February 18, 2025

Modular forms and multiple zeta values
February 17 - 22 (2025), Kindai University, Osaka.

Linear independence of odd zeta values using Siegel's lemma, following Stéphane Fischler

Michel Waldschmidt

Professeur Émérite, Sorbonne Université,
Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

Hermite's proof of the transcendence of e in 1872 and Lindemann's proof of the transcendence of π used explicit auxiliary functions produced by some of Hermite's interpolation integrals. One main ingredient in the solution by Gel'fond and Schneider of Hilbert's 7th problem in 1932 was the use of the Thue–Siegel Lemma, which proves the existence of a suitable auxiliary function without giving an explicit formula. The proof by Apéry in 1976 of the irrationality of $\zeta(3)$, and the proofs by Rivoal, Ball, Zudilin and others on lower bounds for the dimension of the $\mathbb{Q}$–space spanned by odd zeta values rest on explicit constructions. Recently, Fischler showed how to deal with a variant of the Thue–Siegel Lemma, and how this improves some of the earlier results.

Special values of the Riemann zeta function

Leonhard Euler

(1707 – 1783)

Introductio in analysin infinitorum (1748)

Using divergent series, Euler found the values of the Riemann zeta function at negative integers and positive even integers:

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0$$

$$\zeta(-1) = -\frac{1}{12}, \quad \zeta(-3) = \frac{1}{120}, \quad \zeta(-5) = -\frac{1}{252}, \cdots$$

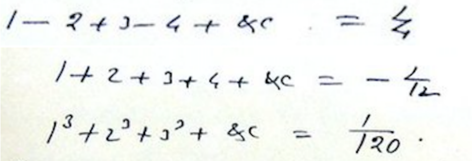
Renormalization of divergent series

$$\zeta(-2n) = 1^{2n} + 2^{2n} + 3^{2n} + 4^{2n} + \cdots = 0 \qquad (n \geqslant 1)$$

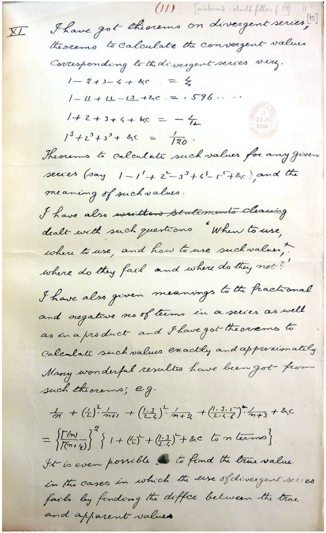
Some of these values were rediscovered by Ramanujan.

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

$$\zeta(-3) = 1^3 + 2^3 + 3^3 + 4^3 + \cdots = \frac{1}{120}$$



Srinivasan Ramanujan


(11) [misbound: should follow f 19] 11 [15]

XI I have got theorems on divergent series, theorems to calculate the convergent values corresponding to the divergent series viz.

$1 - 2 + 3 - 4 + \&c = \frac{1}{4}$

$1 - 1! + 2! - 3! + \&c = .596\ldots$

$1 + 2 + 3 + 4 + \&c = -\frac{1}{12}$

$1^3 + 2^3 + 3^3 + \&c = \frac{1}{120}$.

Theorems to calculate such values for any given series (say $1 - 1^1 + 2^2 - 3^3 + 4^4 - 5^5 + \&c$), and the meaning of such values.

I have also ~~written statements clearing~~ dealt with such questions 'When to use, where to use, and how to use such values, where do they fail and where do they not?'

I have also given meanings to the fractional and negative no of terms in a series as well as in a product and I have got theorems to calculate such values exactly and approximately. Many wonderful results have been got from such theorems; e.g.

$$\frac{1}{n} + \left(\frac{1}{2}\right)^2 \frac{1}{n+1} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{1}{n+2} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{1}{n+3} + \&c$$

$$= \left\{\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}\right\}^2 \left\{1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \&c \text{ to } n \text{ terms}\right\}$$

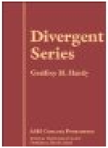
It is even possible to find the true value in the cases in which the use of divergent series fails by finding the diffce between the true and apparent values.

Srinivasan Ramanujan

(1887 - 1920)

Srinivasan Ramanujan's first famous letter in original format (manuscript except for the 1st page) to Prof G H Hardy dated 16 January 1913.

G.H. Hardy: Divergent Series (1949)



Niels Henrik Abel

(1802 – 1829)

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.

Bernoulli numbers

Values of Riemann zeta function at negative integers:

Jacob Bernoulli

(1655 - 1705)

$$\sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}.$$

For $m \geqslant 0$,

$$\zeta(-m) = -\frac{B_{m+1}}{m+1}.$$

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_3 = B_5 = B_7 = \cdots = 0,$$

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \ldots$$

Values at positive even integers

After Euler:

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \ldots$$

$$\zeta(2r) = \frac{(-1)^{r-1} B_{2r} 2^{2r-1}}{(2r)!} \pi^{2r}.$$

Hence

$$\zeta(1-2r) = (-1)^r 2^{1-2r} \pi^{-2r} (2r-1)! \zeta(2r).$$

Special values of the Riemann zeta function

Leonhard Euler

(1707 - 1783)

Ferdinand von Lindemann

(1852 - 1939)

$\zeta(2n)$ is transcendental for $n \geqslant 1$

Conjecture: *The numbers* π, $\zeta(3), \zeta(5), \ldots, \zeta(2n+1)$ *are algebraically independent.*

Credit photo https://mathshistory.st-andrews.ac.uk/Biographies/

Linearization of the problem (Euler)

Introduction of multiple zeta values.

$$\sum_{n_1\geqslant 1}\frac{1}{n_1^{s_1}}\sum_{n_2\geqslant 1}\frac{1}{n_2^{s_2}}=\sum_{n\geqslant 1}\frac{1}{n^{s_1+s_2}}+\sum_{n_1>n_2\geqslant 1}\frac{1}{n_1^{s_1}n_2^{s_2}}+\sum_{n_2>n_1\geqslant 1}\frac{1}{n_1^{s_1}n_2^{s_2}},$$

$$\zeta(s_1)\zeta(s_2)=\zeta(s_1+s_2)+\zeta(s_1,s_2)+\zeta(s_2,s_1).$$

For k, $s_1,\dots,s_k$ positive integers with $s_1\geqslant 2$, we set $\underline{s}=(s_1,\dots,s_k)$ and

$$\zeta(\underline{s})=\sum_{n_1>n_2>\cdots>n_k\geqslant 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}.$$

The product of two zeta values is a linear combination of multiple zeta values; reduces the problem of algebraic independence to a question of linear independence.

The MZV algebra

The $\mathbb{Q}$–vector space $\mathfrak{Z}$ spanned by the numbers $\zeta(\underline{s})$ is also a $\mathbb{Q}$–algebra.

k is the *depth* while $n = s_1 + \cdots + s_k$ is the *weight* of $\underline{s} = (s_1, \ldots, s_k)$.

First Conjecture: *There is no linear relation among multiple zeta values of different weights.*

For $n \geqslant 2$, denote by $\mathfrak{Z}_n$ the $\mathbb{Q}$-subspace of $\mathfrak{Z}$ spanned by the real numbers $\zeta(\underline{s})$ where $\underline{s}$ has weight $s_1 + \cdots + s_k = n$.

Define also $\mathfrak{Z}_0 = \mathbb{Q}$ and $\mathfrak{Z}_1 = \{0\}$.

The *First Conjecture* is

$$\mathfrak{Z} = \bigoplus_{n \geqslant 0} \mathfrak{Z}_n.$$

Second Conjecture

Don Zagier

Denote by d_n the dimension of $\mathfrak{z}_n$.

Conjecture (Zagier). *For* $n \geqslant 3$*, we have*

$$d_n = d_{n-2} + d_{n-3}.$$

$$(d_0,\ d_1,\ d_2, \ldots) = (1, 0, 1, 1, 1, 2, 2, \ldots).$$

$$\sum_{n \geqslant 0} d_n X^n = \frac{1}{1 - X^2 - X^3}.$$

$$d_n = \sharp\{\underline{s} \in \{2,3\}^k \ \mid\ k \geqslant 1\ ;\ s_1 + \cdots + s_k = n\}.$$

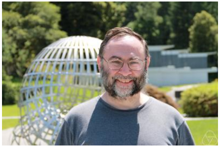
Heuristic for the conjectures

Michael E. Hoffman

The algebra of multiple harmonic series, J. Algebra, vol. **194**.2, 477-495, (1997).

Tomohide Terasoma

Mixed Tate motives and multiple zeta values, Invent. Math., vol. **149**.2, 339–369, (2002).



Alexander B. Goncharov

Galois symmetries of fundamental groupoids and noncommutative geometry, Duke Math. J., vol. **128**.2, 209–284, (2005).

Bourbaki Seminar by Pierre Cartier

Pierre Cartier. – *Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents.* Sém. Bourbaki no. 885 Astérisque **282** (2002), 137-173.

Francis Brown: Motivic zeta values

Upper bound for the dimension – Hoffman's conjectured basis for $\mathfrak{Z}_n$, $\zeta(\underline{s})$ with $s_i \in \{2, 3\}$, is a generating set.

Francis Brown

Irrationality proofs for zeta values, moduli spaces and dinner parties.
http://arxiv.org/abs/1412.6508

On the decomposition of motivic multiple zeta values.
http://lanl.arxiv.org/abs/1102.1310
Mixed Tate motives over $\mathbb{Z}$. Annals of Mathematics **175** (2012), 949–976.
http://dx.doi.org/10.4007/annals.2012.175.2.10

Arithmetic properties of zeta values: state of the art

Known:

- $\zeta(3)$ is irrational.
- Infinitely many odd zeta values are irrational.
- The $\mathbb{Q}$–vector space spanned by them has infinite dimension.

Open problems:

- No other odd value $2n+1 \geqslant 5$ is known for which one can prove that $\zeta(2n+1)$ is irrational.
- Nothing is known concerning the transcendence of $\zeta(3)$.
- No known result so far for MZV which are not a consequence of results for the Riemann zeta function in one variable.

Roger Apéry Journées Arithmétiques Caen 1976

Roger Apéry

Roger Apéry

(1916 – 1994)

https://mathshistory.st-andrews.ac.uk/Biographies/Apery/

Société Mathématique de France
Astérisque 61 (1979) p. 11-13

IRRATIONALITÉ DE $\zeta 2$ ET $\zeta 3$

par

Roger APÉRY

-:-:-:-

Notre méthode de démonstration de l'irrationalité d'un réel α défini par les sommes partielles σ_n d'une série de rationnels, comporte les étapes suivantes :

1. Remplacer la suite $\sigma_n = u_{o,n}$ par une suite de rationnels à deux indices $u_{k,n}$ avec $0 \leqslant k \leqslant n$ telle que pour chaque k la suite $u_{k,n}$ converge plus rapidement vers α que la suite $u_{k-1,n}$.

2. Poser $u_{k,n} = \dfrac{t_{k,n}}{\binom{n+k}{k}}$.

3. Majorer, en fonction de n exclusivement, le dénominateur de $t_{k,n}$ c'est-à dire montrer qu'il existe une suite d'entiers p_n tels que $p_n\, t_{k,n}$ soit entier et que $p_n \leqslant \mu^{n+\varepsilon}$.

4. Effectuer une même combinaison linéaire (dépendant de n) à cœfficients entiers positifs sur la colonne n du tableau des $t_{k,n}$ et du tableau des $\binom{n+k}{h}$.

5. On obtient ainsi une suite $\dfrac{v_n}{u_n}$ de fractions de numérateur rationnel et de dénominateur entier. On détermine la limite commune λ de $\sqrt[n]{v_n}$ et de $\sqrt[n]{u_n}$.

6. Si on a de la chance, $\lambda > \mu$: on peut conclure l'irrationalité. On peut aussi déduire une mesure d'irrationalité : quels que soient les entiers p,q,

$$\left|\frac{p}{q} - \alpha\right| > \frac{1}{q^{\gamma+\varepsilon}}$$

Roger Apéry, Astérique **61** (1979) 11–13

avec $\gamma = \dfrac{2 \log \lambda}{\log \lambda - \log \mu}$

Pour la construction des $u_{n,k}$, nous utilisons le développement suivant : étant donnée une suite de réels $a_1, a_2 \ldots a_k$, toute fonction analytique $f(x)$ par rapport à la variable $\frac{1}{x}$ qui tend vers 0 avec $\frac{1}{x}$ admet un développement (unique) de la forme

$$f(x) \equiv \sum_{k \geq 1} \frac{C_k}{(x+a_1)(x+a_2)\ldots(x+a_k)}$$

(Nous écrivons $\equiv$ au lieu de $=$ pour tenir compte des répugnances des mathématiciens qui considèrent avec Abel, Cauchy et d'Alembert les séries divergentes comme une invention du diable ; en fait, nous n'utilisons jamais qu'une somme finie de termes, mais le nombre de termes croit avec x).

Pour étudier $\zeta 2$, nous posons :

$$\frac{1}{n^2} \equiv \frac{1}{n(n-1)} - \frac{1}{n(n-1)(n-2)} + \ldots \frac{(-1)^{k-1}\, k!}{n(n-1)(n-2)\ldots(n-k-1)} + \ldots$$

$t_{k,n}$ appartient au module

$$\mathbb{Z}(1, \tfrac{1}{4}, \ldots \tfrac{1}{n^2})$$

D'après un résultat classique sur le p.p.c.m. des n premiers entiers, μ est égal à e^2.

La suite u_n s'écrit $(1,3,19,147,1251,11253,\ldots)$
La suite v_n s'écrit $(0,5,\frac{125}{4},\ldots)$

Elles vérifient la récurrence

$$(n+1)^2 u_{n+1} - (11n^2+11n+3)u_n - (n-1)^2 u_{n-1} = 0$$

$$\lambda = \frac{11+5\sqrt{5}}{2}$$

L'irrationalité de $\zeta 2 = \frac{\pi^2}{6}$ est connue depuis Euler, mais notre méthode donne une mesure d'irrationalité de π^2.

Pour étudier $\zeta 3$, nous posons :

$$\frac{1}{n^3} \equiv \frac{1}{n(n^2-1)} - \frac{1}{n(n^2-1)(n^2-4)} + \ldots + \frac{(-1)^k\,(k!)^2}{n(n^2-1)\ldots(n^2-(k+1)^2)} + \ldots$$

L'utilisation de la diagonale $u_{n,n}$ donne la série

$$\zeta 3 = \frac{5}{2} \sum_n \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}$$

qui à défaut de prouver immédiatement l'irrationalité de $\zeta 3$ converge mieux que $\sum \frac{1}{n^3}$.

2.$t_{k,n}$ appartient au module

$$\mathbb{Z}(1, \tfrac{1}{2^3} \ldots \tfrac{1}{n^3} \ldots)$$

μ est égal à e^3.

La suite u_n s'écrit $(1,5,73,1445,33001,\ldots)$
La suite v_n s'écrit $(0,6,\frac{351}{4},\frac{62531}{36},\ldots)$

Les deux suites vérifient la relation de récurrence

$$(n+1)^3 u_{n+1} - (34n^3+51n^2+27n+5)u_n + n^3 u_{n-1} = 0$$

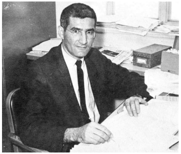
$$\lambda = 17+12\sqrt{2}$$

Roger APERY
Département de Mathématiques
Esplanade de la Paix
14032 CAEN CEDEX

A proof that Euler missed

Alf van der Poorten

(1942 – 2010)



Tom Apostol

(1923 – 2016)

A. van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of* $\zeta(3)$ (An informal report), Math. Intelligencer **1**:4 (1978/79), 195–203

http://www.ift.uni.wroc.pl/~mwolf/Poorten_MI_195_0.pdf

Tom M. Apostol: *A Proof that Euler Missed; Evaluating* $\zeta(2)$ *the EasyWay*,
The Mathematical Intelligencer, **5**, 59–60 (1983).
https://doi.org/10.1007/BF03026576

Further references

Éric Reyssat

Irrationalité de $\zeta(3)$ selon Apéry, Séminaire Delange–Pisot–Poitou, 20ème année (1978–1979), exposé no. 6

Frits Beukers

- *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$*, Bull. London Math. Soc. **11**:3 (1979), 268–272
- *Irrationality proofs using modular forms*, Astérisque **147–148** (1987), 271–283

https://wain.mi-ras.ru/zw/Beukers-Asterisque1987.pdf

Infinitely many irrational numbers

Tanguy Rivoal

Keith Ball

• Tanguy Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris Sér. I Math. **331**:4 (2000), 267–270

http://arxiv.org/abs/math/0008051

• Keith Ball et Tanguy Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. **146**:1 (2001), 193–207

https://doi.org/10.1007/s002220100168

The $\mathbb{Q}$–span has infinite dimension

Keith Ball, Tanguy Rivoal, *The dimension of the* $\mathbb{Q}$*–vector space spanned by the numbers* $1, \zeta(3)$, $\zeta(5)$, $\zeta(7), \ldots, \zeta(s)$ *is at least*

$$\frac{1-\epsilon}{1+\log 2}\log s$$

for any $\epsilon > 0$ *provided that* s *is an odd integer large enough in terms of* ϵ*.*

Tanguy Rivoal, *Irrationalité d'au moins un des neuf nombres* $\zeta(5)$, $\zeta(7)$, $\ldots$, $\zeta(21)$, Acta Arith. **103**.2 (2002), 157–167

Wadim Zudilin

One of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational*, Uspekhi Mat. Nauk [Russian Math. Surveys] **56**:4 (2001), 149–150.

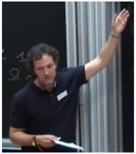
http://wain.mi.ras.ru/PS/zeta5-11$.pdf

Wadim Zudilin

References

- Wadim Zudilin https://www.math.ru.nl/~zudilin/
  https://www.math.ru.nl/~wzudilin/zw/
- Stéphane Fischler: *Irrationalité de valeurs de zêta [d'après Apéry, Rivoal,...]* Séminaire Bourbaki, 55ème année, 2002–2003, n° 910, Novembre 2002. Astérisque **294** (2004), 27–62.
  http://arxiv.org/abs/math/0303066

Stéphane Fischler (2021)



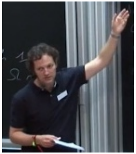
Stéphane Fischler

Among 1 *and the odd zeta values* $\zeta(3), \zeta(5), \dots, \zeta(s)$, *at least* $0.21\sqrt{s/\log s}$ *are linearly independent over the rationals, for any sufficiently large odd integer* s.

The proof is based on Siegel's lemma to construct non–explicit linear forms in odd zeta values, instead of using explicit well–poised hypergeometric series.

Linear independence of odd zeta values using Siegel's lemma,
https://doi.org/10.48550/arXiv.2109.10136
[Submitted on 21 Sep 2021 (v1), last revised 3 Mar 2024 (this version, v2)]

Irrational numbers among the odd zeta values



Stéphane Fischler

Johannes Sprang

Wadim Zudilin

2018: J. Sprang; W. Zudilin.

2019: S. Fischler, J. Sprang and W. Zudilin.

2020: Li Lai and Pin Yu, There are at least $1.19\sqrt{s/\log s}$ irrational numbers among $\zeta(3), \zeta(5), \ldots, \zeta(s)$.

Li Lai and Pin Yu, *A note on the number of irrational odd zeta values*, Compositio Mathematica **156** (2020), no. 8, 1699 – 1717.

Transcendence of π

Charles Hermite

(1822 – 1901)

Ferdinand Lindemann

(1852 – 1939)

Hermite–Lindemann Theorem. *If* α *is algebraic and* $\neq 0$, *then* e^{α} *is transcendental.*

Hermite's method: explicit construction using integral formulae. Auxiliary function:

$$F(z) = A(z)\mathrm{e}^{z} - B(z).$$

Interpolation formulae

George Pólya

(1887 – 1985)

G. Pólya (1914).
The function 2^z is a transcendental function of least growth order mapping the nonnegative integers into the integers.

If f is a transcendental entire function satisfying $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}_{\geqslant 0}$, then

$$\limsup_{R\to\infty} 2^{-R}|f|_R \geqslant 1.$$

Expand 2^z as an interpolation series on $0, 1, 2, \ldots$, involving the polynomials

$$\binom{z}{n} := \frac{z(z-1)\cdots(z-n+1)}{n!}.$$

Interpolation formulae

Alexander O. Gelfond

(1906 – 1968)

Rodion Kuzmin

(1891 – 1949)

- Gel'fond (1929). *Transcendence of*

$$e^{\pi} = 23.140\,692\,632\,779\,269\,005\,72\ldots$$

Expands $e^{i\pi z}$ as an interpolation series on $\mathbb{Z}[i]$.

- Kuzmin (1930). Transcendence of the number

$$2^{\sqrt{2}} = 2.665\,144\,142\,690\,225\,188\,65\ldots$$

(Sequences A039661 and A007507 in OEIS).

Using Dirichlet's box principle

Carl Ludwig Siegel

(1896 – 1981)

1923 The nonzero periods of a Weierstrass elliptic function with algebraic invariants g_2, g_3 are transcendental.

1929

Über einige Anwendungen diophantischer Approximationen

Part I: On transcendental numbers. E and G functions.

Part II: On Diophantine equations. Integer points on curves of genus $\geqslant 1$.

Translation by Clemens Fuchs:

On some applications of Diophantine approximations.

Quaderni Scuola Normale Superiore di Pisa. Monographs 2.

Pisa: Edizioni della Normale 161 p. (2014).

Thue – Siegel Lemma

Carl Ludwig Siegel

(1896 – 1981)

Axel Thue

(1863 - 1922)

Let

$$y_1 = a_{11}x_1 + \cdots + a_{1n}x_n$$

$$\vdots$$

$$y_m = a_{m1}x_1 + \cdots + a_{mn}x_n$$

be m *linear forms in* n *variables with rational integer coefficients. Let* $n > m$*. Let the absolute values of the* mn *coefficients* a_{kl} *be not bigger than a given natural number* A*. Then the homogeneous linear equations* $y_1 = 0, \ldots, y_m = 0$ *are solvable in rational integer numbers* $x_1, \ldots, x_n$*, which are not all zero, but are all smaller than* $(1 + (nA))^{\frac{m}{n-m}}$ *in absolute value.*

Hilbert's 7th Problem

Theodor Schneider

(1911 – 1988)

Alexander O. Gelfond

(1906 – 1968)

Theorem of Gel'fond Schneider (1934). *If* α *and* β *are algebraic numbers with* $\alpha \neq 0$, $\log\alpha \neq 0$ *and* $\beta \notin \mathbb{Q}$, *then the number* $\alpha^\beta := \exp(\beta\log\alpha)$ *is transcendental.*

Examples: $2^{\sqrt{2}}$, also e^π with $\alpha = 1$, $\log\alpha = 2i\pi$, $\beta = -i/2$.

Transcendence method of Gel'fond and Schneider

Transcendence of e^{α} (Hermite – Lindemann)

$$F(z) = P(z, \mathrm{e}^z), \qquad \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^t F(s\alpha) \in \mathbb{Q}(\alpha, \mathrm{e}^{\alpha}).$$

or

$$F(z) = P(z, \mathrm{e}^{\alpha z}), \qquad \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^t F(s) \in \mathbb{Q}(\alpha, \mathrm{e}^{\alpha}).$$

Transcendence of α^{β} (Gel'fond)

$$F(z) = P(\mathrm{e}^z, \mathrm{e}^{\beta z}), \qquad \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^t F(s \log \alpha) \in \mathbb{Q}(\alpha, \beta, \alpha^{\beta}).$$

Transcendence of α^{β} (Schneider)

$$F(z) = P(z, \alpha^z), \qquad F(s_1 + s_2\beta) \in \mathbb{Q}(\alpha, \beta, \alpha^{\beta}).$$

Auxiliary functions in transcendence theory

Plan of transcendence proofs:

• Using the Thue – Siegel Lemma, show the existence of a non-zero polynomial P such that F has many zeroes.
Less equations than unknowns.

• Show the existence of a nonzero value (zero estimate or multiplicity estimate).
More equations than unknowns.

• Analytic estimate (Schwarz lemma):
upper bound for this value.

• Arithmetic estimate (Liouville's inequality):
lower bound for this value.

Polylogarithms

Tanguy Rivoal

Raffaele Marcovecchio

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

For any $\epsilon > 0$ and any sufficiently large s, for any nonzero algebraic number γ such that $|\gamma| < 1$,

$$\dim_{\mathbb{Q}(\gamma)} \mathrm{Span}_{\mathbb{Q}(\gamma)} \bigl(1, \mathrm{Li}_1(\gamma), \dots, \mathrm{Li}_s(\gamma)\bigr) \geqslant \frac{1-\epsilon}{(1+\log 2)[\mathbb{Q}(\gamma):\mathbb{Q}]} \log s.$$

Stéphane Fischler (2024)

$$\dim_{\mathbb{Q}(\gamma)} \mathrm{Span}_{\mathbb{Q}(\gamma)} \bigl(1, \mathrm{Li}_1(\gamma), \dots, \mathrm{Li}_s(\gamma)\bigr) \geqslant \frac{0.26}{[\mathbb{Q}(\gamma):\mathbb{Q}]} \cdot \frac{\sqrt{s}}{\sqrt{\log s}}.$$

Proofs dealing with odd zeta values

Consider a rational function

$$F_n(X) = \sum_{i=1}^{s} \sum_{j=0}^{n} \frac{c_{ij}}{(X+j)^i} \in \mathbb{Q}(X$$

with $c_{ij} \in \mathbb{Z}$.
Ball – Rivoal: for n even and s odd,

$$F_n(X) = d_n^s n!^{s-2r} \frac{(X-rn)_{rn}(X+n+1)_{rn}}{(X)_{n+1}^s}, \qquad r = \left\lfloor \frac{s}{(\log s)^2} \right\rfloor$$

with $d_n = \mathrm{lcm}(1, 2, \dots, n)$ and the Pochhammer symbol

$$(x)_\alpha = x(x+1)\cdots(x+\alpha-1).$$

Connection with well–poised hypergeometric series.

Hypergeometric series

$$ {}_{q+1}F_q \left[\begin{matrix} \alpha_0, \alpha_1, \ldots, \alpha_q \\ \beta_1, \ldots, \beta_q \end{matrix} \; ; \; z \right] = \sum_{k=0}^{\infty} \frac{(\alpha_0)_k (\alpha_1)_k \cdots (\alpha_q)_k}{k! (\beta_1)_k \cdots (\beta_q)_k} \, z^k. $$

- *balanced*: $\alpha_0 + \cdots + \alpha_q + 1 = \beta_1 + \cdots + \beta_q$;
- *nearly–poised*: $\alpha_1 + \beta_1 = \cdots = \alpha_q + \beta_q$;
- *well–poised*: $\alpha_0 + 1 = \alpha_1 + \beta_1 = \cdots = \alpha_q + \beta_q$;
- *very well–poised*: well poised and $\alpha_1 = \frac{1}{2}\alpha_0 + 1$.

Ball – Rivoal:

$$ q = s+1, \; \alpha_i = rn + 1, \; \beta_i = rn + n + 2, \; (1 \leqslant i \leqslant q), $$
$$ \alpha_0 = 2rn + n + 2 = \alpha_i + \beta_i - 1. $$

Ball – Rivoal

Their choice produces a linear combination of 1 and odd zeta values

$$\sum_{t=1}^{\infty} F_n(t) = \varrho_{0,n} + \varrho_{3,n}\zeta(3) + \cdots + \varrho_{s,n}\zeta(s)$$

with $\varrho_{i,n} \in \mathbb{Z}$ such that, as $n \to \infty$,

$$|\varrho_{i,n}| \leqslant \beta^{n(1+o(1))} \quad \text{and} \quad \left|\sum_{t=1}^{\infty} F_n(t)\right| \leqslant \alpha^{n(1+o(1))}$$

with $\alpha < 1 < \beta$.

A linear independence criterion yields a lower bound for the dimension of the $\mathbb{Q}$–span:

$$1 - \frac{\log \alpha}{\log \beta}\cdot$$

Asymptotics

Ball – Rivoal

$$\log \alpha \sim -s \log s, \qquad \log \beta \sim (1 + \log 2)s.$$

Stéphane Fischler

For each n, existence of a family of c_{ij} in $\mathbb{Z}$ with the asymptotics as $s \to \infty$

$$\log \alpha \sim -4.55\sqrt{s \log s}, \qquad \log \beta \sim 20.93 \log s.$$

Output of Fischler's construction

Let $a \in \mathbb{Z}_{\geqslant 1}$, $\omega, \Omega, r \in \mathbb{Q}$ with $a > \Omega > \omega > 0$, $r \geqslant 1$. For any $n \geqslant 1$ such that $rn, \omega n, \Omega n \in \mathbb{Z}$, there exists integers $c_{ij} \in \mathbb{Z}$ ($1 \leqslant i \leqslant a$, $0 \leqslant j \leqslant n$), not all zero, such that, as $|t| \to \infty$, for the rational function

$$F_n(X) = \sum_{i=1}^{a} \sum_{j=0}^{n} \frac{c_{ij}}{(X+j)^i} \in \mathbb{Q}(X),$$

(i)

$$F_n(t) := \sum_{d=1}^{\infty} \frac{\mathfrak{A}_d^{(n)}}{t^d} = O(|t|^{-\omega n}),$$

(ii)

$$|c_{ij}| \leqslant \chi^{n(1+o(1))} \quad \text{with some explicit } \chi,$$

(iii)

$$|\mathfrak{A}_d^{(n)}| \leqslant r^{d-\Omega n} n^d d^a \chi^{n(1+o(1))} \quad \text{for any} \quad d < \Omega n.$$

A lemma from analytic number theory

The construction of the auxiliary function is 16 pages, including 3 pages for his variant of Siegel's Lemma. One of the many auxiliary results is an asymptotic estimate for an lcm.

Let a, N be integers $\geqslant 1$. Denote by $\Delta_{a,N}$ the least common multiple of the products $N_1 \cdots N_\alpha$ where $\alpha \leqslant a$ and $N_1, \ldots, N_\alpha$ are pairwise **distinct integers** *between $-N$ and N such that $\max N_i - \min N_i \leqslant N$. Then for each a, as $N \to \infty$,*

$$\Delta_{a,N} = \exp\left(N\left(\sum_{j=1}^{a} \frac{1}{j} + o(1)\right)\right) \leqslant \left((a+1)\mathrm{e}^{\gamma+o(1)}\right)^N$$

where γ is Euler constant.

A naive version would be $\Delta_{a,N} \leqslant d_N^a \leqslant \mathrm{e}^{Na+o(N)}$.

Computing an lcm

The proof of this lemma is reminiscent of the shuffle product. The exact value of this lcm is

$$\Delta_{a,N} = \prod_{p^e \leqslant N} p^{f_{a,N}(p^e)},$$

where the product is over all pairs (p, e) with p prime, $e \geqslant 1$, $p^e \leqslant N$ and

$$f_{a,N}(p^e) = \min\left(a, \left\lfloor \frac{N}{p^e} \right\rfloor\right).$$

Eliminating the even zeta values

For $i \geqslant 2$

$$\mathrm{Li}_i(-1) = (2^{1-i} - 1)\zeta(i)$$

so that

$$(1 - (-1)^i)\mathrm{Li}_i(-1) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 2(2^{1-i} - 1)\zeta(i) & \text{if } i \text{ is odd.} \end{cases}$$

Extrapolation: derivatives with respect to t

For $p \geqslant 0$, we have

$$F_n^{(p)}(X) = \sum_{i=1}^{a} \sum_{j=0}^{n} \frac{c_{ij}(-1)^p (i)_p}{(X+j)^{i+p}}.$$

Let h satisfy $0 \leqslant h \leqslant a$. Define

$$S_{n,p}(z) = z^{rn} \sum_{t=rn+1}^{\infty} \left(F_n^{(p)}(t) z^{-t} - F_n^{(p)}(-t) z^{t} \right).$$

Then for some polynomial $V_p(z)$ of degree $\leqslant 2rn$,

$$S_{n,p}(z) = V_p(z) + \sum_{i=1}^{a} z^{rn} P_i(z) (-1)^p (i)_p \left(\mathrm{Li}_{i+p}(1/z) - (-1)^{i+p} \mathrm{Li}_{i+p}(z) \right).$$

Extrapolation: derivatives with respect to z

For $k \geqslant 2rn + 2$,

$$S_{n,p}^{(k-1)}(z) = Q_{k,0}^{[p]}(z) + \sum_{i=1}^{a+h} Q_{k,i}^{[p]}(z) \left(\mathrm{Li}_i(1/z) - (-1)^i \mathrm{Li}_i(z)\right)$$

with explicit polynomials $Q_{k,i}^{[p]}(z)$.

For $z = -1$, for each n, with p and k which vary, this produces sufficiently many linear forms in the odd zeta values $\leqslant a + h$ to apply a multiplicity estimate and a linear independence criterion.

Multiplicity estimate

Andrei Borisovich Shidlovskii (1915 – 2007)

Daniel Bertrand

Frits Beukers

Let $A \in \mathrm{Mat}_{N\times N}(\mathbb{C}(z))$, $S_0, \dots, S_{N-1} \in \mathbb{C}[X]$, $\deg S_i \leqslant m$. For $Y = {}^t(y_0, \dots, y_{N-1})$ a solution of the differential system $Y' = AY$ associate the remainder

$$R(Y)(z) = \sum_{i=0}^{N-1} S_i(z) y_i(z).$$

Let Σ be a finite subset of $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ containing ∞. The multiplicity estimate is an upper bound for

$$\sum_{\sigma\in\Sigma} \sum_{j\in J_\sigma} \mathrm{ord}_\sigma(R(Y_j))$$

when $(Y_j)_{j\in J_\sigma}$ is a family of solutions of $Y' = AY$ satisfying suitable assumptions.

System of differential equations

Let h be a parameter in the interval $1 \leqslant h \leqslant a$ and let

$$b = \max \{ i \in \{1, \ldots, a\} \mid \text{there exists } j \in \{0, \ldots, n\}, c_{i,j} \neq 0 \}.$$

For $q = 0, 1, \ldots, h$, the system of differential equations is

$$\begin{cases} y'_{i,q}(z) = -\frac{1}{z} y'_{i,-1q}(z) \text{ for } 1 \leqslant i \leqslant b + h, \quad i \neq h+1, \\ y'_{h+1,q}(z) = \frac{z+1}{z(1-z)} y_{h,q}(z), \\ y'_{0,q}(z) = 0. \end{cases}$$

The multiplicity estimate produces independent linear forms in the odd zeta values which allow to apply a linear independence criterion.

Linear independence criteria

Carl Ludwig Siegel

(1896 – 1981)

Yuri Nesterenko

With explicit constructions, one appeals to a criterion due to Nesterenko. Given real numbers $\theta_0, \dots, \theta_p$, to prove a lower bound for $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\theta_0, \dots, \theta_p)$, it suffices to produce for each sufficiently large n, a linear form in $p+1$ variables with bounded coefficients in $\mathbb{Z}$ which takes at $\theta_0, \dots, \theta_p$ a value which is small but not too small.
A non explicit auxiliary function does not produce a lower bound: one needs to invoke an older criterion due to Siegel.

Siegel linear independence criterion (1929)

Let $\theta_0, \ldots, \theta_p$ be real numbers, not all 0. Let α, β satisfy $0 < \alpha < 1 < \beta$. Assume that for infinitely many $n \geqslant 1$, $\left(\ell_{i,j}^{(n)}\right)_{0\leqslant i,j\leqslant p}$ is a regular matrix with integer coefficients such that, as $n \to \infty$,

$$\max_{0\leqslant i,j\leqslant p} |\ell_{i,j}^{(n)}| \leqslant \beta^{n(1+o(1))}$$

and

$$\max_{0\leqslant j\leqslant p} |\ell_{0,j}^{(n)}\theta_0 + \cdots + \ell_{p,j}^{(n)}\theta_p| \leqslant \alpha^{n(1+o(1))}.$$

Then

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(\theta_0, \ldots, \theta_p) \geqslant 1 - \frac{\log \alpha}{\log \beta}.$$

Laurent interpolation determinants

• Siegel's Lemma, Dirichlet's Box Principle: some system of linear equations has a nontrivial solution.

Linear algebra: upper bound for the rank of a matrix.

• Zero or multiplicity estimate: some system of linear equations has no nontrivial solution.

Linear algebra: lower bound for the rank of a matrix.

Michel Laurent

The zero or multiplicity estimate produces a matrix with maximal rank. An analytic estimate (Schwarz's Lemma) yields an upper bound for the absolute value of a determinant, An arithmetic estimate (Liouville's inequality) yields a lower bound for the same.

Fourier – Borel duality

Joseph Fourier

(1768 - 1830)

Émile Borel

(1871 - 1956)

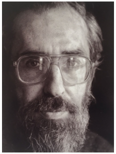
With the interpolation determinants, the methods of Gel'fond and Schneider are *dual* to each other: one transposes the matrix.

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^t \left(z^\lambda \mathrm{e}^{\mu x z}\right)(sy) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\lambda \left(z^t \mathrm{e}^{syz}\right)(\mu x)$$

$$= \sum_{j=0}^{\min\{t,\lambda\}} \frac{t!\lambda!}{j!(t-j)!(\lambda-j)!} (\mu x)^{t-j}(sy)^{\lambda-j}(\mathrm{e}^{xy})^{s\mu}.$$

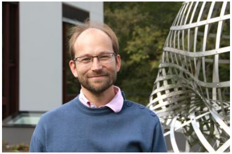
$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^t \left(\mathrm{e}^{(\mu_1+\mu_2\beta)z}\right)(s\log\alpha) = \left(z^t \alpha^{sz}\right)(\mu_1+\mu_2\beta)$$

Bost's slope inequalities



Souren Arakelov

Jean-Benoît Bost



Antoine Chambert-Loir

A matrix, a determinant, are associated with bases of vector spaces. Using Arakelov's Theory one can get rid of bases.

J.-B. Bost. *Evaluation maps, slopes, and algebraicity criteria.* In Proceedings of the International Congress of Mathematicians, Madrid 2006, volume II, pages 537–562. European Mathematical Society, 2006.

J.-B. Bost and A. Chambert-Loir. *Analytic curves in algebraic varieties over number fields.* In Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, volume **269** of Progr. Math., pages 69–124. Birkhäuser, Boston, MA, 2009.

arXiv > math > arXiv:2408.15403

Mathematics > Number Theory

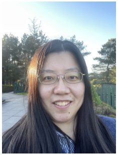
[Submitted on 27 Aug 2024 (v1), last revised 16 Sep 2024 (this version, v2)]

The linear independence of 1, $\zeta(2)$, and $L(2, \chi_{-3})$

Frank Calegari, Vesselin Dimitrov, Yunqing Tang

Frank Calegari

Vesselin I. Dimitrov



Yunqing Tang

$$L(2, \chi_{-3}) = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \cdots = 0.781\,302\,412\ \ldots$$

The argument applies a new kind of arithmetic holonomy bound to a well-known construction of Zagier.

218 pages

A reference

Multiple zeta values

Tasting notes

Wadim Zudilin

IMAPP, Radboud University, PO Box 9010, 6500 GL Nijmegen, Netherlands

E-mail address: w.zudilin@math.ru.nl
URL: http://www.math.ru.nl/~wzudilin/

お誕生日おめでとうございます

Happy Birthday Masanobu

http://www.imj-prg.fr/~michel.waldschmidt/