The Legacy of Srinivasa Ramanujan – Dec. 2012

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From highly composite numbers
to transcendental number theory
by

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Abstract

In a well–known paper published in 1915 in the Proceedings of the London Mathematical Society, Srinivasa Ramanujan defined and studied *highly composite numbers.* A *highly composite number* is a positive integer n with more divisors than any positive integer smaller than n.

This work was pursued in 1944 by L. Alaoglu and P. Erdős, who raised a question which belongs to transcendental number theory.

A simple instance is the following open question: *does there exist a real irrational number t such that 2^t and 3^t are integers?*

We give a short survey of this topic where we point out links with a number of other subjects.

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1 Highly composite and similar numbers

In 1915 [22] (see also [23] and [2]), S. Ramanujan defined a *highly composite number* as a number n such that, for $m < n$, one has $d(m) < d(n)$, where $d(n)$ is the number of divisors of n:

$$d(n) = \sum_{d|n} 1.$$

The sequence of highly composite numbers (reference `A002182` in Sloane's Encyclopaedia of Integer Sequences [32]) starts with

1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, ...

In 1944, L. Alaoglu and P. Erdős [1] defined *highly abundant numbers, super abundant numbers* and *colossally abundant numbers.* A *colossally abundant* *number* is a positive integer n for which there exists $\varepsilon > 0$ such that, for all $k > 1$,

$$\frac{\sigma(n)}{n^{1+\varepsilon}} \geq \frac{\sigma(k)}{k^{1+\varepsilon}}\cdot$$

Here, σ is the function sum of divisors:

$$\sigma(n) = \sum_{d|n} d.$$

The sequence of colossally abundant numbers (reference `A004490` in [32]) starts with

$$2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320, 21621600, \ldots$$

The successive quotients are: $3, 2, 5, 2, 3, 7, 2, 11, 13, 2, 3, 5, \ldots$

In their paper [1], Alaoglu and Erdős write:

> ... this makes q^x rational. It is very likely that q^x and p^x can not be rational at the same time except if x is an integer. This would show that the quotient of two consecutive colossally abundant numbers is a prime. At present we can not show this. Professor Siegel has communicated to us the result that q^x, r^x and s^x cannot be simultaneously rational except if x is an integer. Hence *the quotient of two consecutive colossally abundant numbers is either a prime or the product of two distinct primes.*

This is the origin of the problem that we now consider.

2 Four exponentials Conjecture and six exponentials Theorem

If p and q are distinct primes and if x is a real number such that $p^x = r$ and $q^x = s$ are integers, then we have

$$x = \frac{\log r}{\log p} = \frac{\log s}{\log q}$$

and the matrix

$$\begin{pmatrix} \log p & \log q \\ \log r & \log s \end{pmatrix}$$

has rank 1. One does not know an example where this happens without x being an integer. More generally, if we replace the assumption that p and q are distinct primes by the assumption that they are two multiplicatively independent positive numbers, then the expected conclusion is that x should be rational. Recall that two positive numbers p and q are multiplicatively independent if the relation $p^a q^b = 1$ with a and b rational integers implies $a = b = 0$. Hence the condition of multiplicative independence is equivalent to the condition that the quotient $\log p / \log q$ be irrational.

There are several simple proofs of the weaker statement that if x is a positive real number such that n^x is an integer for all positive integers n, then x is a nonnegative integer [5, 40].

The four exponentials Conjecture was formulated by Th. Schneider [30], S. Lang [7] and K. Ramachandra [18].

Four exponentials Conjecture (matrix form). *Let*

$$M = \begin{pmatrix} \log\alpha_1 & \log\alpha_2 \\ \\ \log\beta_1 & \log\beta_2 \end{pmatrix}$$

be a 2×2 *matrix , the entries of which are logarithms of algebraic numbers. Assume that the two columns of* M *are* $\mathbf{Q}$*–linearly independent and that the two rows of* M *are also* $\mathbf{Q}$*–linearly independent. Then* M *has rank* 2*.*

Four exponentials Conjecture (exponential form). *Let* x_1*,* x_2 *be* $\mathbf{Q}$*–linearly independent complex numbers and* y_1*,* y_2 *be also* $\mathbf{Q}$*–linearly independent complex numbers. Then at least one of the four numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2}$$

is transcendental.

The equivalence between the two statements follows by writing

$$\alpha_j = e^{x_1y_j}, \quad \beta_j = e^{x_2y_j}.$$

The next result was apparently known to C.L. Siegel (at least), its proof was published first by S. Lang [7] and then by K. Ramachandra [18].

Six exponentials Theorem (exponential form). *Let* x_1, x_2 *be two* $\mathbf{Q}$*–linearly independent complex numbers and* y_1, y_2, y_3 *be also* $\mathbf{Q}$*–linearly independent complex numbers. Then at least one of the* 6 *numbers*

$$e^{x_iy_j}, \quad (i=1,2,\ j=1,2,3)$$

is transcendental.

An equivalent form of this statement is the following one:

Six exponentials Theorem (matrix form). *Let*

$$M = \begin{pmatrix} \log\alpha_1 & \log\alpha_2 & \log\alpha_3 \\ \\ \log\beta_1 & \log\beta_2 & \log\beta_3 \end{pmatrix}$$

be a 2×3 *matrix whose entries are logarithms of algebraic numbers. Assume that the three columns are linearly independent over* $\mathbf{Q}$ *and that the two rows are also linearly independent over* $\mathbf{Q}$*. Then the matrix* M *has rank* 2*.*

Proofs of the six exponential Theorem are given in many places including [7, 18, 19, 35, 40, 42]. A consequence of the six exponentials Theorem is that, if t is an irrational number, at least one of the three numbers 2^t, 2^{t^2}, 2^{t^3} is transcendental.

In case t is algebraic, these three numbers are transcendental by Gel'fond–Schneider's Theorem.

If t is a transcendental number and a, b, c are positive integers with $b \neq c$, then of at least one of the numbers

$$2^{t^a}, \quad 2^{t^b}, \quad 2^{t^c}, \quad 2^{t^{a+b}}, \quad 2^{t^{a+c}}$$

is transcendental. For instance if a and b are positive integers, at least one of the numbers

$$2^{t^a}, \quad 2^{t^b}, \quad 2^{t^{a+b}}, \quad 2^{t^{2a+b}}.$$

is transcendental.

A consequence of Schanuel's Conjecture would be that all numbers 2^{π^n} ($n \geq 1$) are transcendental. A special case of the four exponentials Conjecture is that at least one of the two numbers 2^π, 2^{π^2} is transcendental. According to the six exponentials Theorem, at least one of the three numbers 2^π, 2^{π^2}, 2^{π^3} is transcendental.

Algebraic approximations to 2^{π^k} have been investigated by T.N. Shorey [31] and S. Srinivasan [33, 34]. See also [13].

Upper bounds for the number of algebraic numbers among 2^{π^k}, $(1 \leq k \leq N)$ have been obtained by S. Srinivasan [34]. See also [20, 21].

3 Five exponentials Theorem, strong four exponentials Conjecture and strong six exponentials Theorem

The five exponentials Theorem was proved in 1986 [39] Cor. 2.2.

Five exponentials Theorem (Exponential form). *Let x_1, x_2 be two $\mathbf{Q}$–linearly independent complex numbers and y_1, y_2 be also two $\mathbf{Q}$–linearly independent complex numbers. Then at least one of the 5 numbers*

$$e^{x_1y_1}, \; e^{x_1y_2}, \; e^{x_2y_1}, \; e^{x_2y_2}, \; e^{x_2/x_1}$$

is transcendental.

The next result is stronger:

Five exponentials Theorem (Matrix form). *Let M be a 2×3 matrix whose entries are either algebraic numbers or logarithms of algebraic numbers. Assume that the three columns are linearly independent over $\overline{\mathbf{Q}}$ and that the two rows are also linearly independent over $\overline{\mathbf{Q}}$. Then M has rank 2.*

We deduce the exponential form from the matrix form by considering the matrix

$$\begin{pmatrix} \log\alpha_{11} & \log\alpha_{12} & 1 \\ \log\alpha_{21} & \log\alpha_{22} & \log\gamma \end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & 1 \\ x_2y_1 & x_2y_2 & x_2/x_1 \end{pmatrix}.$$

Denote by $\mathcal{L}$ the $\mathbf{Q}$–vector subspace of $\mathbf{C}$ of logarithms of algebraic numbers: it consists of the complex numbers λ for which e^λ is algebraic (say $\lambda = \log \alpha$). The $\mathbf{Q}$–vector space spanned by 1 and $\mathcal{L}$ is $\mathbf{Q} \cup \mathcal{L}$. Further, denote by $\widetilde{\mathcal{L}}$ the $\overline{\mathbf{Q}}$–vector space spanned by 1 and $\mathcal{L}$: hence $\widetilde{\mathcal{L}}$ is the set of linear combinations with algebraic coefficients of logarithms of algebraic numbers:

$$\widetilde{\mathcal{L}} = \{\Lambda = \beta_0 + \beta_1\lambda_1 + \cdots + \beta_n\lambda_n \; ; \; n \geq 0, \beta_i \in \overline{\mathbf{Q}}, \; \lambda_i \in \mathcal{L}\}.$$

Notice that $\widetilde{\mathcal{L}} \supset \overline{\mathbf{Q}} \cup \mathcal{L}$.

The strong six exponentials Theorem was proved by D. Roy in 1992 [25], where he also proposed the strong four exponentials Conjecture.

Strong Six Exponentials Theorem (Exponential form). *If x_1, x_2 are $\overline{\mathbf{Q}}$–linearly independent complex numbers and y_1, y_2, y_3 are $\overline{\mathbf{Q}}$–linearly independent complex numbers, then at least one of the six numbers*

$$x_1y_1, \; x_1y_2, \; x_1y_3, \;\; x_2y_1, \; x_2y_2, \; x_2y_3$$

is not in $\widetilde{\mathcal{L}}$.

An equivalent statement is the next one:

Strong Six Exponentials Theorem (matrix form). *Let*

$$M = \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \\ \Lambda_1 & \Lambda_2 & \Lambda_3 \end{pmatrix}$$

be a 2×3 matrix whose entries are in $\widetilde{\mathcal{L}}$ Assume that the three columns are linearly independent over $\overline{\mathbf{Q}}$ and that the two rows are also linearly independent over $\overline{\mathbf{Q}}$. Then M has rank 2.

Clearly, the strong six exponentials Theorem implies the six exponentials Theorem and the five exponentials Theorem, while the four exponentials Conjecture is a special case of the following strong four exponentials Conjecture.

Strong Four Exponentials Conjecture (exponential form). *If x_1, x_2 are $\overline{\mathbf{Q}}$–linearly independent complex numbers and y_1, y_2 are $\overline{\mathbf{Q}}$–linearly independent complex numbers, then at least one of the four numbers*

$$x_1y_1, \; x_1y_2, \;\; x_2y_1, \; x_2y_2$$

is not in $\widetilde{\mathcal{L}}$.

Again, an equivalent statement is the next one:

Strong Four Exponentials Conjecture (matrix form). *Let*

$$M = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \\ \Lambda_1 & \Lambda_2 \end{pmatrix}$$

be a 2×2 matrix whose entries are in $\widetilde{\mathcal{L}}$ Assume that the two columns are linearly independent over $\overline{\mathbf{Q}}$ and that the two rows are also linearly independent over $\overline{\mathbf{Q}}$. Then M has rank 2.

4 Lower bound for the rank of matrices with entries logarithms of algebraic numbers

The conclusions of the previous results are that some matrices have rank > 1. The six exponentials Theorem has been generalized in 1980 [36] in order to produce lower bounds for the rank of matrices with entries in $\mathcal{L}$ of any size. Under suitable assumptions, the rank r of such a $d \times \ell$ matrix satisfies

$$r \geq \frac{d\ell}{d+\ell}.$$

Hence, when $d = \ell$,

$$r \geq \frac{d}{2},$$

which is half of what is expected. One cannot expect to reach the maximal rank if one only assumes that the columns and the rows are linearly independent: the matrix

$$\begin{pmatrix} 0 & \log 2 & -\log 3 \\ -\log 2 & 0 & \log 5 \\ \log 3 & -\log 5 & 0 \end{pmatrix}$$

has rank 2 only.

The main result of [36] (see also Theorem 12.17 of [42]) states that a $d \times \ell$ matrix of rank n with entries in $\mathcal{L}$ is $\mathbf{Q}$–equivalent to a bloc matrix

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

where C is a $d' \times \ell'$ matrix with $d' > 0$ and

$$\frac{n}{d} \geq \frac{\ell'}{d'+\ell'}.$$

A consequence of this result is the answer to a question from A. Weil [37]: if the values of a Hecke Grössencharacter are algebraic (resp. in a number field), then the character is of type A (resp. A_0). Another consequence is the answer by D. Roy in [26] to a question raised by J.-L. Colliot-Thélène, D. Coray and J.-J. Sansuc: given a number field k with a group of units of rank r, the smallest positive integer m for which there exists a finitely generated subgroup of rank m of $k^\times$ having a dense image in $(\mathbf{R} \otimes_{\mathbf{Q}} k)^\times$ under the canonical embedding is $r + 2$.

There is a version for nonarchimedean valuations, which implies the lower bound $r_p \geq r/2$ for the p–adic rank r_p of the units of an algebraic number field, in terms of the rank r of the group of units [36, 38], while Leopoldt's Conjecture predicts $r_p = r$. See also the results which are proved by M. Laurent [8] and those which are claimed in [14, 15] by P. Mihăilescu.

Also, the ultrametric transcendence result has applications to ℓ–adic representations [6]. Let K be a number field, and $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}/K)$ the Galois group of $\overline{\mathbf{Q}}$ over K. Let E be a number field, λ a finite place of E and ϱ an E_λ–adic representation of G_K, and assume that ϱ is abelian, semisimple and rational over E; then ϱ is locally algebraic. Further, if ϱ is a semisimple ℓ–adic representation of G_K that is unramified except for a finite number of places of K and ramified over $\mathbf{Q}$, then the Lie algebra of the image of ϱ is algebraic.

In [27], D. Roy introduced the definition of *structural rank* of a matrix with entries in $\mathcal{L}$. For

$$M = \bigl(\lambda_{ij}\bigr)_{\substack{1\le i\le d\\ 1\le j\le \ell}}$$

with $\lambda_{ij} \in \mathcal{L}$, select a basis $\{\mu_1, \ldots, \mu_s\}$ of the $\mathbf{Q}$–vector subspace of $\mathcal{L}$ spanned by the $d\ell$ numbers λ_{ij}, write

$$\lambda_{ij} = \sum_{\sigma=1}^{s} a_{ij\sigma}\mu_\sigma$$

with $a_{ij\sigma} \in \mathbf{Q}$ and denote by $r_{\rm srt}(M)$ the rank of the matrix

$$\left(\sum_{\sigma=1}^{s} a_{ij\sigma}X_\sigma\right)_{\substack{1\le i\le d\\ 1\le j\le \ell}} \in \mathrm{Mat}_{d\times\ell}\bigl(\mathbf{Q}(X_1,\ldots,X_s)\bigr).$$

This number $r_{\rm srt}(M)$ does not depend on the basis $\{\mu_1, \ldots, \mu_s\}$ and is called the structural rank of M. It is obvious that the rank $r(M)$ of the matrix M is bounded above by the structural rank. One deduces from Schanuel's Conjecture that for a matrix M with entries in $\mathcal{L}$, the equality $r_{\rm srt}(M) = r(M)$ always holds. The best lower bound so far for $r(M)$ is ([42], Cor. 12.18)

$$r(M) \ge \frac{1}{2} r_{\rm srt}(M).$$

5 Lower bound for the rank of matrices with entries in $\widetilde{\mathcal{L}}$

In 1992, D. Roy extended his strong six exponentials Theorem dealing with 2×3 matrices to matrices of any size with entries $\widetilde{\mathcal{L}}$ (linear combinations of 1 and logarithms of algebraic numbers). He defines the *structural rank* $r_{\rm srt}(M)$ of a matrix

$$M = \bigl(\Lambda_{ij}\bigr)_{\substack{1\le i\le d\\ 1\le j\le \ell}}$$

with entries Λ_{ij} in $\widetilde{\mathcal{L}}$ as the rank of the matrix

$$\left(\sum_{\sigma=1}^{s} \alpha_{ij\sigma}X_\sigma\right)_{\substack{1\le i\le d\\ 1\le j\le \ell}} \in \mathrm{Mat}_{d\times\ell}\bigl(\overline{\mathbf{Q}}(X_1,\ldots,X_s)\bigr),$$

where $\{M_1, \ldots, M_s\}$ is a basis of the $\overline{\mathbf{Q}}$–vector subspace of $\widetilde{\mathcal{L}}$ spanned by the $d\ell$ numbers Λ_{ij} and where $\alpha_{ij\sigma}$ are defined by

$$\Lambda_{ij} = \sum_{\sigma=1}^{s} \alpha_{ij\sigma}M_\sigma.$$

Again, this number $r_{\mathrm{srt}}(M)$ does not depend on the choice of the basis $\{M_1, \ldots, M_s\}$. Further, for a matrix M with entries in $\mathcal{L}$, the two definitions of $r_{\mathrm{srt}}(M)$ coming from the inclusion $\mathcal{L} \subset \widetilde{\mathcal{L}}$, coincide. In [27], D. Roy proves that the rank $r(M)$ of M satisfies:

$$r(M) \geq \frac{1}{2} r_{\mathrm{srt}}(M).$$

6 Schanuel's Conjecture

Schanuel's Conjecture [7] state that *Let $x_1, \ldots, x_n$ be $\mathbf{Q}$–linearly independent complex numbers. Then at least n of the $2n$ numbers $x_1, \ldots, x_n, \; e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.*

One of the most important and open special cases of Schanuel's conjecture is the conjecture on algebraic independence of logarithms of algebraic numbers: *if $\lambda_1, \ldots, \lambda_n$ are linearly independent logarithms of algebraic numbers, then these numbers are algebraically in dependent.*

So far, it is not even known if *there exist two logarithms of algebraic numbers which are algebraically independent.* Baker's result provides a satisfactory answer for the *linear independence of such numbers over the field of algebraic numbers.* But he says nothing about algebraic independence. Even the non–existence of non–trivial quadratic relations among logarithms of algebraic numbers is not yet established. According to the *four exponentials Conjecture, any quadratic relation* $(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$ *is trivial : either* $\log \alpha_1$ *and* $\log \alpha_2$ *are linearly dependent, or else* $\log \alpha_1$ *and* $\log \alpha_3$ *are linearly dependent.*

However something is known on the conjecture of algebraic independence of logarithms of algebraic numbers. Instead of taking linearly independent logarithms of algebraic numbers $\lambda_1, \ldots, \lambda_n$ and asking about the non–vanishing of values $P(\lambda_1, \ldots, \lambda_n)$ of polynomials $P \in \mathbf{Z}[X_1, \ldots, X_n]$, D. Roy looks at this question from another point of view: starting with a nonzero polynomial $P \in \mathbf{Z}[X_1, \ldots, X_n]$, he investigates the tuples $(\lambda_1, \ldots, \lambda_n) \in \mathcal{L}^n$ whose components are logarithms of algebraic numbers such that $P(\lambda_1, \ldots, \lambda_n) = 0$. More generally, he remarked that the conjecture of algebraic independence of logarithms of algebraic numbers is equivalent to the next statement: *if V is an affine algebraic subvariety of $\mathbf{C}^n$, then the set $\mathcal{L}^n \cap V$ is contained in the union of linear subspaces of $\mathbf{C}^n$ rational over V contained in V.* In [27], he proves special cases of this statement. See also the paper [4] by S. Fischler.

The conjecture on algebraic independence of logarithms of algebraic numbers would solve the question of the rank of matrices having entries in the space $\mathbf{Q} \cup \mathcal{L}$ spanned by 1 and the logarithms of algebraic numbers. Conversely, it has been proved by D. Roy that *the conjecture on algebraic independence of logarithms is equivalent to the conjecture that he rank of a matrix with entries in $\mathbf{Q} \cup \mathcal{L}$ is equal to its structural rank.* The key lemma ([24, 27] – see also [42] § 12.1.5) is that if k is a field and $P \in k[X_1, \ldots, X_n]$ a polynomial in n variables, then there exists a square matrix M, whose entries are linear combinations of $1, X_1, \ldots, X_n$ with coefficients in k, such that P is the determinant of M.

A promising strategy for proving Schanuel's Conjecture has been devised by D. Roy in [28] (see also § 15.5.3 of [42]). He proposes a new conjecture which he shows to be equivalent to Schanuel's Conjecture. In a series of recent papers, he proved special cases of his conjecture – see for instance [29].

7 Ellliptic four exponentials Conjecture

The question of algebraic independence of logarithms of algebraic numbers can be generalized by replacing the multiplicative group by other algebraic group, like an elliptic curve (early examples are in [18] which is expanded in [43]) or a commutative group variety. An example of such a generalization occurs in connection with a problem of K. Mahler et Yu.V. Manin on the transcendence of $J(q)$ for algebraic $q = e^{2i\pi\tau}$. This problem has been solved by K. Barré–Siriex, G. Diaz, F. Gramain and G. Philibert in 1996 [3]; see also [44].

Mixed four exponentials Theorem. *Let $\log\alpha$ be a logarithm of a non–zero algebraic number. Let $\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ be a lattice associated with a Weierstrass elliptic curve having algebraic invariants g_2, g_3. Then the matrix*

$$\begin{pmatrix} \omega_1 & \log\alpha \\ & \\ \omega_2 & 2\pi i \end{pmatrix}$$

has rank 2.

Here is a stronger statement. Let $\wp$ be a Weierstraß elliptic function with algebraic invariants g_2, g_3 and E be the corresponding elliptic curve. Denote by $\mathcal{L}_E$ the set of $u \in \mathbf{C}$ which either are poles of $\wp$ or are such that $\wp(u)$ is algebraic.

Mixed four exponentials Conjecture. *Let u_1 and u_2 be two elements in $\mathcal{L}_E$ and $\log\alpha_1$, $\log\alpha_2$ be two logarithms of algebraic numbers. Assume further that the two rows of the matrix*

$$M = \begin{pmatrix} u_1 & \log\alpha_1 \\ u_2 & \log\alpha_2 \end{pmatrix}$$

are linearly independent over $\mathbf{Q}$. Then the matrix M has rank 2.

8 Density questions

As shown in [41], these questions are related with a problem of B. Mazur [9, 10, 11, 12] on the density of rational points on varieties. See also [16, 17]. Just to give an example, a positive answer to the next question would follow from the four exponentials Conjecture (see [43]).

For $\alpha = a + b\sqrt{2} \in \mathbf{Q}(\sqrt{2})$, write $\overline{\alpha} = a - b\sqrt{2}$. Define

$$\alpha_1 := 2\sqrt{2} - 1, \quad \alpha_2 := 3\sqrt{2} - 1, \quad \alpha_3 := 4\sqrt{2} - 1,$$

and let Γ be the set of elements in $(\mathbf{R}^\times)^2$ of the form

$$\left(\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3},\ \overline{\alpha}_1^{a_1}\overline{\alpha}_2^{a_2}\overline{\alpha}_3^{a_3}\right)$$

with $(a_1, a_2, a_3) \in \mathbf{Z}^3$.

Question: *Is this subgroup* Γ *dense in* $(\mathbf{R}^\times)^2$*?*

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