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Interpolation of analytic functions and arithmetic applications.

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Abstract

The very first interpolation formula for analytic functions is given by Taylor series. There are many other ways of interpolating analytic functions. Lagrange interpolation polynomials involve the values of the function at several points; some derivatives may be included. We discuss other types of interpolation formulae, starting with Lidstone interpolation of a function of exponential type $< \pi$ by its derivatives of even order at 0 and 1. A new result is a lower bound for the exponential type of a transcendental entire function having derivatives of even order at two points taking integer values.

The interpolation problem

An entire function is a holomorphic (=analytic) map $\mathbb{C} \to \mathbb{C}$.
The graph $\{(z, f(z)) \mid z \in \mathbb{C}\}$ has the power of continuum.
However, such a function is uniquely determined by a countable set; for instance by the sequence of coefficients of its Taylor series at a given point z_0:

$$f(z) = \sum_{n \geq 0} f^{(n)}(z_0) \frac{(z - z_0)^n}{n!}.$$

Notation:

$$f^{(n)}(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} f(z).$$

There are other sequences of numbers which determine uniquely an entire function, at least if we restrict to some classes of entire functions.

References on interpolation:

Whittaker, J. M. (1935).
Interpolatory function theory, volume 33.
Cambridge University Press, Cambridge.

Gel'fond, A. O. (1952).
Calculus of finite differences.
Hindustan Publishing Corporation. VI (1971).

Boas, Jr., R. P. and Buck, R. C. (1964).
Polynomial expansions of analytic functions.
Ergebnisse der Mathematik und ihrer Grenzgebiete, **19**,
Springer-Verlag .

Interpolation data

Given complex numbers $\{\sigma_i\}_{i\in I}$, $\{a_i\}_{i\in I}$ and nonnegative integers $\{k_i\}_{i\in I}$, the *interpolation problem* is to decide whether there exists an analytic function f satisfying

$$f^{(k_i)}(\sigma_i) = a_i \text{ for all } i \in I.$$

We will consider this question for f analytic everywhere in $\mathbb{C}$ (i.e. f an entire function) and $I = \mathbb{N}$.
The unicity is given by the answer to the same question with $a_i = 0$ for all $i \in I$.
Taylor series: $\sigma_n = 0$ and $k_n = n$ for all $n \geq 0$. The solution, if it exists, is unique

$$f(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}, \qquad f^{(n)}(0) = a_n.$$

Calculus of finite differences

Another classical interpolation problem is given by the data $k_n = 0$ and $\sigma_n = n$ for all $n \geq 0$. Given complex numbers a_n, does there exist an entire function f satisfying

$$f(n) = a_n \text{ for all } n \geq 0?$$

The answer depends on the growth of the sequence $(a_n)_{n \geq 0}$. The example of the function $\sin(\pi z)$ shows that the solution is not unique in general. However we recover unicity by adding a condition on the growth of the solution f.

For the existence, one uses interpolation formulae based on

$$f(z) = f(0) + z f_1(z), \quad f_1(z) = f_1(1) + (z-1) f_2(z),$$
$$f_n(z) = f_n(n) + (z-n) f_{n+1}(z), \quad \dots$$

Further interpolation problems

We are going to consider the following interpolation problems:

- (Lidstone):

$$f^{(2n)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0.$$

- (Whittaker):

$$f^{(2n+1)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0.$$

- (Poritsky): For $m \geq 2$ and $\sigma_0, \ldots, \sigma_{m-1}$ in $\mathbb{C}$,

$$f^{(mn)}(\sigma_j) = a_{nj} \text{ for } n \geq 0 \text{ and } j = 0, 1, \ldots, m-1.$$

- (Gontcharoff): For $(\sigma_n)_{n\geq 0}$ a sequence of complex numbers,

$$f^{(n)}(\sigma_n) = a_n \text{ for } n \geq 0.$$

Lidstone interpolation problem

The following interpolation problem was considered by G.J. Lidstone in 1930.
Given two sequences of complex numbers $(a_n)_{n\geq 0}$ *and* $(b_n)_{n\geq 0}$, *does there exist an entire function* f *satisfying*

$$f^{(2n)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0 \quad ?$$

Is such a function f *unique?*

The answer to unicity is plain: the function $\sin(\pi z)$ satisfies these conditions with $a_n = b_n = 0$, hence there is no unicity, unless we restrict the question to entire functions satisfying some extra condition. Such a condition is a bound on the growth of f.

We start with unicity ($a_n = b_n = 0$) and polynomials.

Even derivatives at 0 and 1: first proof

Lemma. *Let f be a polynomial satisfying*

$$f^{(2n)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

Then $f = 0$.

First proof.

By induction on the degree of the polynomial f.
If f has degree ≤ 1, say $f(z) = a_0 z + a_1$, the conditions $f(0) = f(1) = 0$ imply $a_0 = a_1 = 0$, hence $f = 0$.
If f has degree $\leq n$ with $n \geq 2$ and satisfies the hypotheses, then f'' also satisfies the hypotheses and has degree $< n$, hence by induction $f'' = 0$ and therefore f has degree ≤ 1.
The result follows. □

Even derivatives at 0 and 1: second proof

Second proof.

Let f be a polynomial satisfying

$$f^{(2n)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

The assumption $f^{(2n)}(0) = 0$ for all $n \geq 0$ means that f is an odd function: $f(-z) = -f(z)$. The assumption $f^{(2n)}(1) = 0$ for all $n \geq 0$ means that $f(1-z)$ is an odd function: $f(1-z) = -f(1+z)$. We deduce $f(z+2) = f(1+z+1) = -f(1-z-1) = -f(-z) = f(z)$, hence the polynomial f is periodic, and therefore it is a constant. Since $f(0) = 0$, we conclude $f = 0$. □

Even derivatives at 0 and 1: third proof

Third proof.

Assume

$$f^{(2n)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

Write

$$f(z) = a_1 z + a_3 z^3 + a_5 z^5 + a_7 z^7 + + \cdots + a_{2n+1} z^{2n+1} + \cdots$$

(finite sum). We have $f(1) = f''(1) = f^{(\text{iv})}(1) = \cdots = 0$:

$$\begin{array}{lllllll}
a_1 & +a_3 & +a_5 & +a_7 & +\cdots & +a_{2n+1} & +\cdots = 0 \\
 & 6a_3 & +20a_5 & +42a_7 & +\cdots & +2n(2n+1)a_{2n+1} & +\cdots = 0 \\
 & & 120a_5 & +840a_7 & +\cdots & +\frac{(2n+1)!}{(2n-3)!}a_{2n+1} & +\cdots = 0 \\
 & & & & & \ddots & \vdots
\end{array}$$

The matrix of this system is triangular with maximal rank. □

Even derivatives at 0 and 1

The fact that this matrix has maximal rank means that a polynomial f is uniquely determined by the numbers

$$f^{(2n)}(0) \text{ and } f^{(2n)}(1) \text{ for } n \geq 0.$$

Given numbers a_n and b_n, all but finitely many of them are 0, there is a unique polynomial f such that

$$f^{(2n)}(0) = a_n \text{ and } f^{(2n)}(1) = b_n \text{ for all } n \geq 0.$$

Involution: $z \mapsto 1 - z$:

$$0 \mapsto 1, \quad 1 \mapsto 0, \quad 1 - z \mapsto z.$$

Lidstone expansion of a polynomial

G. J. Lidstone (1930). There exists a unique sequence of polynomials $\Lambda_0(z), \Lambda_1(z), \Lambda_2(z), \dots$ such that any polynomial f can be written as a finite sum

$$f(z) = \sum_{n\geq 0} f^{(2n)}(0)\Lambda_n(1-z) + \sum_{n\geq 0} f^{(2n)}(1)\Lambda_n(z).$$

This is equivalent to

$$\Lambda_n^{(2k)}(0) = 0 \text{ and } \Lambda_n^{(2k)}(1) = \delta_{nk} \text{ for } n \geq 0 \text{ and } k \geq 0.$$

(Kronecker symbol).
A basis of the $\mathbb{Q}$–space of polynomials in $\mathbb{Q}[z]$ of degree $\leq 2n+1$ is given by the $2n+2$ polynomials

$$\Lambda_0(z), \Lambda_1(z), \dots, \Lambda_n(z), \quad \Lambda_0(1-z), \Lambda_1(1-z), \dots, \Lambda_n(1-z).$$

Analogy with Taylor series

Given a sequence $(a_n)_{n\geq 0}$ of complex numbers, the unique analytic solution (if it exists) f of the interpolation problem

$$f^{(n)}(0) = a_n \text{ for all } n \geq 0$$

is given by the Taylor expansion

$$f(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}.$$

The polynomials $z^n/n!$ satisfy

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{z^n}{n!}\right)_{z=0} = \delta_{nk} \text{ for } n \geq 0 \text{ and } k \geq 0.$$

Lidstone polynomials

$\Lambda_0(z) = z$:

$$\Lambda_0(0) = 0, \quad \Lambda_0(1) = 1, \quad \Lambda_0^{(2n)}(0) = 0 \text{ for } n \geq 1.$$

Induction: the sequence of Lidstone polynomials is determined by $\Lambda_0(z) = z$ and

$$\Lambda_n'' = \Lambda_{n-1} \text{ for } n \geq 1$$

with the initial conditions $\Lambda_n(0) = \Lambda_n(1) = 0$ for $n \geq 1$.
Let $L_n(z)$ be any solution of

$$L_n''(z) = \Lambda_{n-1}(z).$$

Define

$$\Lambda_n(z) = -L_n(1)z + L_n(z).$$

Lidstone polynomials

$\Lambda_0(z) = z$,

$$\Lambda_n'' = \Lambda_{n-1}, \quad \Lambda_n(0) = \Lambda_n(1) = 0 \text{ for } n \geq 1.$$

For $n \geq 0$, the polynomial Λ_n is odd, it has degree $2n+1$ and leading term $\frac{1}{(2n+1)!} z^{2n+1}$.

For instance

$$\Lambda_1(z) = \frac{1}{6}(z^3 - z)$$

and

$$\Lambda_2(z) = \frac{1}{120} z^5 - \frac{1}{36} z^3 + \frac{7}{360} z = \frac{1}{360} z(z^2 - 1)(3z^2 - 7).$$

Lidstone polynomials

The polynomial $f(z) = z^{2n+1}$ satisfies

$$f^{(2k)}(0) = 0 \text{ for } k \geq 0, \quad f^{(2k)}(1) = \begin{cases} \frac{(2n+1)!}{(2n-2k+1)!} & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k \geq n+1. \end{cases}$$

One deduces

$$z^{2n+1} = \sum_{k=0}^{n-1} \frac{(2n+1)!}{(2n-2k+1)!} \Lambda_k(z) + (2n+1)! \Lambda_n(z),$$

which yields the induction formula

$$\Lambda_n(z) = \frac{1}{(2n+1)!} z^{2n+1} - \sum_{k=0}^{n-1} \frac{1}{(2n-2k+1)!} \Lambda_k(z).$$

Order and exponential type

Order of an entire function :

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log\log |f|_r}{\log r} \quad \text{where} \quad |f|_r = \sup_{|z|=r} |f(z)|.$$

Exponential type of an entire function :

$$\tau(f) = \limsup_{r\to\infty} \frac{\log |f|_r}{r}.$$

If the exponential type is finite, then f has order ≤ 1. If f has order < 1, then the exponential type is 0.

For $\tau \in \mathbb{C} \setminus \{0\}$, the function $\mathrm{e}^{\tau z}$ has order 1 and exponential type $|\tau|$.

Exponential type

An alternative definition of the exponential type is the following: f is of exponential type $\tau(f)$ if and only if, for all $z_0 \in \mathbb{C}$,

$$\limsup_{n\to\infty} |f^{(n)}(z_0)|^{1/n} = \tau(f).$$

The equivalence between the two definitions follows from Cauchy's inequalities and Stirling's Formula.

Lidstone series : exponential type $< \pi$

Theorem (H. Poritsky, 1932).
Let f be an entire function of exponential type $< \pi$ satisfying $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all sufficiently large n. Then f is a polynomial.

This is best possible: the entire function $\sin(\pi z)$ has exponential type π and satisfies $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$.

Lidstone series : exponential type $< \pi$

Let f be an entire function of exponential type $< \pi$ satisfying $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all sufficiently large n. Then f is a polynomial.

Proof.

Let $\tilde{f} = f - P$, where P is the polynomial satisfying

$$P^{(2n)}(0) = f^{(2n)}(0) \text{ and } P^{(2n)}(1) = f^{(2n)}(1) \text{ for } n \geq 0.$$

We have $\tilde{f}^{(2n)}(0) = \tilde{f}^{(2n)}(1) = 0$ for all $n \geq 0$. The functions $\tilde{f}(z)$ and $\tilde{f}(1-z)$ are odd, hence $\tilde{f}(z)$ is periodic of period 2. Therefore there exists an entire function g such that $\tilde{f}(z) = g(\mathrm{e}^{i\pi z})$. Since $\tilde{f}(z)$ has exponential type $< \pi$, we deduce $\tilde{f} = 0$ and $f = P$. □

Exponential type $< \pi$: Poritsky's expansion

Theorem (H. Poritsky, 1932).

The expansion

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(1-z) + \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(z)$$

holds for any entire function f of exponential type $< \pi$.

We will check this formula for $f_t(z) = e^{tz}$ with $|t| < \pi$, then deduce the general case.

Solution of the Lidstone interpolation problem

Consequence of Poritsky's expansion formula:
Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of complex numbers satisfying

$$\limsup_{n\to\infty} |a_n|^{1/n} < \pi \text{ and } \limsup_{n\to\infty} |b_n|^{1/n} < \pi.$$

Then the function

$$f(z) = \sum_{n=0}^{\infty} a_n \Lambda_n(1-z) + \sum_{n=0}^{\infty} b_n \Lambda_n(z)$$

is the unique entire function of exponential type $< \pi$ satisfying

$$f^{(2n)}(0) = a_n \text{ and } f^{(2n)}(1) = b_n \text{ for all } n \geq 0.$$

Special case: e^{tz} for $|t| < \pi$

Consider Poritsky's expansion formula

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(1-z) + \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(z)$$

for the function $f_t(z) = e^{tz}$ where $|t| < \pi$. Since $f_t^{(2n)}(0) = t^{2n}$ and $f_t^{(2n)}(1) = t^{2n}e^t$ it gives

$$e^{tz} = \sum_{n=0}^{\infty} t^{2n}\Lambda_n(1-z) + e^t \sum_{n=0}^{\infty} t^{2n}\Lambda_n(z).$$

Replacing t with $-t$ yields

$$e^{-tz} = \sum_{n=0}^{\infty} t^{2n}\Lambda_n(1-z) + e^{-t} \sum_{n=0}^{\infty} t^{2n}\Lambda_n(z).$$

Hence

$$e^{tz} - e^{-tz} = (e^t - e^{-t}) \sum_{n=0}^{\infty} t^{2n}\Lambda_n(z).$$

Generating series

Let $t \in \mathbb{C}$, $t \notin i\pi\mathbb{Z}$. The entire function

$$f(z) = \frac{\sinh(tz)}{\sinh(t)} = \frac{e^{tz} - e^{-tz}}{e^{t} - e^{-t}}$$

satisfies

$$f'' = t^2 f, \quad f(0) = 0, \quad f(1) = 1,$$

hence $f^{(2n)}(0) = 0$ and $f^{(2n)}(1) = t^{2n}$ for all $n \geq 0$.
For $0 < |t| < \pi$ and $z \in \mathbb{C}$, we deduce

$$\boxed{\frac{\sinh(tz)}{\sinh(t)} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).}$$

Notice that

$$e^{tz} = \frac{\sinh((1-z)t)}{\sinh(t)} + e^{t}\, \frac{\sinh(tz)}{\sinh(t)}.$$

Special case: e^{tz}

From Poritsky's expansion of an entire function of exponential type $< \pi$ we deduced the formula

$$\frac{\sinh(tz)}{\sinh(t)} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).$$

Let us prove this formula directly.
We will deduce

$$e^{tz} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(1-z) + e^t \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z)$$

for $|t| < \pi$.

Expansion of $F(z,t) = \sinh(tz)/\sinh(t)$

For $z \in \mathbb{C}$ and $|t| < \pi$ let

$$F(z,t) = \frac{\sinh(tz)}{\sinh(t)}$$

with $F(z,0) = z$.

Fix $z \in \mathbb{C}$. The function $t \mapsto F(z,t)$ is analytic in the disc $|t| < \pi$ and is an even function: $F(z,-t) = F(z,t)$. Consider its Taylor series at the origin:

$$F(z,t) = \sum_{n \geq 0} c_n(z) t^{2n}$$

with $c_0(z) = z$.

We have $F(0,t) = 0$ and $F(1,t) = 1$.

Expansion of $F(z,t) = \sinh(tz)/\sinh(t)$

$$F(z,t) = \frac{\mathrm{e}^{tz} - \mathrm{e}^{-tz}}{\mathrm{e}^{t} - \mathrm{e}^{-t}} = \sum_{n\geq 0} c_n(z) t^{2n}.$$

From

$$c_n(z) = \frac{1}{(2n)!} \left(\frac{\partial}{\partial t}\right)^{2n} F(z,0)$$

it follows that $c_n(z)$ is a polynomial.

From

$$\left(\frac{\partial}{\partial z}\right)^{2} F(z,t) = t^2 F(z,t)$$

we deduce

$$c_n'' = c_{n-1} \text{ for } n \geq 1.$$

Since $c_n(0) = c_n(1) = 0$ for $n \geq 1$ we conclude $c_n(z) = \Lambda_n(z)$.

From e^{tz} to exponential type $< \pi$

Hence a special case of the Poritsky's expansion formula

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(1-z) + \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(z),$$

which holds for any entire function f of exponential type $< \pi$, is

$$e^{tz} = \sum_{n=0}^{\infty} t^{2n}\Lambda_n(1-z) + e^t \sum_{n=0}^{\infty} t^{2n}\Lambda_n(z)$$

for $|t| < \pi$.
Conversely, from this special case (that we proved directly) we are going to deduce the general case by means of Laplace transform (R.C. Buck, 1955, *kernel expansion method)*.

Laplace transform

Let

$$f(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n$$

be an entire function of exponential type $\tau(f)$. The Laplace transform of f, viz.

$$F(t) = \sum_{n \geq 0} a_n t^{-n-1},$$

is analytic in the domain $|t| > \tau(f)$. From Cauchy's residue Theorem, it follows that for $r > \tau(f)$ we have

$$f(z) = \frac{1}{2\pi i} \int_{|t|=r} \mathrm{e}^{tz} F(t) \mathrm{d}t.$$

Hence

$$f^{(2n)}(z) = \frac{1}{2\pi i} \int_{|t|=r} t^{2n} \mathrm{e}^{tz} F(t) \mathrm{d}t.$$

Laplace transform

Assume $\tau(f) < \pi$. Let r satisfy $\tau(f) < r < \pi$. For $|t| = r$ we have

$$e^{tz} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(1-z) + e^t \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).$$

We deduce

$$f(z) = \sum_{n\geq 0} \Lambda_n(1-z) \left(\frac{1}{2\pi i} \int_{|t|=r} t^{2n} F(t) \mathrm{d}t \right) + \sum_{n\geq 0} \Lambda_n(z) \left(\frac{1}{2\pi i} \int_{|t|=r} t^{2n} e^t F(t) \mathrm{d}t \right)$$

and therefore

$$f(z) = \sum_{n\geq 0} f^{(2n)}(0) \Lambda_n(1-z) + \sum_{n\geq 0} f^{(2n)}(1) \Lambda_n(z),$$

where the last series are absolutely and uniformly convergent for z on any compact in $\mathbb{C}$.

Entire functions of finite exponential type

Theorem (I.J. Schoenberg, 1936).

Let f *be an entire function of finite exponential type* $\tau(f)$ *satisfying* $f^{(2n)}(0) = f^{(2n)}(1) = 0$ *for all* $n \geq 0$. *Then there exist complex numbers* $c_1, \ldots, c_L$ *with* $L \leq \tau(f)/\pi$ *such that*

$$f(z) = \sum_{\ell=1}^{L} c_\ell \sin(\ell \pi z).$$

Integral formula for Lidstone polynomials

Using Cauchy's residue Theorem, we deduce the integral formula

$$\Lambda_n(z) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{s=1}^{S} \frac{(-1)^s}{s^{2n+1}} \sin\big(s\pi z\big) + \frac{1}{2\pi i} \int_{|t|=(2S+1)\pi/2} t^{-2n-1} \frac{\sinh(tz)}{\sinh(t)} \mathrm{d}t$$

for $S = 1, 2, \ldots$ and $z \in \mathbb{C}$.

In particular, with $S = 1$ we have

$$\Lambda_n(z) = (-1)^n \frac{2}{\pi^{2n+1}} \sin(\pi z) + \frac{1}{2\pi i} \int_{|t|=3\pi/2} t^{-2n-1} \frac{\sinh(tz)}{\sinh(t)} \mathrm{d}t.$$

One deduces that there exists an absolute constant $c > 0$ such that

$$|\Lambda_n|_r \leq c\pi^{-2n} \mathrm{e}^{3\pi r/2}.$$

Odd derivatives at 0 and 1

A polynomial f is determined *up to the addition of a constant* by the numbers

$$f^{(2n+1)}(0) \text{ and } f^{(2n+1)}(1).$$

The interpolation problem related with odd derivatives at 0 and 1 is solved by using Lidstone interpolation for the derivative of f.

Odd derivatives at 0 and even derivatives at 1

Lemma. *Let* f *be a polynomial satisfying*

$$f^{(2n+1)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

Then $f = 0$.

Proofs.

1. By induction.
2. $f(z+4) = f(z)$.
3. Triangular system. □

Whittaker expansion of a polynomial

The Lemma means that a polynomial f is uniquely determined by the numbers

$$f^{(2n+1)}(0) \text{ and } f^{(2n)}(1) \text{ for } n \geq 0.$$

Any polynomial $f \in \mathbb{C}[z]$ has the finite expansion

$$f(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(1) M_n(z) - f^{(2n+1)}(0) M'_{n+1}(1-z) \right),$$

with only finitely many nonzero terms in the series.
A basis of the $\mathbb{Q}$–space of polynomials in $\mathbb{Q}[z]$ of degree $\leq 2n$ is given by the $2n+1$ polynomials

$$M_0(z), M_1(z), \ldots, M_n(z), \quad M'_1(1-z), \ldots, M'_n(1-z).$$

Whittaker polynomials

Following J.M. Whittaker (1935), one defines a sequence $(M_n)_{n\geq 0}$ of even polynomials by induction on n with $M_0 = 1$,

$$M_n'' = M_{n-1}, \quad M_n(1) = M_n'(0) = 0 \text{ for all } n \geq 1.$$

This is equivalent to

$$M_n^{(2k+1)}(0) = 0, \quad M_n^{(2k)}(1) = \delta_{nk} \text{ for } n \geq 0 \text{ and } k \geq 0.$$

For instance

$$M_1(z) = \frac{1}{2}(z^2 - 1), \quad M_2(z) = \frac{1}{24}(z^2 - 1)(z^2 - 5),$$

$$M_3(z) = \frac{1}{720}(z^2 - 1)(z^4 - 14z^2 + 61).$$

Induction formula for Whittaker polynomials

The polynomial $f(z) = z^{2n}$ satisfies

$$f^{(2k+1)}(0) = 0 \text{ for } k \geq 0, \quad f^{(2k)}(1) = \begin{cases} \frac{(2n)!}{(2n-2k)!} & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k \geq n+1. \end{cases}$$

One deduces

$$z^{2n} = \sum_{k=0}^{n-1} \frac{(2n)!}{(2n-2k)!} M_k(z) + (2n)! M_n(z),$$

which yields the following induction formula

$$M_n(z) = \frac{1}{(2n)!} z^{2n} - \sum_{k=0}^{n-1} \frac{1}{(2n-2k)!} M_k(z).$$

Exponential type $< \pi/2$

Theorem (J.M. Whittaker, 1935).

The expansion

$$f(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(1) M_n(z) - f^{(2n+1)}(0) M'_{n+1}(1-z) \right)$$

holds for any entire function f of exponential type $< \pi/2$.

Hence, if such a function satisfies $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all sufficiently large n, then it is a polynomial.

This is best possible: the entire function $\cos(\frac{\pi}{2} z)$ has exponential type $\pi/2$ and satisfies $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$.

Solution of the Whittaker interpolation problem

Consequence of Whittaker's expansion formula:
Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of complex numbers satisfying

$$\limsup_{n\to\infty} |a_n|^{1/n} < \pi \text{ and } \limsup_{n\to\infty} |b_n|^{1/n} < \pi.$$

Then the function

$$f(z) = \sum_{n=0}^{\infty} a_n M_n(z) - \sum_{n=0}^{\infty} b_n M'_{n+1}(1-z)$$

is the unique entire function of exponential type $< \pi$ satisfying

$$f^{(2n)}(1) = a_n \text{ and } f^{(2n+1)}(0) = b_n \text{ for all } n \geq 0.$$

Finite exponential type

Theorem (I.J. Schoenberg, 1936).
Let f be an entire function of finite exponential type $\tau(f)$ satisfying $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$. Then there exist complex numbers $c_1, \ldots, c_L$ with $L \leq 2\tau(f)/\pi$ such that

$$f(z) = \sum_{\ell=0}^{L} c_\ell \cos\left(\frac{(2\ell+1)\pi}{2} z\right).$$

Generating series

For $t \in \mathbb{C}$, $t \notin i\pi + 2i\pi\mathbb{Z}$, the entire function

$$f(z) = \frac{\cosh(tz)}{\cosh(t)} = \frac{\mathrm{e}^{tz} + \mathrm{e}^{-tz}}{\mathrm{e}^{t} + \mathrm{e}^{-t}}$$

satisfies

$$f'' = t^2 f, \quad f(1) = 1, \quad f'(0) = 0,$$

hence $f^{(2n)}(1) = t^{2n}$ and $f^{(2n+1)}(0) = 0$ for all $n \geq 0$.
The sequence $(M_n)_{n\geq 0}$ is also defined by the expansion

$$\boxed{\frac{\cosh(tz)}{\cosh(t)} = \sum_{n=0}^{\infty} t^{2n} M_n(z)}$$

for $|t| < \pi/2$ and $z \in \mathbb{C}$.

Integral formula for Whittaker polynomials

Using Cauchy's residue Theorem, we deduce the integral formula

$$M_n(z) = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \sum_{s=0}^{S-1} \frac{(-1)^s}{(2s+1)^{2n+1}} \cos\left(\frac{(2s+1)\pi}{2} z\right) + \frac{1}{2\pi i} \int_{|t|=S\pi} t^{-2n-1} \frac{\cosh(tz)}{\cosh(t)} \mathrm{d}t$$

for $S = 1, 2, \ldots$ and $z \in \mathbb{C}$.

In particular, with $S = 1$ we obtain

$$M_n(z) = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \cos(\pi z/2) + \frac{1}{2\pi i} \int_{|t|=\pi} t^{-2n-1} \frac{\cosh(tz)}{\cosh(t)} \mathrm{d}t.$$

Lidstone interpolation vs Whittaker interpolation

Let us display horizontally the points and vertically the derivatives.

- ● interpolation values ○ no condition

Lidstone interpolation

	s_0	s_1
$\vdots$	$\vdots$	$\vdots$
$f^{(2n+1)}$	○	○
$f^{(2n)}$	●	●
$\vdots$	$\vdots$	$\vdots$
f''	●	●
f'	○	○
f	●	●

Whittaker interpolation

	s_0	s_1
$\vdots$	$\vdots$	$\vdots$
$f^{(2n+1)}$	●	○
$f^{(2n)}$	○	●
$\vdots$	$\vdots$	$\vdots$
f''	○	●
f'	●	○
f	○	●

Generalizations with 3 points

Poritsky interpolation

	s_0	s_1	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(3n+2)}$	○	○	○
$f^{(3n+1)}$	○	○	○
$f^{(3n)}$	●	●	●
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(\text{iv})}$	○	○	○
f'''	●	●	●
f''	○	○	○
f'	○	○	○
f	●	●	●

Gontcharoff interpolation

	s_0	s_1	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(3n+2)}$	○	○	●
$f^{(3n+1)}$	○	●	○
$f^{(3n)}$	●	○	○
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f^{(\text{iv})}$	○	●	○
f'''	●	○	○
f''	○	○	●
f'	○	●	○
f	●	○	○

Poritsky interpolation

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type.

Theorem (H. Poritsky,1932).

If

$$f^{(mn)}(s_0) = f^{(mn)}(s_1) = \cdots = f^{(mn)}(s_{m-1}) = 0$$

for all sufficiently large n, then f is a polynomial.

For $m = 2$, $s_0 = 0$, $s_1 = 1$, this reduces Poritsky's above mentioned result on Lidstone expansion (up to the exact bound on the exponential type).

Gontcharoff interpolation

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type.

Theorem (W. Gontcharoff 1930, A. J. Macintyre 1954).

If

$$f^{(n)}(s_0)f^{(n)}(s_1)\cdots f^{(n)}(s_{m-1}) = 0$$

for all sufficiently large n, then f is a polynomial.

For $m = 2$, $s_0 = 0$, $s_1 = 1$, this implies Whittaker's above mentioned result for $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$.

Arithmetic result for Poritsky interpolation

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type.

Theorem 1.
If

$$f^{(mn)}(s_j) \in \mathbb{Z}$$

for all sufficiently large n and for $0 \leq j \leq m-1$, then f is a polynomial.

For $m = 2$ with $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$, the assumption on the exponential type $\tau(f)$ of f is

$$\tau(f) < \min\{1, \pi/|s_0 - s_1|\},$$

and this is best possible.

Arithmetic result for Lidstone interpolation

If $\tau(f) < \min\left\{1, \frac{\pi}{|s_0 - s_1|}\right\}$ $f^{(2n)}(s_0) \in \mathbb{Z}$ *and* $f^{(2n)}(s_1) \in \mathbb{Z}$

for all sufficiently large n*, then* f *is a polynomial.*

The function

$$f(z) = \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)}$$

has exponential type 1 and satisfies $f^{(2n)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

The function

$$f(z) = \sin\left(\pi \frac{z - s_0}{s_1 - s_0}\right)$$

has exponential type $\frac{\pi}{|s_1 - s_0|}$ and satisfies $f^{(2n)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

Arithmetic result for Gontcharoff interpolation

Let $s_0, s_1, \ldots, s_{m-1}$ be distinct complex numbers and f an entire function of sufficiently small exponential type.

Theorem 2.

Assume that for each sufficiently large n, one at least of the numbers

$$f^{(n)}(s_j) \quad j = 0, 1, \ldots, m-1$$

is in $\mathbb{Z}$. Then f is a polynomial.

In the case $m = 2$ with $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$, the assumption is

$$\tau(f) < \min\left\{1, \frac{\pi}{2|s_0 - s_1|}\right\},$$

and this is best possible.

Arithmetic result for Whittaker interpolation

If $\tau(f) < \min\left\{1, \frac{\pi}{2|s_0 - s_1|}\right\}, \quad f^{(2n+1)}(s_0) \in \mathbb{Z}$ *and* $f^{(2n)}(s_1) \in \mathbb{Z}$

for each sufficiently large n*, then* f *is a polynomial.*

The function

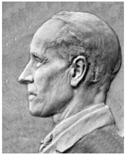
$$f(z) = \frac{\sinh(z - s_1)}{\cosh(s_0 - s_1)}$$

has exponential type 1 and satisfies $f^{(2n+1)}(s_0) = 1$ and $f^{(2n)}(s_1) = 0$ for all $n \geq 0$.

The function

$$f(z) = \cos\left(\frac{\pi}{2} \cdot \frac{z - s_0}{s_1 - s_0}\right)$$

has exponential type $\frac{\pi}{2|s_1 - s_0|}$ and satisfies $f^{(2n+1)}(s_0) = f^{(2n)}(s_1) = 0$ for all $n \geq 0$.



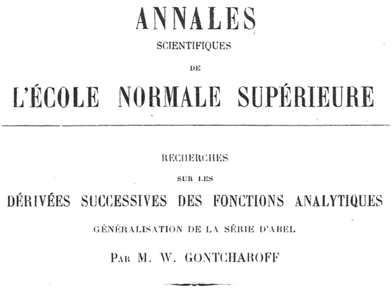
George James Lidstone
(1870 – 1952)

Lidstone, G. J. (1930).
Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types.
Proc. Edinb. Math. Soc., II. Ser., 2:16–19.

Interpolation problem for

$$f^{(2n)}(0) \text{ and } f^{(2n)}(1), \quad n \geq 0.$$

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Lidstone.html



ANNALES
SCIENTIFIQUES
DE
L'ÉCOLE NORMALE SUPÉRIEURE

RECHERCHES
SUR LES
DÉRIVÉES SUCCESSIVES DES FONCTIONS ANALYTIQUES
GÉNÉRALISATION DE LA SÉRIE D'ABEL
PAR M. W. GONTCHAROFF

Gontcharoff, W. (1930).
Recherches sur les dérivées successives des fonctions analytiques. Généralisation de la série d'Abel.
Ann. Sci. Éc. Norm. Supér. (3), 47:1–78.

Interpolation problem for

$$f^{(n)}(\sigma_n), \quad n \geq 0.$$

Example:

$$f^{(nm+j)}(s_j), \quad n \geq 0, \quad 0 \leq j \leq m-1.$$

Hillel Poritsky
(1898 — 1990)
Ph.D. Cornell University 1927
Topics in Potential Theory.
Wallie Abraham Hurwitz
(student of David Hilbert)

Poritsky, H. (1932).
On certain polynomial and other approximations to analytic functions.
Trans. Amer. Math. Soc., 34(2):274–331.

Interpolation problem for

$$f^{(nm)}(s_j), \quad n \geq 0, \quad 0 \leq j \leq m-1.$$

https://pt.wikipedia.org/wiki/Hillel_Poritsky
https://www.genealogy.math.ndsu.nodak.edu/id.php?id=41924

John Macnaghten Whittaker
(1905 – 1984)

Whittaker, J. M. (1933).
On Lidstone's series and two-point expansions of analytic functions.
Proc. Lond. Math. Soc. (2), 36:451–469.

Standard sets of polynomials: complete, indeterminate, redundant.

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Whittaker_John.html

John Macnaghten Whittaker
(1905 – 1984)

Whittaker, J. M. (1935). *Interpolatory function theory*, volume 33. Cambridge University Press, Cambridge.

Chap. III. Properties of successive derivatives.

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Whittaker_John.html

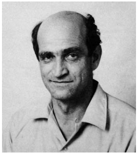
Isaac Jacob Schoenberg
(1903 – 1990)

Schoenberg, I. J. (1936).
On certain two-point expansions of integral functions of exponential type.
Bull. Am. Math. Soc., 42:284–288.

Interpolation problem for

$$f^{(2n+1)}(0) \text{ and } f^{(2n)}(1), \quad n \geq 0.$$

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Schoenberg.html



Ernst Gabor Straus
(1922 – 1983)

Straus, E. G. (1950).
On entire functions with algebraic derivatives at certain algebraic points.
Ann. of Math. (2), 52:188–198.

Connection with transcendental number theory.

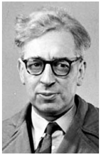
http://www-groups.dcs.st-and.ac.uk/history/Biographies/Straus.html

Aleksandr Osipovich Gelfond
(1906 – 1968)

Chapitre 3: construction d'une fonction entière à partir d'éléments donnés.

Gel'fond, A. O. (1952). Calculus of finite differences. Authorized English translation of the 3rd Russian edition. International Monographs on Advanced Mathematics and Physics. Delhi, India: Hindustan Publishing Corporation. VI,451 p. (1971).

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Gelfond.html



Archibald James Macintyre
(1908 – 1967)

Macintyre, A. J. (1954).
Interpolation series for integral functions of exponential type.
Trans. Amer. Math. Soc., 76:1–13.

Interpolation problem for

$$f^{(nm+b_j)}(s_j), \quad n \geq 0, \quad 0 \leq j \leq m-1.$$

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Macintyre_Archibald.html

Ralph Philip Boas Jr
(1912 – 1992)

Robert Creighton Buck
(1920 – 1998)

Boas, Jr., R. P. and Buck, R. C. (1964).
Polynomial expansions of analytic functions.
Second printing, corrected. Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., Bd. 19. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin.

Chap. I § 3: the method of the kernel expansion.

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Boas.html
https://en.wikipedia.org/wiki/Robert_Creighton_Buck

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Interpolation of analytic functions and arithmetic applications.

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