$F(z, t) = \sum_{n \geq 0} c_n(z)t^{2n}$$

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$$F(z, t) = \sum_{n \geq 0} c_n(z) t^{2n}$$

with $c_0(z) = z$.

We have $F(0, t) = 0$ and $F(1, t) = 1$. 
Expansion of $F(z, t) = \frac{\sinh(tz)}{\sinh(t)}$

$$F(z, t) = \frac{e^{tz} - e^{-tz}}{e^t - e^{-t}} = \sum_{n \geq 0} c_n(z) t^{2n}.$$  

From

$$c_n(z) = \frac{1}{(2n)!} \left( \frac{\partial}{\partial t} \right)^{2n} F(z, 0)$$

it follows that $c_n(z)$ is a polynomial.

From

$$\left( \frac{\partial}{\partial z} \right)^2 F(z, t) = t^2 F(z, t)$$

we deduce

$$c_n'''' = c_{n-1} \text{ for } n \geq 1.$$  

Since $c_n(0) = c_n(1) = 0$ for $n \geq 1$ we conclude $c_n(z) = \Lambda_n(z)$.  

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From $e^{tz}$ to exponential type $< \pi$

Hence a special case of the Poritsky’s expansion formula

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(0) \Lambda_n(1-z) + \sum_{n=0}^{\infty} f^{(2n)}(1) \Lambda_n(z),$$

which holds for any entire function $f$ of exponential type $< \pi$, is

$$e^{tz} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(1-z) + e^t \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z)$$

for $|t| < \pi$.

Conversely, from this special case (that we proved directly) we are going to deduce the general case by means of Laplace transform (R.C. Buck, 1955, kernel expansion method).
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Conversely, from this special case (that we proved directly) we are going to deduce the general case by means of Laplace transform (R.C. Buck, 1955, *kernel expansion method*).
Laplace transform

Let

\[ f(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n \]

be an entire function of exponential type \( \tau(f) \). The Laplace transform of \( f \), viz.

\[ F(t) = \sum_{n \geq 0} a_n t^{-n-1}, \]

is analytic in the domain \( |t| > \tau(f) \). From Cauchy’s residue Theorem, it follows that for \( r > \tau(f) \) we have

\[ f(z) = \frac{1}{2\pi i} \int_{|t|=r} e^{tz} F(t) dt. \]

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\[ f^{(2n)}(z) = \frac{1}{2\pi i} \int_{|t|=r} t^{2n} e^{tz} F(t) dt. \]
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We deduce

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where the last series are absolutely and uniformly convergent for $z$ on any compact in $\mathbb{C}$. 
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**Theorem** (I.J. Schoenberg, 1936).

Let $f$ be an entire function of finite exponential type $\tau(f)$ satisfying $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$. Then there exist complex numbers $c_1, \ldots, c_L$ with $L \leq \tau(f)/\pi$ such that

$$f(z) = \sum_{\ell=1}^{L} c_{\ell} \sin(\ell \pi z).$$
Integral formula for Lidstone polynomials

Using Cauchy’s residue Theorem, we deduce the integral formula

\[
\Lambda_n(z) = (-1)^n \sum_{s=1}^{S} \frac{(-1)^s}{s^{2n+1}} \sin(s\pi z) + \frac{1}{2\pi i} \int_{|t|=(2S+1)\pi/2} t^{-2n-1} \frac{\sinh(tz)}{\sinh(t)} dt
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for \( S = 1, 2, \ldots \) and \( z \in \mathbb{C} \).

In particular, with \( S = 1 \) we have

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One deduces that there exists an absolute constant \( c > 0 \) such that

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|\Lambda_n|_r \leq c \pi^{-2n} e^{3\pi r/2}.
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A polynomial $f$ is determined up to the addition of a constant by the numbers

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Lemma. Let \( f \) be a polynomial satisfying
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Then \( f = 0 \).

Proofs.
1. By induction.
2. \( f(z + 4) = f(z) \).
3. Triangular system.
Odd derivatives at 0 and even derivatives at 1

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Whittaker expansion of a polynomial

The Lemma means that a polynomial $f$ is uniquely determined by the numbers

$$f^{(2n+1)}(0) \text{ and } f^{(2n)}(1) \text{ for } n \geq 0.$$ 

Any polynomial $f \in \mathbb{C}[z]$ has the finite expansion

$$f(z) = \sum_{n=0}^{\infty} \left( f^{(2n)}(1) M_n(z) - f^{(2n+1)}(0) M'_{n+1}(1 - z) \right),$$

with only finitely many nonzero terms in the series.

A basis of the $\mathbb{Q}$–space of polynomials in $\mathbb{Q}[z]$ of degree $\leq 2n$ is given by the $2n + 1$ polynomials

$$M_0(z), M_1(z), \ldots, M_n(z), \quad M'_1(1 - z), \ldots, M'_{n}(1 - z).$$
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Following J.M. Whittaker (1935), one defines a sequence $(M_n)_{n \geq 0}$ of even polynomials by induction on $n$ with $M_0 = 1,$

\[ M''_n = M_{n-1}, \quad M_n(1) = M'_n(0) = 0 \text{ for all } n \geq 1. \]

This is equivalent to

\[ M^{(2k+1)}_n(0) = 0, \quad M^{(2k)}_n(1) = \delta_{nk} \text{ for } n \geq 0 \text{ and } k \geq 0. \]

For instance

\[ M_1(z) = \frac{1}{2}(z^2 - 1), \quad M_2(z) = \frac{1}{24}(z^2 - 1)(z^2 - 5), \]

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Induction formula for Whittaker polynomials

The polynomial \( f(z) = z^{2n} \) satisfies

\[
f^{(2k+1)}(0) = 0 \quad \text{for} \quad k \geq 0, \quad f^{(2k)}(1) = \begin{cases} 
\frac{(2n)!}{(2n-2k)!} & \text{for} \quad 0 \leq k \leq n, \\
0 & \text{for} \quad k \geq n + 1.
\end{cases}
\]

One deduces

\[
z^{2n} = \sum_{k=0}^{n-1} \frac{(2n)!}{(2n-2k)!} M_k(z) + (2n)!M_n(z),
\]

which yields the following induction formula

\[
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Exponential type $< \pi/2$

**Theorem (J.M. Whittaker, 1935).**

The expansion

$$f(z) = \sum_{n=0}^{\infty} \left( f^{(2n)}(1) M_n(z) - f^{(2n+1)}(0) M'_{n+1}(1 - z) \right)$$

holds for any entire function $f$ of exponential type $< \pi/2$.

Hence, if such a function satisfies $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all sufficiently large $n$, then it is a polynomial.

This is best possible: the entire function $\cos(\frac{\pi}{2} z)$ has exponential type $\pi/2$ and satisfies $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$. 
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This is best possible: the entire function \( \cos(\pi/2 z) \) has exponential type \( \pi/2 \) and satisfies \( f^{(2n+1)}(0) = f^{(2n)}(1) = 0 \) for all \( n \geq 0 \).
Solution of the Whittaker interpolation problem

Consequence of Whittaker’s expansion formula:
Let \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) be two sequences of complex numbers satisfying

\[
\limsup_{n \to \infty} |a_n|^{1/n} < \pi \quad \text{and} \quad \limsup_{n \to \infty} |b_n|^{1/n} < \pi.
\]

Then the function

\[
f(z) = \sum_{n=0}^{\infty} a_n M_n(z) - \sum_{n=0}^{\infty} b_n M'_{n+1}(1 - z)
\]

is the unique entire function of exponential type \(< \pi\) satisfying

\[
f^{(2n)}(1) = a_n \quad \text{and} \quad f^{(2n+1)}(0) = b_n \quad \text{for all} \quad n \geq 0.
\]
Finite exponential type

Theorem (I.J. Schoenberg, 1936).

Let $f$ be an entire function of finite exponential type $\tau(f)$ satisfying $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$. Then there exist complex numbers $c_1, \ldots, c_L$ with $L \leq 2\tau(f)/\pi$ such that

$$f(z) = \sum_{\ell=0}^{L} c_\ell \cos \left( \frac{(2\ell + 1)\pi}{2} z \right).$$
Generating series

For $t \in \mathbb{C}$, $t \not\in i\pi + 2i\pi\mathbb{Z}$, the entire function

$$f(z) = \frac{\cosh(tz)}{\cosh(t)} = \frac{e^{tz} + e^{-tz}}{e^t + e^{-t}}$$

satisfies

$$f'' = t^2 f, \quad f(1) = 1, \quad f'(0) = 0,$$

hence $f^{(2n)}(1) = t^{2n}$ and $f^{(2n+1)}(0) = 0$ for all $n \geq 0$. The sequence $(M_n)_{n \geq 0}$ is also defined by the expansion

$$\frac{\cosh(tz)}{\cosh(t)} = \sum_{n=0}^{\infty} t^{2n} M_n(z)$$

for $|t| < \pi/2$ and $z \in \mathbb{C}$. 
Integral formula for Whittaker polynomials

Using Cauchy’s residue Theorem, we deduce the integral formula

\[
M_n(z) = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \sum_{s=0}^{S-1} \frac{(-1)^s}{(2s + 1)^{2n+1}} \cos \left( \frac{(2s + 1)\pi}{2} z \right) \\
+ \frac{1}{2\pi i} \int_{|t|=S\pi} t^{-2n-1} \frac{\cosh(tz)}{\cosh(t)} \, dt
\]

for \( S = 1, 2, \ldots \) and \( z \in \mathbb{C} \).

In particular, with \( S = 1 \) we obtain

\[
M_n(z) = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \cos(\pi z / 2) + \frac{1}{2\pi i} \int_{|t|=\pi} t^{-2n-1} \frac{\cosh(tz)}{\cosh(t)} \, dt.
\]
**Lidstone interpolation vs Whittaker interpolation**

Let us display horizontally the points and vertically the derivatives.

- **Lidstone interpolation values**
- **no condition**

### Lidstone interpolation

<table>
<thead>
<tr>
<th>( f )</th>
<th>( f'(2n+1) )</th>
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<th>( f'' )</th>
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### Generalizations with 3 points

#### Poritsky interpolation

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#### Gontcharoff interpolation

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Poritsky interpolation

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and $f$ an entire function of sufficiently small exponential type.

**Theorem (H. Poritsky, 1932).**

If

$$f^{(mn)}(s_0) = f^{(mn)}(s_1) = \cdots = f^{(mn)}(s_{m-1}) = 0$$

for all sufficiently large $n$, then $f$ is a polynomial.

For $m = 2$, $s_0 = 0$, $s_1 = 1$, this reduces Poritsky’s above mentioned result on Lidstone expansion (up to the exact bound on the exponential type).
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Let \( s_0, s_1, \ldots, s_{m-1} \) be distinct complex numbers and \( f \) an entire function of sufficiently small exponential type.

**Theorem (W. Gontcharoff 1930, A. J. Macintyre 1954).**

If

\[
f^{(n)}(s_0)f^{(n)}(s_1)\cdots f^{(n)}(s_{m-1}) = 0
\]

for all sufficiently large \( n \), then \( f \) is a polynomial.

For \( m = 2, s_0 = 0, s_1 = 1 \), this implies Whittaker’s above mentioned result for \( f^{(2n+1)}(0) = f^{(2n)}(1) = 0 \).
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Arithmetic result for Poritsky interpolation

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and $f$ an entire function of sufficiently small exponential type.

**Théorème 1.**

If

$$f^{(mn)}(s_j) \in \mathbb{Z}$$

for all sufficiently large $n$ and for $0 \leq j \leq m - 1$, then $f$ is a polynomial.

For $m = 2$ with $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$, the assumption on the exponential type $\tau(f)$ of $f$ is

$$\tau(f) < \min\{1, \pi/|s_0 - s_1|\},$$

and this is best possible.
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Arithmetic result for Lidstone interpolation

If \( \tau(f) < \min \left\{ 1, \frac{\pi}{|s_0 - s_1|} \right\} \) \( f^{(2n)}(s_0) \in \mathbb{Z} \) and \( f^{(2n)}(s_1) \in \mathbb{Z} \)

for all sufficiently large \( n \), then \( f \) is a polynomial.

The function

\[
f(z) = \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)}
\]

has exponential type 1 and satisfies \( f^{(2n)}(s_0) = 1 \) and \( f^{(2n)}(s_1) = 0 \) for all \( n \geq 0 \).

The function

\[
f(z) = \sin \left( \pi \frac{z - s_0}{s_1 - s_0} \right)
\]

has exponential type \( \frac{\pi}{|s_1 - s_0|} \) and satisfies

\( f^{(2n)}(s_0) = f^{(2n)}(s_1) = 0 \) for all \( n \geq 0 \).
Arithmetic result for Lidstone interpolation

If $\tau(f) < \min \left\{ 1, \frac{\pi}{|s_0 - s_1|} \right\}$, then $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large $n$, then $f$ is a polynomial.

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The function

$$f(z) = \sin \left( \pi \frac{z - s_0}{s_1 - s_0} \right)$$

has exponential type $\frac{\pi}{|s_1 - s_0|}$ and satisfies $f^{(2n)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$. 
Arithmetic result for Lidstone interpolation

If $\tau(f) < \min \left\{ 1, \frac{\pi}{|s_0 - s_1|} \right\}$, then

$$f^{(2n)}(s_0) \in \mathbb{Z} \quad \text{and} \quad f^{(2n)}(s_1) \in \mathbb{Z}$$

for all sufficiently large $n$, then $f$ is a polynomial.

The function

$$f(z) = \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)}$$

has exponential type $1$ and satisfies $f^{(2n)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

The function

$$f(z) = \sin \left( \pi \frac{z - s_0}{s_1 - s_0} \right)$$

has exponential type $\frac{\pi}{|s_1 - s_0|}$ and satisfies $f^{(2n)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$. 
Let \( s_0, s_1, \ldots, s_{m-1} \) be distinct complex numbers and \( f \) an entire function of sufficiently small exponential type.

**Théorème 2.**

Assume that for each sufficiently large \( n \), one at least of the numbers

\[
f^{(n)}(s_j) \quad j = 0, 1, \ldots, m - 1
\]

is in \( \mathbb{Z} \). Then \( f \) is a polynomial.

In the case \( m = 2 \) with \( f^{(2n+1)}(s_0) \in \mathbb{Z} \) and \( f^{(2n)}(s_1) \in \mathbb{Z} \), the assumption is

\[
\tau(f) < \min \left\{ 1, \frac{\pi}{2|s_0 - s_1|} \right\},
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and this is best possible.
Arithmetic result for Gontcharoff interpolation

Let \( s_0, s_1, \ldots, s_{m-1} \) be distinct complex numbers and \( f \) an entire function of sufficiently small exponential type.

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Arithmetic result for Whittaker interpolation

If \( \tau(f) < \min \left\{ 1, \frac{\pi}{2|s_0 - s_1|} \right\} \), \( f^{(2n+1)}(s_0) \in \mathbb{Z} \) and \( f^{(2n)}(s_1) \in \mathbb{Z} \) for each sufficiently large \( n \), then \( f \) is a polynomial.

The function

\[
f(z) = \frac{\cosh(z - s_1)}{\cosh(s_0 - s_1)}
\]

has exponential type 1 and satisfies \( f^{(2n+1)}(s_0) = 1 \) and \( f^{(2n)}(s_1) = 0 \) for all \( n \geq 0 \).

The function

\[
f(z) = \cos \left( \frac{\pi}{2} \cdot \frac{z - s_0}{s_1 - s_0} \right)
\]

has exponential type \( \frac{\pi}{2|s_1 - s_0|} \) and satisfies \( f^{(2n+1)}(s_0) = f^{(2n)}(s_1) = 0 \) for all \( n \geq 0 \).
Arithmetic result for Whittaker interpolation

If $\tau(f) < \min \left\{ 1, \frac{\pi}{2|s_0 - s_1|} \right\}$, $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for each sufficiently large $n$, then $f$ is a polynomial.

The function

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If $\tau(f) < \min \left\{ 1, \frac{\pi}{2|s_0 - s_1|} \right\}$, \( f^{(2n+1)}(s_0) \in \mathbb{Z} \) and \( f^{(2n)}(s_1) \in \mathbb{Z} \)

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Historical survey and annotated references

George James Lidstone
(1870 – 1952)


Interpolation problem for

\[ f^{(2n)}(0) \text{ and } f^{(2n)}(1), \quad n \geq 0. \]

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Interpolation problem for 

\[ f^{(n)}(\sigma_n), \quad n \geq 0. \]

Example:

\[ f^{(nm+j)}(s_j), \quad n \geq 0, \quad 0 \leq j \leq m - 1. \]
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Hillel Poritsky
(1898 — 1990)
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\[ f^{(nm)}(s_j), \quad n \geq 0, \quad 0 \leq j \leq m - 1. \]

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John Macnaghten Whittaker
(1905 – 1984)

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John Macnaghten Whittaker (1905 – 1984)


Chap. III. Properties of successive derivatives.

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Isaac Jacob Schoenberg
(1903 – 1990)


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\[ f^{(2n+1)}(0) \text{ and } f^{(2n)}(1), \quad n \geq 0. \]

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Ernst Gabor Straus (1922 – 1983)


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Aleksandr Osipovich Gelfond
(1906 – 1968)

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http://www-groups.dcs.st-and.ac.uk/history/Biographies/Gelfond.html
Archibald James Macintyre
(1908 – 1967)

Interpolation problem for

\[ f^{(nm+b_j)}(s_j), \quad n \geq 0, \quad 0 \leq j \leq m - 1. \]


http://www-groups.dcs.st-and.ac.uk/history/Biographies/Macintyre_Archibald.html
Historical survey and annotated references


Chap. I § 3: the method of the kernel expansion.

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Boas.html
Interpolation of analytic functions
and arithmetic applications.

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Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/