SASTRA International Conference on "NUMBER THEORY & COMBINATORICS" Kumbakonam, 19 - 22 DECEMBER 2006

On the work of S.S. Pillai

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S.Sivasankaranarayana Pillai (1901–1950)

http ://www.geocities.com/thangadurai_kr/PILLAI.html

Collected works of S. S. Pillai,
ed. R. Balasubramanian and R. Thangadurai, 2007.

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On m consecutive integers (number theory)

• Any two consecutive integers are relatively prime.
• Given three consecutive integers
for instance 3, 4, 5 : any two of them are relatively prime
for 2, 3, 4 : only 3 is prime to 2 and to 4.
In the general case n, $n+1$, $n+2$, the middle term is relatively prime to each other.
• Given four consecutive integers n, $n+1$, $n+2$, $n+3$, the odd number among $n+1$, $n+2$ is relatively prime to the three remaining integers. Hence one at least of the four numbers is relatively prime to the three others.

On m consecutive integers

• Given five consecutive integers

$$n,\ n+1,\ n+2,\ n+3,\ n+4$$

the only possible common prime factors between two of them are 2 and 3 , and one at least of the odd elements is not divisible by 3. Hence again one at least of the five numbers is relatively prime to the four others.
• After 2, 3, 4, 5, continue with $6, 7, 8\dots$ up to 16 – done by S.S. Pillai in 1940.

On 17 consecutive integers, following S.S. Pillai

- In every set of not more than 16 consecutive integers there is a number which is prime to all the others.
- This is not true for 17 consecutive numbers : take $n = 2184$ and consider the 17 consecutive integers $2184, \dots, 2200$. Then any two of them have a gcd > 1.
- One produces infinitely many such sets of 17 consecutive numbers by taking

$$n+N, n+N+1, \dots, n+N+16$$

or

$$N-n-16, n-N-15, \dots, N-n$$

where N is a multiple of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30\,030$.

On m consecutive integers (continued)

S.S. Pillai, 1940.

- **In every set of not more than 16 consecutive integers there is a number which is prime to all the others.**
- **For any $m \geq 17$ there exists a set of m consecutive integers that has not this property.**
- Application to the Diophantine equation

$$n(n+1)\cdots(n+m-1) = y^r$$

(See more recent work, esp. by T.N. Shorey).

First decimals of $\sqrt{2}$ (Combinatorics)

1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343276415727350138462309122970249248360558507372126441214970999358314132226659275055927557999505011527820605714701095599716059702745345968620147285174186408891986095523292304843087143214508397626036279952514079896872533965463318088296406206152583523950547457502877599617298355752203375318570113543746034084988471603868999706990048150305440277903164542478230684929369186215805784631115966687130130156185689872372352885092648612494977154218334204285686060146824720771435854874155657069677653720226485447015858801620758474922657226002085584466521458398893944370926591800311388246468157082630100594858704003186480342194897278290641045072636881313739855256117322040245091227700226941127573627280495738108967504018369868368450725799364729060762996941380475654823728997180326802474420629269124859052181004459842150591120249441341728531478105803603371077309182869314710171111683916581726889419758716582152128229518488472089694633862891562882765952635140542267653239694617511291602408715510135150455381287560052631468017127402653969470240300517495318862925631385188163478001569369176881852378684052287837629389214300655869568685964595155501644724509836896036887323114389415576651040883914292338113206052433629485317049915771756228549741438999188021762430965206564211827316726257539594717255934637238632261482742622208671155839599926521176252698917540988159348640083457085181472231814204070426509056532333398436457865796796519267292399875366617215982578860263363617827495994219403777753681426217738799194551397231274066898329989895386728822856378697749662519966583525776198939322845344735694794962952168891485492538904755828834526096524096542889394538646625744927872401…

First binary digits of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.011010100000100111100110011001111111001110111100110010010000
10001011001011111011000100110110011011101010100101010111110100
11111000111010110111101100000101110101000100100111011101010000
10011001110110100010111101011001000010110000011001100111001100
10001010101001010111111001000001100000100001110101011100010100
01011000011101010001011000111111110011011111101110010000011110
11011001110010000111101110100101010000101111001000011100111000
11110110100101001111000000000100100001110011011000111101111101
00010011101101000110100100010000000101110100001110100001010101
11100011111010011100101001100000101100111000110000000010001101
11100001100110111101111001010101100011011110010010001000101101
00010000100010110001010010001100000101010111100011100100010111
10111110001001110001100111100011011010101101010001010001110001
01110110111111010011101110011001011001010100110001101000011001
10001111100111100100001001101111101010010111100010010000011111
00000110110111001011000001011101110101010100100101000001000100
110010000001000000011001010010010101000000010011100101001010

Le fabuleux destin de $\sqrt{2}$

- *The fabulous destiny of* $\sqrt{2}$
Benoît Rittaud, Éditions *Le Pommier*, 2006.
http://www.math.univ-paris13.fr/~rittaud/RacineDeDeux

- Computation of decimals of $\sqrt{2}$:
1 542 computed by hand by Horace Uhler in 1951
14 000 decimals computed in 1967
1 000 000 decimals in 1971
$137 \cdot 10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.
- Motivation : computation of π.

Complexity of the g–ary expansion of an irrational algebraic real number

Let $g \geq 2$ be an integer.

- É. Borel (1909 and 1950) : *the g–ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (with respect to Lebesgue's measure).*
- In particular *each digit should occur*, hence *each given sequence of digits should occur infinitely often*.
- There is no explicitly known example of a triple (g, a, x), where $g \geq 3$ is an integer, a a digit in $\{0, \ldots, g-1\}$ and x an algebraic irrational number, for which one can claim that the digit a occurs infinitely often in the g–ary expansion of x.

Conjecture 1 (Émile Borel)

- Rendiconti del Circolo matematico di Palermo, **27** (1909), 24–271.
  Comptes Rendus de l'Académie des Sciences de Paris **230** (1950), 591–593.
- **Conjecture 1.** *Let x be an irrational algebraic real number, $g \geq 3$ a positive integer and a an integer in the range $0 \leq a \leq g-1$. Then the digit a occurs at least once in the g–ary expansion of x.*
- If a real number x satisfies Conjecture 1 for all g and a, then it follows that for any g, each given sequence of digits occurs infinitely often in the g–ary expansion of x.
- This is easy to see by considering powers of g.

Borel's Conjecture 1

- For instance, Conjecture 1 with $g = 4$ implies that each of the four sequences $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$ should occur infinitely often in the binary expansion of each irrational algebraic real number x.
- K. Mahler : *For any $g \geq 2$ and any $n \geq 1$, there exist algebraic irrational numbers x such that any block of n digits occurs infinitely often in the g–ary expansion of x.*

Normal Numbers

- A real number x is called *simply normal in base g* if each digit occurs with frequency $1/g$ in its g–ary expansion.
- A real number x is called *normal in base g* or *g–normal* if it is simply normal in base g^m for all $m \geq 1$.
- Hence a real number x is normal in base g if and only if, for any $m \geq 1$, each sequence of m digits occurs with frequency $1/g^m$ in its g–ary expansion.
- A real number is called *normal* if it is normal in any base $g \geq 2$.
- Hence a real number is normal if and only if it is simply normal in any base $g \geq 2$.

Borel's Conjecture 2

- **Conjecture 2.** *Let x be an irrational algebraic real number. Then x is normal.*
- Almost all real numbers (for Lebesgue's measure) are normal.
- Examples of computable normal numbers have been constructed (W. Sierpinski, H. Lebesgue, V. Becher and S. Figueira) but the known algorithms to compute such examples are fairly complicated ("ridiculously exponential", according to S. Figueira).

Example of normal numbers

An example of a 2–normal number (Champernowne 1933, Bailey and Crandall 2001) is the *binary Champernowne number*, obtained by the concatenation of the sequence of integers

$$0.\,1\,10\,11\,100\,101\,110\,111\,1000\,1001\,1010\,1011\,1100\,\dots$$

$$=\sum_{k\geq 1} k2^{-c_k} \quad \text{with} \quad c_k = k+\sum_{j=1}^{k}[\log_2 j].$$

Proof : Pillai, 1939 and 1940.

Further examples of normal numbers

- (Korobov, Stoneham ...) : *if a and g are coprime integers > 1, then*

$$\sum_{n\geq 0} a^{-n} g^{-a^n}$$

  is normal in base g.
- A.H. Copeland and P. Erdős (1946) : a normal number in base 10 is obtained by concatenation of the sequence of prime numbers

$$0.2\,3\,5\,7\,11\,13\,17\,19\,23\,29\,31\,37\,41\,43\,47\,53\,59\,61\,67\ \ldots$$

On Waring's Problem : $g(6) = 73$

S.S. Pillai, 1940.

- **Any positive integer N is sum of at most 73 sixth powers : $N = x_1^6 + \cdots + x_s^6$ with $s \leq 73$.**
- Since $2^6 = 64$, the integer $N = 63 = 1^6 + \cdots + 1^6$ requires at least 63 terms x_i.
- Any decomposition of an integer $N \leq 728 = 3^6 - 1$ as a sum of sixth powers involves only 1 and 2^6
- The decomposition as a sum of sixth powers of any integer $N \leq 728$ of the form $63 + k64$ requires at least $63 + k$ terms.
- The number $703 = 63 + 64 \times 10$ requires $63 + 10 = 73$ terms.

Waring's Problem

In 1770, a few months before J.L. Lagrange solved a conjecture of Bachet and Fermat by proving that every positive integer is the sum of at most four squares of integers, E. Waring wrote :

"Every integer is a cube or the sum of two, three, ...nine cubes ; every integer is also the square of a square, or the sum of up to nineteen such ; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree."

The number $g(k)$

- Waring's function g is defined as follows : For any integer $k \geq 2$, $g(k)$ is the least positive integer s such that any positive integer N can be written $x_1^k + \cdots + x_s^k$.
- Hence Pillai's above mentioned result is $g(6) = 73$.

Results on Waring's Problem

$g(2) = 4$	J-L. Lagrange	(1770)
$g(3) = 9$	A. Wieferich	(1909)
$g(4) = 19$	R. Balasubramanian, J-M. Deshouillers, F. Dress	(1986)
$g(5) = 37$	J. Chen	(1964)
$g(6) = 73$	S.S. Pillai	(1940)
$g(7) = 143$	L.E. Dickson	(1936)

The ideal Waring's Theorem

For each integer $k \geq 2$, define $I(k) = 2^k + [(3/2)^k] - 2$. It is easy to show that $g(k) \geq I(k)$. Indeed, write

$$3^k = 2^k q + r \quad \text{with} \quad 0 < r < 2^k, \quad q = [(3/2)^k],$$

and consider the integer

$$N = 2^k q - 1 = (q-1)2^k + (2^k - 1)1^k.$$

Since $N < 3^k$, writing N as a sum of k-th powers can involve no term 3^k, and since $N < 2^k q$, it involves at most $(q-1)$ terms 2^k, all others being 1^k; hence it requires a total number of at least $(q-1) + (2^k - 1) = I(k)$ terms.

The ideal Waring's Theorem

L.E. Dickson and S.S. Pillai proved independently in 1936 that $g(k) = I(k)$, provided that $r = 3^k - 2^k q$ satisfies

$$r \le 2^k - q - 2.$$

The condition $r \le 2^k - q - 2$ is satisfied for $3 \le k \le 471\ 600\ 000$, and (K. Mahler) also for all sufficiently large k.
The conjecture, dating back to 1853, is $g(k) = I(k) = 2^k + [(3/2)^k] - 2$ for any $k \ge 2$. This is true as soon as

$$\left\| \left(\frac{3}{2}\right)^k \right\| \ge \left(\frac{3}{4}\right)^k,$$

where $\|\cdot\|$ denote the distance to the nearest integer.

On Waring's Problem with exponents $\ge n$

S.S. Pillai, 1940.

- For any integer $n \ge 2$, denote by $g_2(n)$ the least positive integer s such that any positive integer N can be written $x_1^{m_1} + \cdots + x_s^{m_s}$ with $m_i \ge n$.
- S.S. Pillai (1940) : explicit formula for $g_2(n)$, $n \ge 32$.
- Proof of the lower bound $g(n) \ge 2^n + h - 1$ if $2^{n+h} \le 3^n$.

Lower bound for $g_2(n)$

• The lower bound $g_2(n) \geq 2^n - 1$ is trivial : take $N = 2^n - 1$.
• Any decomposition $N = x_1^{m_1} + \cdots + x_s^{m_s}$ with $m_i \geq n$ of a positive integer $N < 3^n$ has $x_i \in \{1, 2\}$.
• Let $h \geq 1$ satisfy $2^{n+h} \leq 3^n$. Consider the integer $N = 2^{n+h} - 1$. Its binary expansion is

$$N = 2^{n+h-1} + 2^{n+h-2} + \cdots + 2 + 1,$$

hence it can be written

$$N = 2^{n+h-1} + 2^{n+h-2} + \cdots + 2^n + (2^n - 1),$$

which is a sum of h numbers 2^m with $m \geq n$ and $2^n - 1$ powers of 1.

Value of $g_2(n)$ for $n \geq 32$

One easily deduces $g_2(n) \geq 2^n + h - 1$ as soon as h satisfies $2^{n+h} \leq 3^n$.
This condition on h is $2^h \leq (3/2)^n$, which means $2^h \leq I_n$ with $I_n = [(3/2)^n]$.

Define

$$h_n = [\log I_n / \log 2] \quad \text{where} \quad I_n = [(3/2)^n].$$

Pillai's Theorem : *For* $n \geq 32$, $g_2(n) = 2^n + h_n - 1$.

Square, cubes...

- A perfect power is an integer of the form a^b where $a \geq 1$ and $b > 1$ are positive integers.
- Squares :

$$1,\ 4,\ 8,\ 9,\ 16,\ 25,\ 27,\ 36,\ 49,\ 64,\ 81,\ 100,\ 121,\ 125,\ 128,\ \dots$$

- Cubes :

$$1,\ 8,\ 27,\ 64,\ 125,\ 216,\ 343,\ 512,\ 729,\ 1\,000,\ 1\,331\dots$$

- Fifth powers :

$$1,\ 32,\ 243,\ 1\,024,\ 3\,125,\ 7\,776,\ 16\,807,\ 32\,768\dots$$

Consecutive integers
Normal Numbers
Waring's Problem
Diophantine equations

Pillai's early work

In 1936 Pillai proved that for any fixed positive integers a and b, both at least 2, the number of solutions (x, y) of the Diophantine inequality $0 < a^x - b^y \leq c$ is asymptotically equal to

$$\frac{(\log c)^2}{2 \log a \log b}$$

as c tends to infinity.

References :

PILLAI, S. S. – *On some Diophantine equations*, J. Indian Math. Soc., XVIII (1930), 291-295.
PILLAI, S. S. – *On* $A^x - B^y = C$, J. Indian Math. Soc. (N.S.), II (1936), 119–122.

http://www.math.jussieu.fr/~miw/

Connexion with Ramanujan's work

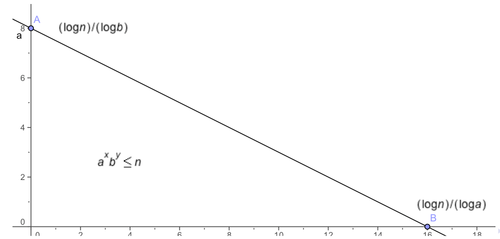
It is remarkable that this asymptotic value is related to another problem which Pillai studied later and which originates in the following claim by Ramanujan :
The number of numbers of the form $2^u \cdot 3^v$ *less than* n *is*

$$\frac{\log(2n)\log(3n)}{2\log 2\log 3}.$$

Number of integers $a^u b^v \leq n$

The number of numbers of the form $a^u \cdot b^v$ less than n is asymptotically

$$\frac{(\log n)^2}{2\log a\log b}.$$



Perfect powers

The sequence of perfect powers starts with :

1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, 196, 216, 225, 243, 256, 289, 324, 343, 361, 400, 441, 484, 512, 529, 576, 625, 676, 729, 784 . . .

Write this sequence as

$$a_1 = 1,\ a_2 = 4,\ a_3 = 8,\ a_4 = 9,\ a_5 = 16,\ a_6 = 25,\ a_7 = 27,\ \ldots$$

Taking only the squares into account, we deduce

$$a_n \le n^2 \quad \text{for all} \quad n \ge 1.$$

Lower bound for a_n

We want also a lower bound for a_n. For this we need an upper bound for the number of perfect powers a^x bounded by a_n which are not squares. We do it in a crude way : if $a^x \le N$ with $a \ge 2$ and $x \ge 3$ then $x \le (\log N)/(\log 2)$ and $a \le N^{1/3}$, hence the number of such a^x is less than

$$\frac{1}{\log 2} \cdot N^{1/3} \log N$$

Hence the number of elements in the sequence of perfect powers which are less than N is at most

$$\sqrt{N} + \frac{1}{\log 2} \cdot N^{1/3} \log N.$$

The sequence of perfect powers

The upper bound

$$n \le \sqrt{a_n} + \frac{1}{\log 2} \cdot a_n^{1/3} \log a_n.$$

together with $a_n \ge n^2$ yields

$$a_n \ge n^2 - \frac{2}{\log 2} \cdot n^{2/3} \log n,$$

and one checks that this estimate is true as soon as $n \ge 8$.
As a consequence

$$\limsup (a_{n+1} - a_n) = +\infty.$$

Consecutive elements in the sequence of perfect powers

- Difference 1 : $(8, 9)$
- Difference 2 : $(25, 27)$
- Difference 3 : $(1, 4)$, $(125, 128)$
- Difference 4 : $(4, 8)$, $(32, 36)$, $(121, 125)$
- Difference 5 : $(4, 9)$, $(27, 32)$...

Two conjectures

- Catalan's Conjecture : In the sequence of perfect powers, $8, 9$ is the only example of consecutive integers.
- Pillai's Conjecture : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.
- Alternatively : Let k be a positive integer. The equation

$$x^p - y^q = k,$$

where the unknowns x, y, p and q take integer values, all ≥ 2, has only finitely many solutions (x, y, p, q).

Consecutive integers
Normal Numbers
Waring's Problem
Diophantine equations

Pillai's conjecture

Pillai, S. S. – *On the equation* $2^x - 3^y = 2^X + 3^Y$, Bull. Calcutta Math. Soc. 37, (1945). 15–20.

I take this opportunity to put in print a conjecture which I gave during the conference of the Indian Mathematical Society held at Aligarh.

Arrange all the powers of integers like squares, cubes etc. in increasing order as follows :

1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, . . .

Let a_n *be the* n*-th member of this series so that* $a_1 = 1$, $a_2 = 4$, $a_3 = 8$, $a_4 = 9$, *etc. Then*

Conjecture :

$$\liminf (a_n - a_{n-1}) = \infty.$$

http://www.math.jussieu.fr/~miw/

Indian Science Congress 1949

"The audience may be a little disappointed at the scanty reference to Indian work. ⋯ However, we need not feel dejected. Real research in India started only after 1910 and India has produced Ramanujan and Raman"

This was the statement of Dr. S. Sivasankaranarayana Pillai in the 36th Annual session of the Indian Science Congress on 3rd January, 1949 at Allahabad university.

http ://www.geocities.com/thangadurai_kr/PILLAI.html

http ://www.geocities.com/thangadurai_kr/PILLAI.html

The tragic end

For his achievements, he was invited to visit the Institute of Advance Studies, Princeton, USA for a year. Also, he was invited to participate in the International Congress of Mathematicians at Harvard University as a delegate of Madras University. So, he proceeded to USA by air in the august 1950. But due to the air crash near Cairo on August 31, 1950, Indian Mathematical Community lost one of the best known mathematicians.

Results

- P. Mihăilescu, 2002. Catalan was right : *the equation* $x^p - y^q = 1$ *where the unknowns* x, y, p *and* q *take integer values, all* ≥ 2, *has only one solution* $(x, y, p, q) = (3, 2, 2, 3)$.
  Previous partial results : J.W.S. Cassels, R. Tijdeman, M. Mignotte…
- Higher values of k : nothing known.
- Pillai's conjecture as a consequence of the *abc* conjecture :
$$|x^p - y^q| \geq c(\epsilon) \max\{x^p, y^q\}^{\kappa-\epsilon}$$
  with
$$\kappa = 1 - \frac{1}{p} - \frac{1}{q}\cdot$$

The *abc* Conjecture

- For a positive integer n, we denote by
$$R(n) = \prod_{p|n} p$$
  the *radical* or *square free part* of n.
- The *abc* Conjecture resulted from a discussion between D. W. Masser and J. Œsterlé in the mid 1980's.
- Conjecture (*abc* Conjecture). *For each* $\varepsilon > 0$ *there exists* $\kappa(\varepsilon)$ *such that, if* a, b *and* c *in* $\mathbf{Z}_{>0}$ *are relatively prime and satisfy* $a + b = c$, *then*
$$c < \kappa(\varepsilon) R(abc)^{1+\varepsilon}.$$

Waring's Problem and the *abc* Conjecture

S. David : the estimate

$$\left\| \left(\frac{3}{2}\right)^k \right\| \geq \left(\frac{3}{4}\right)^k,$$

(for sufficiently large k) follows not only from Mahler's estimate, but also from the *abc* Conjecture!

Hence the ideal Waring Theorem $g(k) = 2^k + [(3/2)^k] - 2.$ would follow from an explicit solution of the *abc* Conjecture.

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