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Journal of Computational and Applied Mathematics 160 (2003) 323–333
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Algebraic values of analytic functions

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Received 24 October 2002; received in revised form 16 April 2003

Abstract

Given an analytic function of one complex variable f, we investigate the arithmetic nature of the values of f at algebraic points. A typical question is whether $f(\alpha)$ is a transcendental number for each algebraic number α. Since there exist transcendental entire functions f such that $f^{(t)}(\alpha) \in \mathbb{Q}[\alpha]$ for any $t \geqslant 0$ and any algebraic number α, one needs to restrict the situation by adding hypotheses, either on the functions, or on the points, or else on the set of values.

Among the topics we discuss are recent results due to Andrea Surroca on the number of algebraic points where a transcendental analytic function takes algebraic values, new transcendence criteria by Daniel Delbos concerning entire functions of one or several complex variables, and Diophantine properties of special values of polylogarithms.

Keywords: Arithmetic functions; Algebraic values; Transcendence criterion; Diophantine analysis; Transcendental functions

1. Introduction

At the end of XIXth century, after the proof by Hermite and Lindemann of the transcendence of e^{α} for nonzero algebraic α, the question arose (see [13]):

(*) *Does a transcendental analytic function usually takes transcendental values at algebraic points?*

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0377-0427/$ - see front matter
doi:10.1016/S0377-0427(03)00637-X

In the example of the exponential function e^z, the word “usually” stands for avoiding the exception $\alpha=0$. Recall also that a *transcendental* function is a function f (here a complex valued function in a single complex variable) such that, for any nonzero polynomial $P\in\mathbb{C}[X,Y]$, the function $P(z,f(z))$ is not the zero function. If f is meromorphic in all of $\mathbb{C}$, this just means that f is not a rational function. If f is an *entire* function, namely a function which is analytic in $\mathbb{C}$, to say that f is a transcendental function amounts to say that it is not a polynomial.

However in 1886 Weierstrass found that a positive answer to the initial question $(*)$ can hold only for restricted classes of functions: he gave an example of a transcendental entire function which takes rational values at all rational points. He also suggested that there exist transcendental entire functions which take algebraic values at any algebraic point. After the early works of Strauss and Stäckel at the end of XIXth century, one knows now that for each countable subset $\Sigma\subset\mathbb{C}$ and each dense subset $T\subset\mathbb{C}$ there is a transcendental entire function f such that $f(\Sigma)\subset T$. According to [9], in case the countable set Σ is contained in $\mathbb{R}$, then the same conclusion holds also for a dense subset $T\subset\mathbb{R}$.

Denote by $\bar{\mathbb{Q}}$ the field of complex algebraic number ($\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ into $\mathbb{C}$). Another construction due to Stäckel produces an entire function f whose derivatives $f^{(t)}$, for $t=0,1,\ldots$, all map $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}$. Furthermore, Faber refined the result in $f^{(t)}(\bar{\mathbb{Q}})\subset\mathbb{Q}(i)$ for any $t\geqslant 0$. Later (1968), A.J. van der Poorten constructed a transcendental entire function f such that $f^{(t)}(\alpha)\in\mathbb{Q}(\alpha)$ for any $t\geqslant 0$ and any $\alpha\in\bar{\mathbb{Q}}$. Surroca [16] recently revisited this construction of van der Poorten by providing, for such a function f, a sharp lower bound for the number of $\alpha\in\bar{\mathbb{Q}}$ of bounded degree and height such that $f(\alpha)$ has also a bounded height (see Section 2).

One may notice that all these constructions may be worked out so that the growth of the constructed function f is as small as possible (see [9]): given any transcendental entire function φ, one may require

$$|f|_R\leqslant|\varphi|_R$$

for all sufficiently large R, where

$$|f|_R:=\sup_{|z|=R}|f(z)|.$$

On the other hand Elkies and Surroca give, for a transcendental analytic function f, upper bounds for the number of $\alpha\in\bar{\mathbb{Q}}$ of bounded degree and height such that $f(\alpha)\in\bar{\mathbb{Q}}$ has also bounded degree and height. We discuss this topic in Section 2.

In view of such results, one needs to restrict the initial question $(*)$. Most often the restriction is on the class of analytic functions: for instance one requires that the considered function satisfies some differential equation. The case of entire functions satisfying a linear differential equation provides the strongest results, related with Siegel’s E functions [7], Chapter 5 Ref. [7] contains most references to the subject before 1998, including Siegel E and G functions. Philippon K-functions are introduced in Ref. [14]. Another class of functions, which are analytic only in a neighborhood of the origin, have also been introduced by Siegel under the name of G-functions [7], Chapter 5, Section 7. More recently, generalizations of transcendence and independence results related to modular functions to more general classes of functions have been introduced by Philippon with his new class of K-functions [14]: typical such functions are Ramanujan P, Q and R functions.

Another type of differential equation is related with the solution, by Gel’fond, of Hilbert’s seventh problem, and gives rise to the Schneider-Lang criterion. Once again this criterion started from a transcendence result, it provides a general statement on the values of analytic functions, and then it yields new transcendence results (here in connection with algebraic groups). A number of variations have been produced. In Section 3 we quote a new result by Delbos which is a variant of a result by Bombieri [1] dealing with functions of several variables satisfying algebraic differential equations.

A fashionable topic nowadays is the study of the arithmetic nature of special values of polylogarithms. Despite the fact that very few information is available so far on the values of Riemann zeta function, one may expect that extending the investigations to multiple zeta values will prove to be a fruitful direction. We consider this topic in Section 4 where we notice that the open problem of algebraic independence of logarithms of algebraic numbers reduces to a linear independence question on the values of multiple polylogarithms at algebraic points: the point is that for $n \geqslant 1$, $(\log(1-z))^n$ is a multiple polylogarithm.

2. Algebraic values of analytic functions, following Surroca

In the course of his Diophantine investigations, Elkies [6] devoted attention to the number of rational points of bounded height lying on a transcendental curve $\mathscr{C}$. Assume $\mathscr{C}$ is a planar curve, in $\mathbb{C}^2$, and denote by $\mathscr{C}(\mathbb{Q})$ the intersection $\mathscr{C} \cap \mathbb{Q}^2$. One expects indeed the number of $(\alpha, \beta) \in \mathscr{C}(\mathbb{Q})$ with $h(\alpha) \leqslant N$ and $h(\beta) \leqslant N$ to be quite small compared with the number of $\alpha \in \mathbb{Q}$ with $h(\alpha) \leqslant N$. Here, $h(p/q) = \max\{\log|p|, \log q\}$ for $p/q \in \mathbb{Q}$ with $\gcd(p,q) = 1$ and $q > 0$. In particular, given an interval $\mathscr{I} \subset \mathbb{R}$ (the problem is local), the number of $p/q \in \mathbb{Q} \cap \mathscr{I}$ with $h(p/q) \leqslant N$ is not too far from e^N.

A special case of Elkies result reads as follows:

Let f be a transcendental real analytic function on an open subset of $\mathbb{R}$ containing an interval $\mathscr{I}$. For $N > 0$, consider the set

$$S_N = \{x \in \mathbb{Q} \cap \mathscr{I};\ f(x) \in \mathbb{Q},\ h(x) \leqslant N,\ h(f(x)) \leqslant N\}.$$

Then, for each $\varepsilon > 0$, there exists $N_0 > 0$ such that

$$|S_N| \leqslant \mathrm{e}^{\varepsilon N} \quad \text{for} \quad N \geqslant N_0.$$

Several questions then arise: is this estimate best possible? Is it possible to improve it for infinitely many N (in place of all sufficiently large N)? Do similar results hold for *algebraic points* on a transcendental curve, in place of rational points?

These questions, and much more, are addressed by Surroca [16]. Denote by $h(\alpha)$ the absolute logarithmic height of an algebraic number α:

$$h(\alpha) = \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \log M(\alpha),$$

where

$$M(\alpha) = \prod_v \max\{1, |\alpha|_v\}$$

is Mahler's measure of α (and v ranges over the set of normalized absolute values of $\mathbb{Q}(\alpha)$). For $\alpha = p/q \in \mathbb{Q}$, one recovers the previous definition of $h(p/q)$.

The number of algebraic numbers α with $[\mathbb{Q}(\alpha):\mathbb{Q}] \leqslant D$ and $h(\alpha) \leqslant N$ has been investigated by many a mathematician, including Schanuel, Evertse, Schmidt, Loher and Masser (see for instance [11,12], where the question of counting points of bounded height is dealt with, and further references are given). Loosely speaking, a rough estimate is $e^{D(D+1)N}$.

In [16] and in her thesis, A. Surroca shows that Elkies' result is not far from best possible, to a certain extent:

Theorem 1. *Let ϕ be a positive valued real function such that $\phi(x)/x \to 0$ as $x \to \infty$. There exist a transcendental entire function f satisfying*

$$f^{(t)}(\alpha) \in \mathbb{Q}(\alpha) \quad \textit{for all} \quad t \geqslant 0 \quad \textit{and} \quad \alpha \in \bar{\mathbb{Q}}$$

and such that, for any positive integer D, there are infinitely many $N \geqslant 0$, for which the set

$$S_N = \{\alpha \in \bar{\mathbb{Q}}; |\alpha| \leqslant 1,\ [\mathbb{Q}(\alpha):\mathbb{Q}] \leqslant D, h(\alpha) \leqslant N,\ h(f(\alpha)) \leqslant N\}$$

has cardinality

$$|S_N| \geqslant e^{D(D+1)\phi(N)}.$$

In the other direction, using transcendence arguments, Surroca [16] proves the following result, which is exponentially sharper than Elkies' one, but is valid only for infinitely many N:

Theorem 2. *Let $\mathscr{U}$ be a connected open set in $\mathbb{C}$ and $\mathscr{K}$ a compact in $\mathscr{U}$. There exists a positive real number $c > 0$ such that, for any transcendental complex analytic function f in $\mathscr{U}$ and any positive integer D, there are infinitely many integers N for which the set*

$$S_N = \{\alpha \in \bar{\mathbb{Q}} \cap \mathscr{K}; f(\alpha) \in \bar{\mathbb{Q}}, [\mathbb{Q}(\alpha, f(\alpha)):\mathbb{Q}] \leqslant D, h(\alpha) \leqslant N, h(f(\alpha)) \leqslant N\}$$

has

$$|S_N| \leqslant cD^3N^2.$$

An explicit value for c follows from [16]. Further results are given in [16].

3. Transcendence criteria for entire functions of one or several complex variables

In this section we concentrate on *entire functions* in one or several complex variables: the local question of analytic functions in an open subset is only briefly discussed at the end of Section 3. For an entire function f in $\mathbb{C}^n$, we denote

$$|f|_R = \sup_{|z_1|=\cdots=|z_n|=R} |f(z_1,\ldots,z_n)|$$

the maximum modulus of f on a polydisc of radius R (any other norm would do), and we say that f has order $\leqslant \varrho$ if

$$\limsup_{R\to\infty} \frac{1}{R^{\varrho}} \log |f|_R < \infty.$$

We start with the easier case $n=1$. A simple case of Schneider–Lang Transcendence Criterion in one variable ([10] Chapter III, Theorem 1) is:

Theorem 3. *Let f_1, f_2 be two algebraically independent entire functions of finite order in $\mathbb{C}$ and let K be a number field. Assume*

$$f_j' \in K[f_1, f_2] \quad \text{for } j=1 \text{ and } j=2.$$

Then the set

$$S = \{w \in \mathbb{C}; f_j(w) \in K \quad \text{for } j=1 \text{ and } j=2\}$$

is finite.

Upper bounds for the number of elements in S are known—but here we are just interested in the finiteness result.

One deduces Hermite–Lindemann's Theorem on the transcendence of e^β for algebraic $\beta \neq 0$ by considering

$$f_1(z) = z, \qquad f_2(z) = e^z, \qquad S = \{m\beta; m \in \mathbb{Z}\}.$$

Notice that when $f_1(z) = z$, the assumption that f_1, f_2 are algebraically independent just means that f_2 is a transcendental function.

Another consequence of Theorem 3 is Gel'fond–Schneider's Theorem on the transcendence of α^β for algebraic α and β with $\alpha \neq 0$, $\beta \notin \mathbb{Q}$, $\log\alpha \neq 0$ and $\alpha^\beta = \exp\{\beta\log\alpha\}$: just consider

$$f_1(z) = e^z, \qquad f_2(z) = e^{\beta z}, \qquad S = \{m\log\alpha; m \in \mathbb{Z}\}.$$

Theorem 3 is only a special case of Schneider–Lang's criterion in one variable in [10], Chapter IV: the full statement deals with meromorphic functions and more general differential equations; in particular it applies to elliptic and even to abelian functions.

An extension of Theorem 3 to several complex variables has also been considered by Schneider and Lang; it deals with Cartesian products ([10] Chapter IV, Theorem 1 and [18], Chapter 4, Section 4.1). Here is a simplified statement, which is sufficient for us.

Theorem 4. *Let $f_1, \dots, f_{n+1}$ be algebraically independent entire functions of finite order in $\mathbb{C}^n$, K a number field, $(e_1, \dots, e_n)$ a basis of $\mathbb{C}^n$ over $\mathbb{C}$ and $S_1, \dots, S_n$ subsets of $\mathbb{C}$. Assume*

$$(\partial/\partial z_\nu) f_j' \in K[f_1, \dots, f_{n+1}] \quad \text{for } 1 \leqslant j \leqslant n+1 \text{ and } 1 \leqslant \nu \leqslant n.$$

Assume also

$$f_j(w_1 e_1 + \cdots + w_n e_n) \in K$$

for any $j = 1, \dots, n+1$ and any $(w_1, \dots, w_n) \in S_1 \times \cdots \times S_n$. Then one at least of the sets $S_1, \dots, S_n$ is finite.

This topic was first investigated in 1941 by Th. Schneider when he proved the transcendence of the values of the beta function $B(a,b)$ at rational numbers a and b with a, b and $a+b$ not in $\mathbb{Z}$.

It was further studied by S. Lang around 1964 in connection with transcendence results on algebraic varieties.

In 1980, Bertrand and Masser pointed out that Baker's result on the linear independence of logarithms of algebraic numbers was in fact a corollary of Theorem 4 (see Chapter 4 of [18]).

According to [10] Chapter IV, Nagata suggested that under the assumptions of Theorem 4, the set

$$S = \{w \in \mathbb{C}^n;\, f_j(w) \in K \quad \text{for } 1 \leqslant j \leqslant n+1\}$$

is contained in an algebraic hypersurface: this is obviously a stronger statement than Theorem 4. This suggestion turns out to be right, as shown by Bombieri in 1970 [1].

Theorem 5. *Let $f_1,\ldots,f_{n+1}$ be algebraically independent entire functions of finite order in $\mathbb{C}^n$ and K a number field. Assume*

$$(\partial/\partial z_v) f_j' \in K[f_1,\ldots,f_{n+1}] \quad \textit{for } 1 \leqslant j \leqslant n+1 \textit{ and } 1 \leqslant v \leqslant n.$$

Then the set

$$S = \{w \in \mathbb{C}^n;\, f_j(w) \in K \quad \textit{for } 1 \leqslant j \leqslant n+1\}$$

is contained in an algebraic hypersurface.

Bombieri's conclusion deals with the source set $S \subset \mathbb{C}^n$. Another type of result has just been produced by Delbos: under the same hypotheses as Theorem 5, the conclusion deals with the range set $\underline{f}(S) \subset \mathbb{C}^{n+1}$, where

$$\underline{f} = (f_1,\ldots,f_{n+1}) : \mathbb{C}^n \to \mathbb{C}^{n+1}.$$

Theorem 6. *Let $f_1,\ldots,f_\ell$ be entire functions of finite order in $\mathbb{C}^n$ and K a number field. Assume*

$$(\partial/\partial z_v) f_j' \in K[f_1,\ldots,f_{n+1}] \quad \textit{for } 1 \leqslant j \leqslant \ell \textit{ and } 1 \leqslant v \leqslant n.$$

Let S be a finite subset of $\mathbb{C}$ such that

$$f_j(w) \in K \quad \textit{for all } w \in S \textit{ and } 1 \leqslant j \leqslant \ell.$$

Then there exists a positive constant c and a positive integer M such that, for each integer $N \geqslant M$, there is a nonzero polynomial Q_N in $\mathbb{C}[X_1,\ldots,X_\ell]$ of degree $\leqslant cN^{\ell/n}$ such that the function $F_N = Q_N(f_1,\ldots,f_\ell)$ has a zero of multiplicity $\geqslant N$ at each point of S.

Again this result contains Baker's Theorem on the linear independence of logarithms of algebraic numbers: starting with a nontrivial linear relation

$$\beta_0 + \beta_1 \log\alpha_1 + \cdots + \beta_{n-1}\log\alpha_{n-1} = \log\alpha_n$$

with algebraic α's and β's, one considers the functions

$$z_0, \mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_{n-1}}, \mathrm{e}^{\beta_0+\beta_1 z_1+\cdots+\beta_{n-1}z_{n-1}}.$$

The proof of Baker's result along these lines is more natural than the approach by Bertrand–Masser using Theorem 4, but it requires a zero-estimate. On the otherhand, in contrast with the proof by

Bertrand and Masser, it is effective and yields quantitative results (measures of linear independence for logarithms of algebraic numbers.)

Other criteria are available from [5]. One may point out that such criteria also apply to real analytic functions, by means of the "elementary approach" of Gel'fond and Linnik in [8]. See [2,3] for the methods of Schneider and Gel'fond respectively. However, according to Delbos, the hypotheses which are necessary for the elementary method to work (involving Rolle Theorem for real functions as a substitute to Schwarz' Lemma for complex functions) usually imply that the functions are just restrictions to the real line of complex analytic functions of finite order.

Further, most results in this section extend easily to meromorphic functions of several variables (essentially, one only needs to avoid singularities). Furthermore, extensions are possible to functions which are defined only locally, say in a polydisc in $\mathbb{C}^n$, but then the results are usually weaker: this is the main reason for which p-adic results are sometimes weaker than their complex analogues. A witness of this difficulty is the open problem of proving a p-adic analogue of the Lindemann–Weierstrass Theorem on the algebraic independence of $e^{\alpha_1},\dots,e^{\alpha_n}$ for linearly independent algebraic α's.

4. Polylogarithms

The classical polylogarithms

$$\mathrm{Li}_s(z) = \sum_{n\geqslant 1}\frac{z^n}{n^s}$$

for $s=1,2,\dots$ and $|z|\leqslant 1$ with $(s,z)\neq(1,1)$, are ubiquitous. The study of the arithmetic nature of their special values is a fascinating subject [4]: very few is known.

Several recent investigations concern the values of these functions at $z=1$: these are the values at the positive integers of Riemann zeta function

$$\zeta(s) = \mathrm{Li}_s(1) = \sum_{n\geqslant 1}\frac{1}{n^s}$$

for $s=2,3,\dots$.

One knows that $\zeta(3)$ is irrational [21], and that infinitely many values $\zeta(2n+1)$ of the zeta function at odd integers are irrational (see the lectures by Rivoal and Zudilin at this conference).

A folklore conjecture is that the numbers

$$\zeta(2),\zeta(3),\zeta(5),\dots,\zeta(2n+1)$$

do not satisfy any nontrivial algebraic relation with rational coefficients: this amounts to say that *the values at the odd integers of Riemann zeta function*, *namely* $\zeta(3),\zeta(5),\dots,$ *are algebraically independent over the field* $\mathbb{Q}(\pi)$.

It is far easier to prove a statement of linear independence rather than a statement of algebraic independence. For instance, according to the Lindemann–Weierstrass' Theorem on the algebraic independence of values of the exponential function, the numbers $e^{\alpha_1},e^{\alpha_2},e^{\alpha_3},\dots$ are *algebraically independent* if the algebraic numbers $\alpha_1,\alpha_2,\alpha_3\dots$ are *linearly independent* over the rational number field. However most proofs establish the equivalent statement that the numbers $e^{\beta_1},e^{\beta_2},e^{\beta_3},\dots$ are *linearly independent* over the field of rational numbers if the algebraic numbers $\beta_1, \beta_2, \beta_3 \dots$ are *pairwise distinct*. In the same way, the algebraic independence problem for the values of the Riemann zeta function boils down to a linear independence result for the values of multiple polylogarithms in one single variable, namely

$$\zeta(\underline{s}) = \mathrm{Li}_{\underline{s}}(1) = \sum_{n_1 > n_2 > \cdots > n_k \geqslant 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

for $\underline{s} = (s_1, \dots, s_k)$ with $s_1 \geqslant 2$, where

$$\mathrm{Li}_{\underline{s}}(z) = \sum_{n_1 > n_2 > \cdots > n_k \geqslant 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}}$$

for $z \in \mathbb{C}$, $|z| \leqslant 1$ with $(z, s_1) \neq (1, 1)$ (see [19]).

There are plenty of linear relations among these numbers, and a conjecture of Zagier [4,19,20] predicts a precise value for the dimension d_n of the vector space over the rational number field spanned by these numbers $\zeta(\underline{s})$ restricted to $s_1 + \cdots + s_k = n$, namely

$$d_n = d_{n-2} + d_{n-3}$$

with $d_1 = 0$, $d_2 = 1$. It is known that $d_3 = 1$ because $\zeta(3) = \zeta(2, 1)$, also $d_4 = 1$ because

$$\zeta(4) = \zeta(2, 1, 1) = 4\zeta(3, 1) = \tfrac{4}{3}\,\zeta(2, 2),$$

but $d_5 = 2$ is equivalent to the open problem to prove that $\zeta(5)/\zeta(2)\zeta(3)$ is irrational.

The fact that the integers d_n are bounded from above by the numbers defined by this inductive formula has just been proved by Terasoma in [17]. As pointed out in [17], the same result was announced by Goncharov in his preprints `AG/0005069` and `AG/0103059`. See the related paper by Deligne and Goncharov `NT/0302267`.

Several related conjectures are explained by Cartier [4]. Also the values at roots of unity of the functions $\mathrm{Li}_{\underline{s}}(z)$ have been considered by a number of authors: in this case it is appropriate to consider polylogarithms in several variables

$$\sum_{n_1 > n_2 > \cdots > n_k \geqslant 1} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}$$

for $\underline{z} = (z_1, \dots, z_k) \in \mathbb{C}$, $|z_i| \leqslant 1$ with $(z_1, s_1) \neq (1, 1)$. However here we shall consider only multiple polylogarithms in a single complex variable.

We describe now another example of linearization of an algebraic independence question. One of the main conjectures in transcendental number theory states that *linearly independent numbers ℓ for which e^ℓ are algebraic* (one says usually that such ℓ is a *logarithm of an algebraic number*) *are algebraically independent*. Let us show how this question reduces to a linear independence problem on special values, at algebraic points, of multiple polylogarithms.

For simplicity let $\alpha_1, \dots, \alpha_m$ be *positive real* algebraic numbers such that the numbers $\ell_1 = \log \alpha_1, \dots, \ell_m = \log \alpha_m$ are linearly independent.

We introduce the notation $\{1\}_n$ for the sequence $(1, 1, \ldots, 1)$ with n occurrences of 1. It is easy to check that

$$\mathrm{Li}_{\{1\}_n}(z) = \frac{(-1)^n}{n!}(\log(1-z))^n$$

for $n \geqslant 1$ and $|z| \leqslant 1$ with $z \neq 1$ (this is formula (1.1) of [10]).

Let m and d be two positive integers and Z_{ij} be variables, with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant \binom{d+1}{m}$. The determinant

$$\Delta(\underline{Z}) = (Z_{1j}^{i_1} \cdots Z_{mj}^{i_m}),$$

with $i_1 + \cdots + i_m \leqslant d$ and $1 \leqslant j \leqslant \binom{d+1}{m}$, as a polynomial in the variables Z_{ij}, does not vanish identically. Let a_{ij} be rational numbers such that the determinant Δ does not vanish at the corresponding point. Writing

$$(X_0 + a_{1j}X_1 + \cdots + a_{mj}X_m)^d$$

as a linear form in the monomials $X_0^{i_0} \cdots X_m^{i_m}$ with $i_0 + \cdots + i_m = d$, we deduce that each of these monomials is a linear combination with rational coefficients of $(X_0 + a_{1j}X_1 + \cdots + a_{mj}X_m)^d$ for $1 \leqslant j \leqslant \binom{d+1}{m}$.

Since

$$\log u + \log v = \log(uv)$$

for $u, v > 0$, it follows that the question of algebraic independence of logarithms of positive algebraic numbers $\ell_1, \ldots, \ell_m$ can be reduced to a question of linear independence of values of the multiple polylogarithms $\mathrm{Li}_{\{1\}_n}$ $(n \geqslant 1)$ (in a single variable), at algebraic points

$$1 - \ell_1^{b_1} \cdots \ell_m^{b_m}$$

where $b_1, \ldots, b_m$ are positive integers.

For instance we translate the (real case of the) four exponentials conjecture, namely:

- *If* a, b, c, d *are algebraic numbers in the range* $0 < x < 1$ *such that*

  $$(\log a)(\log b) = (\log c)(\log d),$$

  then there exists $(p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ *such that*

  $$a^p = c^q \quad \textit{and} \quad b^q = d^p.$$

as follows:

- *Define* $f(x) = \mathrm{Li}_{11}(1-x) = (1/2)(\log x)^2$ *for* $0 < x < 1$. *If* a, b, c, d *are algebraic numbers in the range* $0 < x < 1$ *such that*

  $$f(ab) - f(a) - f(b) = f(cd) - f(c) - f(d),$$

  then there exists $(p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ *such that*

  $$a^p = c^q \quad \textit{and} \quad b^q = d^p.$$

Polylogarithms in one variable are related by differential equation, namely they have a generating series which satisfies the Knizhnik–Zamolodchikov equation

$$y' = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right) y,$$

where x_0 and x_1 are two noncommuting variables (see [4]).

Hence in the previous discussion differential equations are always there. However in the hypotheses of Ramachandra's criterion (transcendence criteria) [15], no differential equation is required, but functional equations (like $f(u+v) = f(u)f(v)$ for the usual exponential function) are there. Also, q-analogues of some the above questions have been extensively studied by Tschakaloff, Lototskii, Bundschuh, Wallisser, Popov and others (see for instance [7] Chapter 2, Section 8). A typical example is the entire function

$$f(z) = \prod_{n \geqslant 1} (1 + zq^{-n})$$

for $q \in \mathbb{C}$, $|q| > 1$, which is solution of the functional equation

$$f(qz) = (1+z)f(z).$$

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