Abstract

The Landau–Ramanujan constant $\alpha$ is defined as follows: for $N \rightarrow \infty$, the number of positive integers $\leq N$ which are sums of two squares is asymptotically

$$\frac{\alpha N}{\sqrt{\log N}}$$

In a joint work with Etienne Fouvry and Claude Levesque, we replace the quadratic form $\Phi_4(X,Y) = X^2 + Y^2$, which is the homogeneous version of the cyclotomic polynomial $\phi_4(t) = t^2 + 1$, with other binary forms.

This is a joint work with Étienne Fouvry and Claude Levesque

Representation of integers by cyclotomic binary forms.
Dedicated to Rob Tijdeman. arXiv: 712.09019 [math.NT]
On binary cyclotomic polynomials

Étienne Fouvry

We study the number of nonzero coefficients of cyclotomic polynomials \( \Phi_w \), where \( w \) is the product of two distinct primes.

Joint work with Claude Levesque:

*Representation of positive integers by binary cyclotomic forms*

The Landau–Ramanujan constant

Edmund Landau 1877 – 1938
Srinivasa Ramanujan 1887 – 1920

The number of positive integers \( \leq N \) which are sums of two squares is asymptotically \( C_{\Phi_4} N (\log N)^{-\frac{1}{2}} \), where

\[
C_{\Phi_4} = \frac{1}{2^{\frac{3}{2}}} \prod_{p \equiv 3 \mod 4} \left( 1 - \frac{1}{p^2} \right)^{-\frac{1}{2}}.
\]

Online Encyclopedia of Integer Sequences

https://oeis.org/A064533

[OEIS A064533] Decimal expansion of Landau–Ramanujan constant:

\[
C_{\Phi_4} = 0.764223653589220\ldots
\]

* Ph. Flajolet and I. Vardi, Zeta function expansions of some classical constants, Feb 18 1996.
* Xavier Gourdon and Pascal Sebah, Constants and records of computation.
* David E. G. Hare, 125 079 digits of the Landau–Ramanujan constant.
The Landau–Ramanujan constant

References: https://oeis.org/A064533

• B. C. Berndt, Ramanujan’s notebook part IV, Springer-Verlag, 1994
• G. H. Hardy, ”Ramanujan, Twelve lectures on subjects suggested by his life and work”, Chelsea, 1940.
• Institute of Physics, Constants - Landau-Ramanujan Constant
• Simon Plouffe, Landau Ramanujan constant
• Eric Weisstein’s World of Mathematics, Ramanujan constant

Sums of two squares

If $a$ and $q$ are two integers, we denote by $N_{a,q}$ any integer $\geq 1$ satisfying the condition

$$p \mid N_{a,q} \iff p \equiv a \mod q.$$ 

An integer $m \geq 1$ can be written as

$$m = \Phi_4(x, y) = x^2 + y^2$$

if and only if there exist integers $a \geq 0$, $N_{3,4}$ and $N_{1,4}$ such that

$$m = 2^a N_{3,4}^2 N_{1,4}.$$ 

Positive definite quadratic forms

Let $F \in \mathbb{Z}[X, Y]$ be a positive definite quadratic form. There exists a positive constant $C_F$ such that, for $N \to \infty$, the number of positive integers $m \in \mathbb{Z}$, $m \leq N$ which are represented by $F$ is asymptotically $C_F N (\log N)^{-\frac{2}{3}}$.


http://www.ethlife.ethz.ch/archive_articles/102097_bernays_fm/
Paul Bernays (1888 – 1977)


• 1912, Ph.D. in mathematics, University of Göttingen, On the analytic number theory of binary quadratic forms (Advisor : Edmund Landau).

• 1913, Habilitation, University of Zürich, On complex analysis and Picard’s theorem, advisor Ernst Zermelo.

• 1912 – 1917, Zürich ; work with Georg Pólya, Albert Einstein, Hermann Weyl.

• 1917 – 1933, Göttingen, with David Hilbert. Studied with Emmy Noether, Bartel Leendert van der Waerden, Gustav Herglotz.

• 1935 – 1936, Institute for Advanced Study, Princeton. Lectures on mathematical logic and axiomatic set theory.

• 1936 —, ETH Zürich.


The quadratic form $x^2 + xy + y^2$

A prime number is represented by the quadratic form $x^2 + xy + y^2$ if and only if it is either 3 or else congruent to 1 modulo 3.

Product of two numbers represented by the quadratic form $x^2 + xy + y^2$:

$$(a^2 + ab + b^2)(c^2 + cd + d^2) = e^2 + ef + f^2$$

with $$e = ac - bd, \ f = ad + bd + bc.$$  

The quadratic cyclotomic field $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, $1 + \zeta_3 + \zeta_3^2 = 0$:

$$a^2 + ab + b^2 = \text{Norm}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(a - \zeta_3 b).$$  

Specific binary forms

• Sums of cubes, biquadrates, . . .

Notice that $X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)$

We start with the quadratic form $\Phi_3(X, Y) = X^2 + XY + Y^2$ which is the homogeneous version of the cyclotomic polynomial $\phi_3(t) = t^2 + t + 1$.

Notice that

$$\Phi_6(X, Y) = \Phi_3(X, -Y) = X^2 - XY + Y^2$$

Also

$$\Phi_8(X, Y) = X^4 + Y^4.$$  

Loeschian numbers : $m = x^2 + xy + y^2$

An integer $m \geq 1$ can be written as

$$m = \Phi_3(x, y) = \Phi_6(x, -y) = x^2 + xy + y^2$$

if and only if there exist integers $b \geq 0$, $N_{2,3}$ and $N_{1,3}$ such that

$$m = 3^b N_{2,3} N_{1,3}.$$  

The number of positive integers $\leq N$ which are represented by the quadratic form $x^2 + xy + y^2$ is asymptotically

$$C_{\Phi_3} N (\log N)^{-\frac{3}{2}},$$

where

$$C_{\Phi_3} = \frac{1}{2^{2 + 3 + 3}} \cdot \prod_{p \equiv 2 \mod 3} \left(1 - \frac{1}{p^2}\right)^{-\frac{3}{2}}$$

The first decimal digits of $C_{3}$ are

$$C_{3} = 0.63890940544\ldots$$

Loeschian numbers which are sums of two squares

An integer $m \geq 1$ is simultaneously of the forms

$$m = \Phi_4(x, y) = x^2 + y^2 \quad \text{and} \quad m = \Phi_3(u, v) = u^2 + uv + v^2$$

if and only if there exist integers $a, b \geq 0$, $N_{5,12}$, $N_{7,12}$, $N_{11,12}$ and $N_{1,12}$ such that

$$m = \left(2^a 3^b N_{5,12} N_{7,12} N_{11,12}\right)^2 N_{1,12}.$$ 

The number of Loeschian integers $\leq N$ which are sums of two squares is asymptotically $\beta N (\log N)^{-3/4}$, where

$$\beta = \frac{3\frac{1}{2}}{2^2} \cdot \pi^{\frac{1}{2}} \cdot (\log(2 + \sqrt{3}))^{\frac{3}{4}} \cdot \frac{1}{\Gamma(1/4)} \cdot \prod_{p \equiv 5, 7, 11 \mod 12} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}}.$$
OEIS A301430 $\beta = 0.30231614235\ldots$

[OEIS A301430] Decimal expansion of an analog of the Landau-Ramanujan constant for Loeschian numbers which are sums of two squares.

\[
\beta = \frac{3^4}{2^4} \cdot \pi^2 \cdot (\log(2+\sqrt{3}))^4 \cdot \frac{1}{\Gamma(1/4)}\cdot \prod_{p\equiv 5,7,11 \mod 12} \left(1-\frac{1}{p^2}\right)^{-\frac{1}{2}}
\]

\[
\beta = \frac{3^4}{2^4} \cdot \pi^2 \cdot (\log(2+\sqrt{3}))^4 \cdot \frac{1}{\Gamma(1/4)}\cdot \prod_{p\equiv 5,7,11 \mod 12} \left(1-\frac{1}{p^2}\right)^{-\frac{1}{2}}
\]

\[
\beta = 0.30231614235706563794
7769900480199715602412
7951893696454588678412
8886544875241051089948
7467189792728567765
913272591066837135863\ldots
\]

Bill Allombert

Cyclotomic polynomials

\[\phi_n(t) = \frac{t^n - 1}{\prod_{d|n, d|n} \phi_d(t)}\]

For instance

\[
\phi_4(t) = \frac{t^4 - 1}{t^2 - 1} = t^2 + 1 = \phi_2(t^2),
\]

\[
\phi_6(t) = \frac{t^6 - 1}{(t^3 - 1)(t + 1)} = \frac{t^3 + 1}{t + 1} = t^2 - t + 1 = \phi_3(-t).
\]

The degree of \(\phi_n(t)\) is \(\varphi(n)\), where \(\varphi\) is the Euler totient function.

Cyclotomic polynomials and roots of unity

For \(n \geq 1\), if \(\zeta\) is a primitive \(n\)-th root of unity,

\[\phi_n(t) = \prod_{\gcd(j,n)=1} (t - \zeta^j).
\]

For \(n \geq 1\), \(\phi_n(t)\) is the irreducible polynomial over \(\mathbb{Q}\) of the primitive \(n\)-th roots of unity,

Let \(K\) be a field and let \(n\) be a positive integer. Assume that \(K\) has characteristic either 0 or else a prime number \(p\) prime to \(n\). Then the polynomial \(\phi_n(t)\) is separable over \(K\) and its roots in \(K\) are exactly the primitive \(n\)-th roots of unity which belong to \(K\).
Properties of $\phi_n(t)$

- For $n \geq 2$ we have
  $$\phi_n(t) = t^{\varphi(n)} \phi_n(1/t)$$

- Let $n = 2^{e_0} p_1^{e_1} \cdots p_r^{e_r}$ where $p_1, \ldots, p_r$ are different odd primes, $e_0 \geq 0$, $e_i \geq 1$ for $i = 1, \ldots, r$ and $r \geq 1$. Denote by $R$ the radical of $n$, namely
  $$R = \begin{cases} 2p_1 \cdots p_r & \text{if } e_0 \geq 1, \\ p_1 \cdots p_r & \text{if } e_0 = 0. \end{cases}$$
  Then,
  $$\phi_n(t) = \phi_R(t^{n/R}).$$

- Let $n = 2m$ with $m$ odd $\geq 3$. Then
  $$\phi_n(t) = \phi_m(-t).$$

$\phi_n(-1)$

For $n \geq 3$,

$$\phi_n(-1) = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ \phi_{n/2}(1) & \text{if } n \text{ is even.} \end{cases}$$

In other terms, for $n \geq 3$,

$$\phi_n(-1) = \begin{cases} p & \text{if } n = 2p^r \text{ with } p \text{ a prime and } r \geq 1; \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\phi_n(-1) = 1$ when $n$ is odd or when $n = 2m$ where $m$ has at least two distinct prime divisors.

$\phi_n(1)$

For $n \geq 2$, we have $\phi_n(1) = e^{\Lambda(n)}$, where the von Mangoldt function is defined for $n \geq 1$ as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

In other terms we have

$$\phi_n(1) = \begin{cases} p & \text{if } n = p^r \text{ with } p \text{ prime and } r \geq 1; \\ 1 & \text{otherwise.} \end{cases}$$

Lower bound for $\phi_n(t)$

For $n \geq 3$, the polynomial $\phi_n(t)$ has real coefficients and no real root, hence it takes only positive values (and its degree $\varphi(n)$ is even).

For $n \geq 3$ and $t \in \mathbb{R}$, we have

$$\phi_n(t) \geq 2^{-\varphi(n)}.$$
\( \phi_n(t) \geq 2^{-\varphi(n)} \) for \( n \geq 3 \) and \( t \in \mathbb{R} \)

**Proof.**
Let \( \zeta_n \) be a primitive \( n \)-th root of unity in \( \mathbb{C} \);

\[
\phi_n(t) = N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(t - \zeta_n) = \prod_{\sigma}(t - \sigma(\zeta_n)),
\]

where \( \sigma \) runs over the embeddings \( \mathbb{Q}(\zeta_n) \to \mathbb{C} \). We have

\[
|t - \sigma(\zeta_n)| \geq |\text{Im}(\sigma(\zeta_n))| > 0,
\]

\[
(2i)\text{Im}(\sigma(\zeta_n)) = \sigma(\zeta_n) - \overline{\sigma(\zeta_n)} = \sigma(\zeta_n) - \overline{\zeta_n}.
\]
Now \( (2i)\text{Im}(\zeta_n) = \zeta_n - \overline{\zeta_n} \in \mathbb{Q}(\zeta_n) \) is an algebraic integer :

\[
2^{\varphi(n)}\phi_n(t) \geq |N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}((2i)\text{Im}(\zeta_n))| \geq 1.
\]

**Refinement (FLW)**

Let \( c_n = \inf_{t \in \mathbb{R}} \phi_n(t) \).

Refinement of the lower bound \( c_n \geq 2^{-\varphi(n)} \):

For \( n \geq 3 \)

\[
c_n \geq \left( \frac{\sqrt{3}}{2} \right)^{\varphi(n)}.
\]

Equality for \( n = 3 \) and \( n = 6 \).

For \( n \) a power of \( 2 \), \( c_n = 1 \).

Otherwise, if \( n \) has \( r \) distinct primes \( p_1, \ldots, p_r \) with \( p_1 \) the smallest, then

\[
c_n = c_{p_1 \cdots p_r} \geq p_1^{-2^{r-2}}.
\]

**Generalization to CM fields**


**The cyclotomic binary forms**

For \( n \geq 2 \), define

\[
\Phi_n(X, Y) = Y^{\varphi(n)}\phi_n(X/Y).
\]

This is a binary form in \( \mathbb{Z}[X, Y] \) of degree \( \varphi(n) \).

Consequence of the lower bound \( c_n \geq 2^{-\varphi(n)} \):

for \( n \geq 3 \) and \( (x, y) \in \mathbb{Z}^2 \),

\[
\Phi_n(x, y) \geq 2^{-\varphi(n)} \max\{|x|, |y|\}^{\varphi(n)}.
\]

Therefore, if \( \Phi_n(x, y) = m \), then

\[
\max\{|x|, |y|\} \leq 2m^{1/\varphi(n)}.
\]

If \( \max\{|x|, |y|\} \geq 3 \), then \( n \) is bounded :

\[
\varphi(n) \leq \frac{\log m}{\log(3/2)}.
\]
Binary cyclotomic forms (EF–CL–MW 2018)

Let $m$ be a positive integer and let $n, x, y$ be rational integers satisfying $n \geq 3$, $\max\{|x|, |y|\} \geq 2$ and $\Phi_n(x, y) = m$. Then

$$\max\{|x|, |y|\} \leq \frac{2}{\sqrt{3}} m^{1/\varphi(n)}, \quad \text{hence} \quad \varphi(n) \leq \frac{2}{\log 3} \log m.$$ 

These estimates are optimal, since for $\ell \geq 1$,

$$\Phi_3(\ell, -2\ell) = 3\ell^2.$$ 

If we assume $\varphi(n) > 2$, namely $\varphi(n) \geq 4$, then

$$\varphi(n) \leq \frac{4}{\log 11} \log m$$

which is best possible since $\Phi_5(1, -2) = 11$.

**OEIS A299214**

https://oeis.org/A299214

Number of representations of integers by cyclotomic binary forms.

The sequence $(a_m)_{m \geq 1}$ starts with

0, 0, 8, 16, 8, 0, 24, 4, 16, 8, 8, 12, 40, 0, 0, 40, 16, 4, 24, 8, 24, 0, 0, 24, 8, 12, 24, 8, 0, 32, 8, 8, 8, 0, 16, 32, 0, 24, 8, 8, 0, 32, 8, 0, 12, 40, 12, 0, 32, 8, 0, 8, 0, 32, 8, 0, 48, 0, 24, 40, 16, 0, 24, 8, 0, 0, 0, 4, 48, 8, 12, 24, ...
OEIS A293654

https://oeis.org/A293654
Integers not represented by cyclotomic binary forms.

\[ a_m = 0 \text{ for } m = 1, 2, 6, 14, 15, 22, 23, 24, 30, 33, 35, 38, 42, 44, 46, 47, 51, 54, 56, 59, 60, 62, 66, 69, 70, 71, 77, 78, 83, 86, 87, 88, 92, 94, 95, 96, 99, 102, 105, 107, 110, 114, 115, 118, 119, 120, 123, 126, 131, 132, 134, 135, 138, 140, 141, 142, 143, 150, \ldots \]

Higher degree

The situation for quadratic forms of degree \( \geq 3 \) is different for several reasons.

- If a positive integer \( m \) is represented by a positive definite quadratic form, it usually has many such representations; while if a positive integer \( m \) is represented by an irreducible binary form of degree \( d \geq 3 \), it usually has few such representations.

- If \( F \) is a positive definite quadratic form, the number of \((x, y)\) with \( F(x, y) \leq N \) is asymptotically a constant times \( N \), but the number of \( F(x, y) \) is much smaller.

- If \( F \) is an irreducible binary form of degree \( d \geq 3 \), the number of \((x, y)\) with \( F(x, y) \leq N \) is asymptotically a constant times \( N^{\frac{d}{2}} \), the number of \( F(x, y) \) is also asymptotically a constant times \( N^{\frac{d}{2}} \).

Numbers represented by a cyclotomic binary form of degree \( \geq 2 \)

For \( N \geq 1 \), the number of \( m \leq N \) for which there exists \( n \geq 3 \) and \((x, y) \in \mathbb{Z}^2\) with \( \max(|x|, |y|) \geq 2 \) and \( m = \Phi_n(x, y) \), is asymptotically

\[
(C_{\Phi_4} + C_{\Phi_3}) \frac{N}{(\log N)^2} = \frac{N}{(\log N)^2} + O\left( \frac{N}{(\log N)^2} \right)
\]

as \( N \to \infty \).

\[ C_{\Phi_4} + C_{\Phi_3} = 1.403133059034 \ldots \beta = 0.30231614235 \ldots \]


*Dedicated to Rob Tijdeman.* arXiv: 712.09019 [math.NT]

Higher degree

A quadratic form has infinitely many automorphisms, an irreducible binary form of higher degree has a finite group of automorphisms.

Sums of \(k\)--th powers

If a positive integer \(m\) is a sum of two squares, there are many such representations.
Indeed, the number of \((x, y)\) in \(\mathbb{Z} \times \mathbb{Z}\) with \(x^2 + y^2 \leq N\) is asymptotic to \(\pi N\), while the number of values \(\leq N\) taken by the quadratic form \(\Phi_4\) is asymptotic to \(C_{\Phi_4} N / \sqrt{\log N}\) where \(C_{\Phi_4}\) is the Landau–Ramanujan constant. Hence \(\Phi_4\) takes each of these values with a high multiplicity, on the average \((\pi / C_{\Phi_4}) \sqrt{\log N}\).

On the opposite, given an integer \(k \geq 3\), that a positive integer is a sum of two \(k\)--th powers in more than one way (not counting symmetries) is

- rare for \(k = 3\),
- extremely rare for \(k = 4\),
- maybe impossible for \(k \geq 5\).

The sequence of Taxicab numbers

[OEIS A001235] Taxi-cab numbers: sums of 2 cubes in more than 1 way.

\[1729 = 10^3 + 9^3 = 12^3 + 1^3, \quad 4104 = 2^3 + 16^3 = 9^3 + 15^3, \ldots\]

\[1729, 4104, 13832, 20683, 32832, 39312, 40033, 46683, 64232, 65728, 110656, 110808, 134379, 149389, 165464, 171288, 195841, 216027, 216125, 262656, 314496, 320264, 327763, 373464, 402597, 439101, 443889, 513000, 513856, 515375, 525824, 558441, 593047, \ldots\]

If \(n\) is in this sequence, then \(nk^3\) also, hence this sequence is infinite.

1729: the taxicab number

The smallest positive integer which is sum of two cubes in two essentially different ways:

\[1729 = 10^3 + 9^3 = 12^3 + 1^3.\]

Godfrey Harold Hardy
1877–1947

Srinivasa Ramanujan
1887 – 1920

1657: Fréicle de Bessy (1605 ? – 1675)

Another sequence of Taxicab numbers (Fermat)

[OEIS A011541] Hardy-Ramanujan numbers: the smallest number that is the sum of 2 positive integral cubes in \(n\) ways.
http://mathworld.wolfram.com/TaxicabNumber.html

\(T_a(1) = 2,\)
\(T_a(2) = 1729 = 10^3 + 9^3 = 12^3 + 1^3,\)
\(T_a(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3,\)
\(T_a(4) = 6963472309248 = 2421^3 + 19083^3 = 5436^3 + 1898^3 = 10 \, 200^3 + 18 \, 072^3 = 13 \, 322^3 + 16 \, 630^3,\)
\(T_a(5) = 48988659276962496,\)
\(T_a(6) = 24153319581254312065344.\)
Hardy and Wright,
An Introduction of Theory of Numbers

Fermat proved that numbers expressible as a sum of two positive integral cubes in \( n \) different ways exist for any \( n \).

Pierre de Fermat
1607 (?) – 1665

2003 : C. S. Calude, E. Calude and M. J. Dinneen,
With high probability,

\[ T_a(6) = 24153319581254312065344. \]

Cubefree taxicab numbers

Stuart Gascoigne and Duncan Moore (2003) :
1 801 049 058 342 701 083 = 92227^3 + 1216500^3 = 136635^3 + 121610^3 = 341995^3 + 120760^3 = 600259^3 + 116584^3

[OEIS A080642] Cubefree taxicab numbers: the smallest cubefree number that is the sum of 2 cubes in \( n \) ways.

https://en.wikipedia.org/wiki/Taxicab_number

Taxicabs and Sums of Two Cubes

If the sequence \( (a_n) \) of cubefree taxicab numbers with \( n \) representations is infinite, then the Mordell-Weil rank of the elliptic curve \( x^3 + y^3 = a_n \) tends to infinity with \( n \).

J. H. Silverman, Taxicabs and Sums of Two Cubes, Amer.
635 318 657 = 158^4 + 59^4 = 134^4 + 133^4.

The smallest integer represented by $x^4 + y^4$ in two essentially different ways was found by Euler, it is

$635 318 657 = 41 \times 113 \times 241 \times 569.$

[OEIS A216284] Number of solutions to the equation

$x^4 + y^4 = n$ with $x \geq y > 0.$

An infinite family with one parameter is known for non trivial solutions to $x_1^4 + x_2^4 = x_3^4 + x_4^4.$


Sums of $k$--th powers

One conjectures that given $k \geq 5$, if an integer can be written as $x^k + y^k$, there is essentially a unique such representation. But there is no value of $k$ for which this has been proved.

Binary cyclotomic forms of higher degree

The situation for binary cyclotomic forms is different when the degree is 2 or when it is $> 2$ also for the following reason.

A necessary and sufficient condition for a number $m$ to be represented by one of the quadratic forms $\Phi_3$, $\Phi_4$, is given by a congruence.

By contrast, consider the quartic binary form

$\Phi_8(X,Y) = X^4 + Y^4.$ On the one hand, an odd integer represented by $\Phi_8$ is of the form $N_{1,8}(N_{3,8}N_{5,8}N_{7,8})^4.$

On the other hand, there are many integers of this form which are not represented by $\Phi_8.$

[OEIS A004831] Numbers that are the sum of at most 2 nonzero 4th powers.

0, 1, 2, 16, 17, 32, 81, 82, 97, 162, 256, 257, 272, 337, 512, 625, ...

Quartan primes

[OEIS A002645] Quartan primes: primes of the form

$x^4 + y^4$, $x > 0$, $y > 0.$

The list of prime numbers represented by $\Phi_8$ start with

2, 17, 97, 257, 337, 641, 881, 1297, 2417, 2657, 3697, 4177, 4721, 6577, 10657, 12401, 14657, 14897, 15937, 16561, 28817, 38561, 39041, 49297, 54721, 65537, 65617, 66161, 66977, 80177, 83537, 83777, 89041, 105601, 107377, 119617, ...

It is not known whether this list is finite or not.

The largest known quartan prime is currently the largest known generalized Fermat prime: The 1353 265--digit $(145 310^{65 536})^4 + 1^4.$
Primes of the form \( x^{2^k} + y^{2^k} \)

- [OEIS A002313] primes of the form \( x^2 + y^2 \),
- [OEIS A002645] primes of the form \( x^4 + y^4 \),
- [OEIS A006686] primes of the form \( x^8 + y^8 \),
- [OEIS A100266] primes of the form \( x^{16} + y^{16} \),
- [OEIS A100267] primes of the form \( x^{32} + y^{32} \).

Primes of the form \( X^2 + Y^4 \)

However, it is known that there are infinitely many prime numbers of the form \( X^2 + Y^4 \).

K. Mahler (1933)

Let \( F \) be a binary form of degree \( d \geq 3 \) with nonzero discriminant.
Denote by \( A_F \) the area (Lebesgue measure) of the domain
\[
\{(x, y) \in \mathbb{R}^2 \mid F(x, y) \leq 1\}.
\]

For \( Z > 0 \) denote by \( N_F(Z) \) the number of \( (x, y) \in \mathbb{Z}^2 \) such that \( 0 < |F(x, y)| \leq Z \).
Then
\[
N_F(Z) = A_F Z^{\frac{d}{2}} + O(Z^{\frac{d}{2} - 1})
\]
as \( Z \to \infty \).

Kurt Mahler

"Über die mittlere Anzahl der Darstellungen grosser Zahlen durch binäre Formen,
Acta Math. 62 (1933), 91-166.
https://carma.newcastle.edu.au/mahler/biography.html"
Let $F$ be a binary form of degree $d \geq 3$ with nonzero discriminant. There exists a positive constant $C_F > 0$ such that the number of integers of absolute value at most $N$ which are represented by $F(X, Y)$ is asymptotic to $C_F N^{\frac{d}{2}} + O(N^{\beta_d})$ with $\beta_d < \frac{2}{3}$.

Cyclotomic binary forms of degree 4

(Joint work with Étienne Fouvry - in progress).

$\Phi_5(X, Y) = X^4 + X^3Y + X^2Y^2 + XY^3 + Y^4$.

$\Phi_8(X, Y) = X^4 + Y^4$.

$\Phi_{12}(X, Y) = X^4 - X^2Y^2 + Y^4$.

Also

$\Phi_{10}(X, Y) = \Phi_5(X, -Y) = X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4$.

For $n \in \{5, 8, 12\}$, the number of positive integers $m \leq N$ which can be written as $m = \Phi_n(x, y)$ is asymptotic to $C_{\Phi_n} N^{\frac{1}{2}}$.

Numbers represented by two cyclotomic binary forms of degree 4

The number of integers $\leq N$ which are represented by two of the three quartic cyclotomic binary forms $\Phi_5$, $\Phi_8$ and $\Phi_{12}$ is bounded by $O(N^{\frac{5}{2}+\varepsilon})$.

Consequence: the number of integers $\leq N$ which are represented by a cyclotomic binary form of degree 4 is asymptotic to

$$C_4N^{\frac{1}{2}} + O(N^{\frac{5}{2}+\varepsilon}),$$

where

$$C_4 = C_{\Phi_5} + C_{\Phi_8} + C_{\Phi_{12}}.$$
Numbers represented by a cyclotomic binary form of degree $\geq d$

Any prime number $p$ is represented by a cyclotomic binary form: $\Phi_p(1, 1) = p$.

Given an integer $d \geq 2$, we consider the set of positive integers $m$ which can be written as $m = \Phi_n(x, y)$ with $n \geq d$ and $(x, y) \in \mathbb{Z}^2$ satisfying $\max(|x|, |y|) \geq 2$.

Isomorphic cyclotomic binary forms

Recall that the cyclotomic polynomials $\phi_n(t) \in \mathbb{Z}[t]$ satisfy $\phi_{2n}(t) = \phi_n(-t)$ for odd $n \geq 3$.

For $n_1$ and $n_2$ positive integers with $n_1 < n_2$, the following conditions are equivalent:

1. $\varphi(n_1) = \varphi(n_2)$ and the two binary forms $\Phi_{n_1}$ and $\Phi_{n_2}$ are isomorphic.
2. The two binary forms $\Phi_{n_1}$ and $\Phi_{n_2}$ represent the same integers.
3. $n_1$ is odd and $n_2 = 2n_1$.

The list of even integers which are not values of Euler $\varphi$ function (i.e., for which $C_d = 0$) starts with

14, 26, 34, 38, 50, 62, 68, 74, 76, 86, 90, 94, 98, 114, 118, 122, 124, 134, 142, 146, 152, 154, 158, 170, 174, 182, 186, 188, 194, 202, 206, 214, 218, 230, 234, 236, 242, 244, 246, 248, 254, 258, 266, 274, 278, 284, 286, 290, 298, 302, 304, 308, 314, 318, ...  

[OEIS A005277] Nontotients: even $n$ such that $\varphi(m) = n$ has no solution.
Numbers represented by two cyclotomic binary forms of the same degree

Given two binary cyclotomic forms of the same degree and not isomorphic, and given \( \epsilon > 0 \), for \( N \to \infty \) the number of positive integers \( \leq N \) which are represented by these two forms is bounded by

\[
\begin{cases}
O_\epsilon(N^{\frac{3}{d} + \epsilon}) & \text{for } d = 4, 6, 8, \\
O_{d,\epsilon}(N^{\frac{1}{d} + \epsilon}) & \text{for } d \geq 10.
\end{cases}
\]

Further developments (work in progress)

Representation of integers by other binary forms

- Representation of integers by the binary forms \( X^n + Y^n \), \( X^n - Y^n \) and \( F_n(X, Y) \), where
  \[
  F_n(X, Y) = X^n + X^{n-1}Y + \cdots + XY^{n-1} + Y^n.
  \]

A weak but uniform bound

For \( d \geq 2 \) and \( N \to \infty \), the number of \( m \leq N \) for which there exists \( n \geq d \) and \((x, y) \in \mathbb{Z}^2\) with \( \max(|x|, |y|) \geq 2 \) and \( m = \Phi_n(x, y) \) is bounded by

\[
29N^{\frac{2}{d}}(\log N)^{1.161}.
\]

Suggestion of Florian Luca (RNTA 2018)

Study the representation of integers by the polynomials

**Dickson polynomials of the first and second kind**

- The sequence of Dickson polynomials of the first kind \((D_n)_{n \geq 0}\) (resp. second kind \((E_n)_{n \geq 0}\)) is defined by
  \[
  D_n(X + Y, XY) = X^n + Y^n
  \]
  (resp.
  \[
  E_n(X + Y, XY) = F_n(X, Y).
  \]

Dickson polynomials: representation of integers by \( X^n + Y^n \) and \( X^n - Y^n \) when \( x + y \) and \( xy \) are integers (\( x \) and \( y \) are quadratic integers).
Cyclotomic Dickson polynomials

- For $n \geq 2$, define
  \[ \Psi_n(X + Y, XY) = \Phi_n(X, Y). \]

Study the representation of integers by the polynomials $\Psi_n$.

Representation of integers by $\Phi_n(X, Y)$ where $x + y$ and $xy$ are integers.

Dickson polynomials are not homogeneous.

Work in progress...