

19th International Mathematics Conference Bangladesh Mathematical Society

Linear recurrence sequences: an introduction

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Abstract

Linear recurrence sequences are ubiquitous. They occur in biology, economics, computer science (analysis of algorithms), digital signal processing. We give a survey of this subject, together with connections with linear combinations of powers. This lecture will include new results, arising from a joint work with [Claude Levesque](#), involving families of Diophantine equations, with explicit examples related to some units of [L. Bernstein](#) and [H. Hasse](#).

Applications of linear recurrence sequences

- Combinatorics
- Elimination
- Symmetric functions
- Hypergeometric series
- Language
- Communication, shift registers
- Finite difference equations
- Logic
- Approximation
- Pseudo-random sequences

Leonardo Pisano (Fibonacci)

Fibonacci sequence $(F_n)_{n \geq 0}$,
0, 1, 1, 2, 3, 5, 8, 13, 21,
34, 55, 89, 144, 233, ...

is defined by

$$F_0 = 0, F_1 = 1,$$

$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0.$$

<http://oeis.org/A000045>

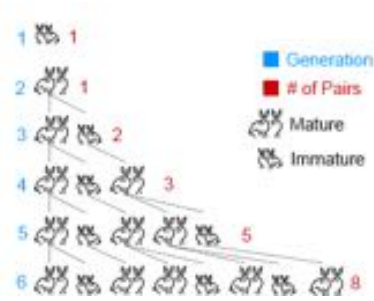
Leonardo Pisano (Fibonacci)
(1170–1250)



Fibonacci rabbits

Fibonacci considered the growth of a rabbit population.

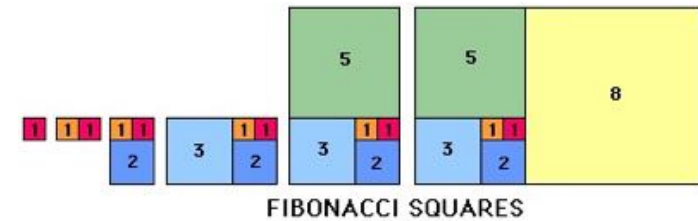
A newly born pair of rabbits, a male and a female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces



one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was : how many pairs will there be in one year?

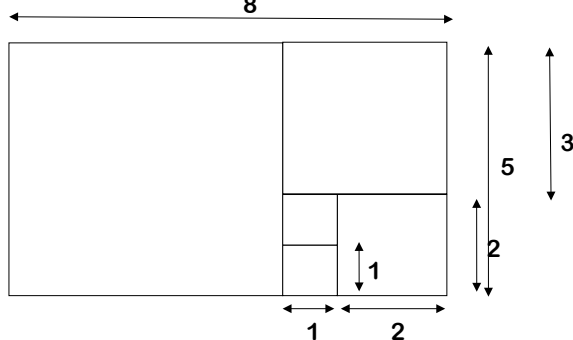
Answer : $F_{12} = 144$.

Fibonacci squares



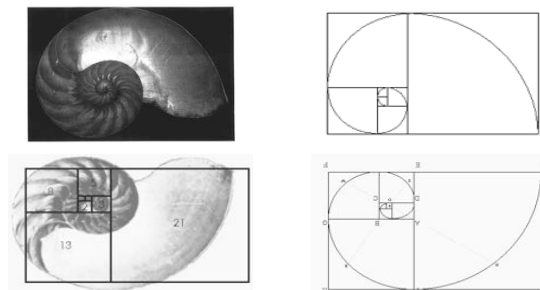
<http://mathforum.org/dr.math/faq/faq.golden.ratio.html>

Geometric construction of the Fibonacci sequence



Fibonacci numbers in nature

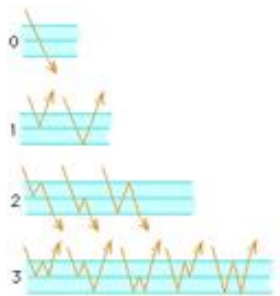
Ammonite (Nautilus shape)



Reflections of a ray of light

Consider three parallel sheets of glass and a ray of light which crosses the first sheet. Each time it touches one of the sheets, it can cross it or reflect on it.

Denote by p_n the number of different paths with the ray going out of the system after n reflections.



$$p_0 = 1,$$

$$p_1 = 2,$$

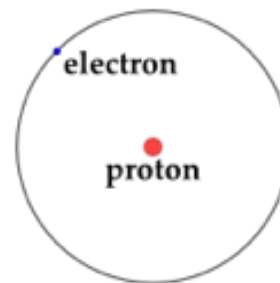
$$p_2 = 3,$$

$$p_3 = 5.$$

In general, $p_n = F_{n+2}$.

Levels of energy of an electron of an atom of hydrogen

An atom of hydrogen can have three levels of energy, 0 at the ground level when it does not move, 1 or 2. At each step, it alternatively gains and loses some level of energy, either 1 or 2, without going sub 0 nor above 2. Let ℓ_n be the number of different possible scenarios for this electron after n steps.



We have $\ell_0 = 1$ (initial state level 0)

$\ell_1 = 2$: state 1 or 2, scenarios 01 or 02.

$\ell_2 = 3$: scenarios 010, 021 or 020.

$\ell_3 = 5$: scenarios 0101, 0102, 0212, 0201 or 0202.

In general, $\ell_n = F_{n+2}$.

Rhythmic patterns

The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody. Several Indian scholars, Pingala (200 BC), Virahanka (c. 700 AD), Gopāla (c. 1135), and the Jain scholar Hemachandra (c. 1150) studied rhythmic patterns that are formed from one-beat notes (or short syllables, ti in Morse Alphabet) : ● and two-beat notes (or long syllables, ta ta in Morse) : ■■.

1 beat, 1 pattern : ●

2 beats, 2 patterns : ●● and ■■

3 beats, 3 patterns : ●●●, ●■■ and ■■■

4 beats, 5 patterns :

●●●●, ■■●●, ●■■●, ●●■■, ■■■■

n beats, F_{n+1} patterns

Fibonacci sequence and the Golden ratio

For $n \geq 0$, the Fibonacci number F_n is the nearest integer to

$$\frac{1}{\sqrt{5}} \Phi^n,$$

where Φ is the Golden Ratio : <http://oeis.org/A001622>

$$\Phi = \frac{1 + \sqrt{5}}{2} = \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = 1.6180339887499\dots$$

which satisfies

$$\Phi = 1 + \frac{1}{\Phi}.$$

Binet's formula

For $n \geq 0$,

$$F_n = \frac{\Phi^n - (-\Phi)^{-n}}{\sqrt{5}}$$

$$= \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}},$$

Jacques Philippe Marie Binet
(1843)



$$\Phi = \frac{1 + \sqrt{5}}{2}, \quad -\Phi^{-1} = \frac{1 - \sqrt{5}}{2},$$

$$X^2 - X - 1 = (X - \Phi)(X + \Phi^{-1}).$$

The so-called Binet Formula

Formula of A. De Moivre (1718, 1730), Daniel Bernoulli (1726), L. Euler (1728, 1765), J.P.M. Binet (1843) : for $n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

Abraham de Moivre
(1667-1754)



Daniel Bernoulli
(1700-1782)



Leonhard Euler
(1707-1783)



Jacques P.M. Binet
(1786-1856)



Generating series

A single series encodes all the Fibonacci sequence :

$$\sum_{n \geq 0} F_n X^n = X + X^2 + 2X^3 + 3X^4 + 5X^5 + \dots + F_n X^n + \dots$$

Fact : this series is the Taylor expansion of a rational fraction :

$$\sum_{n \geq 0} F_n X^n = \frac{X}{1 - X - X^2}.$$

Proof : the product

$$(X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \dots)(1 - X - X^2)$$

is a telescoping series

$$\begin{aligned} & X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \dots \\ & -X^2 - X^3 - 2X^4 - 3X^5 - 5X^6 - \dots \\ & -X^3 - X^4 - 2X^5 - 3X^6 - \dots \\ & = X. \end{aligned}$$

Generating series of the Fibonacci sequence

Remark. The denominator $1 - X - X^2$ in the right hand side of

$$X + X^2 + 2X^3 + 3X^4 + \dots + F_n X^n + \dots = \frac{X}{1 - X - X^2}$$

is $X^2 f(X^{-1})$, where $f(X) = X^2 - X - 1$ is the irreducible polynomial of the Golden ratio Φ .

Fibonacci and powers of matrices

The Fibonacci linear recurrence relation $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ can be written

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}.$$

By induction one deduces, for $n \geq 0$,

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

An equivalent formula is, for $n \geq 1$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

Fibonacci sequence and the Golden ratio (continued)

For $n \geq 1$, $\Phi^n \in \mathbb{Z}[\Phi] = \mathbb{Z} + \mathbb{Z}\Phi$ is a linear combination of 1 and Φ with integer coefficients, namely

$$\Phi^n = F_{n-1} + F_n \Phi$$

Characteristic polynomial

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is

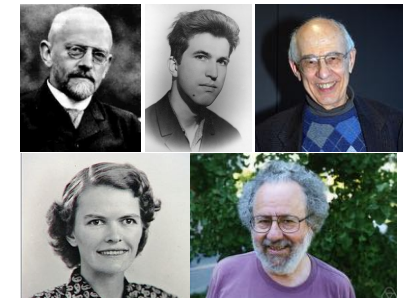
$$\det(XI - A) = \det \begin{pmatrix} X & -1 \\ -1 & X - 1 \end{pmatrix} = X^2 - X - 1,$$

which is the irreducible polynomial of the Golden ratio Φ .

Fibonacci sequence and Hilbert's 10th problem

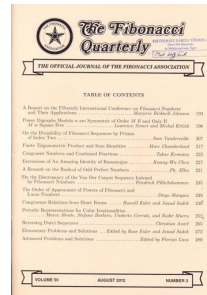
Yuri Matiyasevich (1970) showed that there is a polynomial P in n, m , and a number of other variables x, y, z, \dots having the property that $n = F_{2m}$ iff there exist integers x, y, z, \dots such that $P(n, m, x, y, z, \dots) = 0$.

This completed the proof of the impossibility of the tenth of Hilbert's problems (*does there exist a general method for solving Diophantine equations?*) thanks to the previous work of Hilary Putnam, Julia Robinson and Martin Davis.



The Fibonacci Quarterly

The Fibonacci sequence satisfies a lot of very interesting properties. Four times a year, the *Fibonacci Quarterly* publishes an issue with new properties which have been discovered.



Lucas sequence

<http://oeis.org/000032>

The Lucas sequence $(L_n)_{n \geq 0}$ satisfies the same recurrence relation as the Fibonacci sequence, namely

$$L_{n+2} = L_{n+1} + L_n \quad \text{for } n \geq 0,$$

only the initial values are different :

$$L_0 = 2, L_1 = 1.$$

The sequence of Lucas numbers starts with

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \dots$$

A closed form involving the Golden ratio Φ is

$$L_n = \Phi^n + (-\Phi)^{-n},$$

from which it follows that for $n \geq 2$, L_n is the nearest integer to Φ^n .

François Édouard Anatole Lucas (1842 - 1891)

Edouard Lucas is best known for his results in number theory. He studied the Fibonacci sequence and devised the test for Mersenne primes still used today.



<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Lucas.html>

Generating series of the Lucas sequence

The generating series of the Lucas sequence

$$\sum_{n \geq 0} L_n X^n = 2 + X + 3X^2 + 4X^3 + \dots + L_n X^n + \dots$$

is nothing else than

$$\frac{2 - X}{1 - X - X^2}.$$

The Lucas sequence and power of matrices

From the linear recurrence relation $L_{n+2} = L_{n+1} + L_n$ one deduces, (as we did for the Fibonacci sequence), for $n \geq 0$,

$$\begin{pmatrix} L_{n+1} \\ L_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} L_n \\ L_{n+1} \end{pmatrix},$$

hence

$$\begin{pmatrix} L_n \\ L_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Any one of the three sequences $(F_n)_{n \geq 0}$, $(L_n)_{n \geq 0}$, $(\Phi^n)_{n \geq 0}$ can be written as a linear combination of the two others.

Perrin sequence

<http://oeis.org/A001608>

The Perrin sequence (also called *skiponacci sequence*) is the linear recurrence sequence $(P_n)_{n \geq 0}$ defined by

$$P_{n+3} = P_{n+1} + P_n \quad \text{for } n \geq 0,$$

with the initial conditions

$$P_0 = 3, P_1 = 0, P_2 = 2.$$

It starts with

3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, ...

François Olivier Raoul Perrin (1841-1910) :
https://en.wikipedia.org/wiki/Perrin_number

Plastic (or silver) constant

<https://oeis.org/A060006>

The ratio of successive terms in the Perrin sequence approaches the plastic number ϱ , which is the minimal Pisot-Vijayaraghavan number, real root of

$$x^3 - x - 1$$

which has a value of approximately 1.324718.

This constant is equal to

$$\varrho = \frac{\sqrt[3]{108 + 12\sqrt{69}} + \sqrt[3]{108 - 12\sqrt{69}}}{6}.$$

Perrin sequence and the plastic constant

Decompose the polynomial $X^3 - X - 1$ into irreducible factors over \mathbb{C}

$$X^3 - X - 1 = (X - \varrho)(X - \rho)(X - \bar{\rho})$$

and over \mathbb{R}

$$X^3 - X - 1 = (X - \varrho)(X^2 + \varrho X + \varrho^{-1}).$$

Hence ρ and $\bar{\rho}$ are the roots of $X^2 + \varrho X + \varrho^{-1}$. Then, for $n \geq 0$,

$$P_n = \varrho^n + \rho^n + \bar{\rho}^n.$$

It follows that, for $n \geq 0$, P_n is the nearest integer to ϱ^n .

Generating series of the Perrin sequence

The generating series of the Perrin sequence

$$\sum_{n \geq 0} P_n X^n = 3 + 2X^2 + 3X^3 + 2X^4 + \dots + P_n X^n + \dots$$

is nothing else than

$$\frac{3 - X^2}{1 - X^2 - X^3}.$$

The denominator $1 - X^2 - X^3$ is $X^3 f(X^{-1})$ where $f(X) = X^3 - X - 1$ is the irreducible polynomial of ϱ .

Perrin sequence and power of matrices

From

$$P_{n+3} = P_{n+1} + P_n$$

we deduce

$$\begin{pmatrix} P_{n+1} \\ P_{n+2} \\ P_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.$$

Characteristic polynomial

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is

$$\det(XI - A) = \det \begin{pmatrix} X & -1 & 0 \\ 0 & X & -1 \\ -1 & -1 & X \end{pmatrix} = X^3 - X - 1,$$

which is the irreducible polynomial of the plastic constant ϱ .

Perrin pseudoprimes

<https://oeis.org/A013998>

If p is prime, then p divides P_p .

The smallest composite n such that n divides P_n is $521^2 = 271441$.

The number P_{271441} has 33 150 decimal digits (the number c which satisfies $10^c = \varrho^{271441}$ is $c = 271441(\log \varrho)/(\log 10)$).

Also for the composite number $n = 904631 = 7 \times 13 \times 9941$, the number n divides P_n .

Jon Grantham has proved that there are infinitely many Perrin pseudoprimes.

The website www.Perrin088.org maintained by Richard Turk is devoted to Perrin numbers. See OEISA113788.

Padovan sequence

<https://oeis.org/A000931>

The Padovan sequence $(p_n)_{n \geq 0}$ satisfies the same recurrence

$$p_{n+3} = p_{n+1} + p_n$$

as the Perrin sequence but has different initial values :

$$p_0 = 1, \quad p_1 = p_2 = 0.$$

It starts with

1, 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, ...

Richard Padovan

<http://mathworld.wolfram.com/LinearRecurrenceEquation.html>

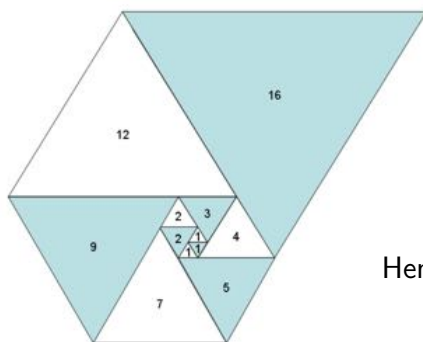
Generating series and power of matrices

$$1 + X^3 + X^5 + \dots + p_n X^n + \dots = \frac{1 - X^2}{1 - X^2 - X^3}.$$

For $n \geq 0$,

$$\begin{pmatrix} p_n \\ p_{n+1} \\ p_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Padovan triangles



$$p_n = p_{n-2} + p_{n-3}$$

$$p_{n-1} = p_{n-3} + p_{n-4}$$

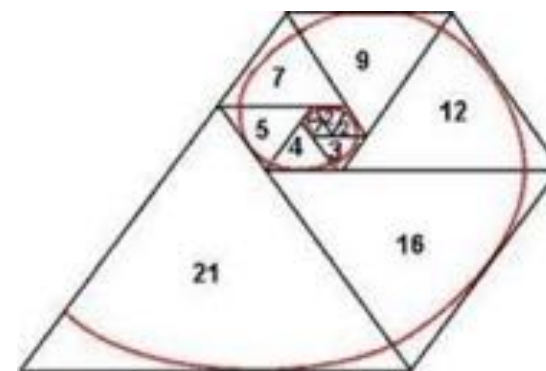
$$p_{n-2} = p_{n-4} + p_{n-5}$$

Hence

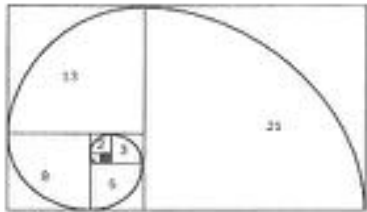
$$p_n - p_{n-1} = p_{n-5}$$

$$p_n = p_{n-1} + p_{n-5}$$

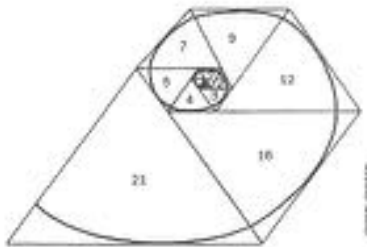
Padovan triangles



Padovan triangles vs Fibonacci squares



C_2 curve, not C_3



C_1 curve, not C_2

Padovan, Euler, Zagier and Brown

For $n \geq 0$, the number of compositions $\underline{s} = (s_1, \dots, s_k)$ with $s_i \in \{2, 3\}$ and $s_1 + \dots + s_k = n$ is p_{n+3} . This is (an upper bound for) the dimension of the space spanned by the multiple zeta values of weight n of Euler and Zagier.



Narayana sequence

<https://oeis.org/A000930>

Narayana sequence is defined by the recurrence relation

$$C_{n+3} = C_{n+2} + C_n$$

with the initial values $C_0 = 2, C_1 = 3, C_2 = 4$.

It starts with

2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, ...

Real root of $x^3 - x^2 - 1$

$$\frac{\sqrt[3]{\frac{29 + 3\sqrt{93}}{2}} + \sqrt[3]{\frac{29 - 3\sqrt{93}}{2}} + 1}{3} = 1.465571231876768 \dots$$

Generating series and power of matrices

$$2 + 3X + 4X^2 + 6X^3 + \dots + C_n X^n + \dots = \frac{2 + X + X^2}{1 - X - X^3}$$

For $n \geq 0$,

$$\begin{pmatrix} C_n \\ C_{n+1} \\ C_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Narayana's cows

Narayana was an Indian mathematician in the 14th century who proposed the following problem :
A cow produces one calf every year. Beginning in its fourth year each calf produces one calf at the beginning of each year.
How many calves are there altogether after, for example, 17 years?

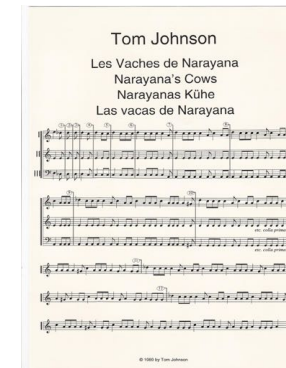
Music :

<http://www.pogus.com/21033.html>

In working this out, Tom Johnson found a way to translate this into a composition called *Narayana's Cows*.

Music : Tom Johnson

Saxophones : Daniel Kientzy



Narayana's cows

<http://webusers.imj-prg.fr/~michel.waldschmidt/>

Year	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Original Cow	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Second generation	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Third generation	0	0	0	1	3	6	10	15	21	28	36	45	55	66
Fourth generation	0	0	0	0	0	0	1	4	10	20	35	56	84	120
Fifth generation	0	0	0	0	0	0	0	0	0	1	5	15	35	70
Sixth generation	0	0	0	0	0	0	0	0	0	0	0	0	1	6
Total	2	3	4	6	9	13	19	28	41	60	88	129	189	277

Jean-Paul Allouche and Tom Johnson



<http://webusers.imj-prg.fr/~jean-paul.allouche/bibliorecente.html>

<http://www.math.jussieu.fr/~allouche/johnson1.pdf>

Cows, music and morphisms

Jean-Paul Allouche and Tom Johnson

- Narayana's Cows and Delayed Morphisms
In 3èmes Journées d'Informatique Musicale (JIM '96), Ile de Tatihou, Les Cahiers du GREYC (1996 no. 4), pages 2-7, May 1996.

<http://kalvos.org/johnness1.html>

- Finite automata and morphisms in assisted musical composition,
Journal of New Music Research, no. 24 (1995), 97 – 108.
<http://www.tandfonline.com/doi/abs/10.1080/09298219508570676>
http://web.archive.org/web/19990128092059/www.swets.nl/jnmr/vol24_2.html

Linear recurrence sequences : examples

- Constant sequence : $u_n = u_0$.
Linear recurrence sequence of order 1 : $u_{n+1} = u_n$.
Characteristic polynomial : $f(X) = X - 1$.
Generating series :

$$\sum_{n \geq 0} X^n = \frac{1}{1 - X}$$

- Geometric progression : $u_n = u_0 \gamma^n$.
Linear recurrence sequence of order 1 : $u_n = \gamma u_{n-1}$.
Characteristic polynomial $f(X) = X - \gamma$.
Generating series :

$$\sum_{n \geq 0} u_0 \gamma^n X^n = \frac{u_0}{1 - \gamma X}$$

Linear recurrence sequences : definitions

A *linear recurrence sequence* is a sequence of numbers $\mathbf{u} = (u_0, u_1, u_2, \dots)$ for which there exist a positive integer d together with numbers a_1, \dots, a_d with $a_d \neq 0$ such that, for $n \geq 0$,

$$(*) \quad u_{n+d} = a_1 u_{n+d-1} + \dots + a_d u_n.$$

Here, a *number* means an element of a field \mathbb{K} of zero characteristic.

Given $\underline{a} = (a_1, \dots, a_d) \in \mathbb{K}^d$, the set $E_{\underline{a}}$ of linear recurrence sequences $\mathbf{u} = (u_n)_{n \geq 0}$ satisfying $(*)$ is a \mathbb{K} -vector subspace of dimension d of the space $\mathbb{K}^{\mathbb{N}}$ of all sequences .

The characteristic (or companion) polynomial of the linear recurrence is

$$f(X) = X^d - a_1 X^{d-1} - \dots - a_d.$$

Linear recurrence sequences : examples

- $u_n = n$. Linear recurrence sequence of order 2 :

$$n + 2 = 2(n + 1) - n.$$

Characteristic polynomial

$$f(X) = X^2 - 2X + 1 = (X - 1)^2.$$

Generating series

$$\sum_{n \geq 0} n X^n = \frac{1}{1 - 2X + X^2}.$$

Power of matrices :

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^n = \begin{pmatrix} -n + 1 & n \\ -n & n + 1 \end{pmatrix}.$$

Linear recurrence sequences : examples

- $u_n = f(n)$, f polynomial of degree d . Linear recurrence sequence of order $d + 1$.

Proof. The sequences

$$(f(n))_{n \geq 0}, (f(n+1))_{n \geq 0}, \dots, (f(n+k))_{n \geq 0}$$

are \mathbb{K} -linearly independent in $\mathbb{K}^{\mathbb{N}}$ for $k = d - 1$ and linearly dependent for $k = d$.

A basis of the space of polynomials of degree d is given by the $d + 1$ polynomials

$$f(X), f(X + 1), \dots, f(X + d).$$

Linear sequences which are ultimately recurrent

The sequence

$$(1, 0, 0, \dots)$$

is not a linear recurrence sequence.

The condition

$$u_{n+1} = u_n$$

is satisfied only for $n \geq 1$.

The relation

$$u_{n+2} = u_{n+1} + 0u_n$$

with $d = 2$, $a_d = 0$ does not fulfill the requirement $a_d \neq 0$.

Order of a linear recurrence sequence

If $\mathbf{u} = (u_n)_{n \geq 0}$ satisfies the linear recurrence, the characteristic polynomial of which is f , then, for any monic polynomial $g \in \mathbb{K}[X]$ with $g(0) \neq 0$, this sequence \mathbf{u} also satisfies the linear recurrence, the characteristic polynomial of which is fg .

Example : for $g(X) = X - \gamma$ with $\gamma \neq 0$, from

$$(\star) \quad u_{n+d} - a_1 u_{n+d-1} - \dots - a_d u_n = 0$$

we deduce

$$u_{n+d+1} - a_1 u_{n+d} - \dots - a_d u_{n+1} - \gamma(u_{n+d} - a_1 u_{n+d-1} - \dots - a_d u_n) = 0.$$

The *order* of a linear recurrence sequence is the smallest d such that (\star) holds for all $n \geq 0$.

Generating series of a linear recurrence sequence

Let $\mathbf{u} = (u_n)_{n \geq 0}$ be a linear recurrence sequence

$$(\star) \quad u_{n+d} = a_1 u_{n+d-1} + \dots + a_d u_n \quad \text{for } n \geq 0$$

with characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \dots - a_d.$$

Denote by f^- the reciprocal polynomial of f :

$$f^-(X) = X^d f(X^{-1}) = 1 - a_1 X - \dots - a_d X^d.$$

Then

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)},$$

where r is a polynomial of degree less than d determined by the initial values of \mathbf{u} .

Generating series of a linear recurrence sequence

Assume

$$u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for } n \geq 0.$$

Then

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)}.$$

Proof. Comparing the coefficients of X^n for $n \geq d$ shows that

$$f^-(X) \sum_{n=0}^{\infty} u_n X^n$$

is a polynomial of degree less than d

Taylor coefficients of rational functions

Conversely, the coefficients the Taylor expansion of any rational fraction $a(X)/b(X)$ with $\deg a < \deg b$ satisfies the recurrence relation with characteristic polynomial $f \in K[X]$ given by $f(X) = b^-(X)$.

Therefore a sequence $\mathbf{u} = (u_n)_{n \geq 0}$ satisfies the recurrence relation (\star) with characteristic polynomial $f \in K[X]$ if and only if

$$\sum_{n=0}^{\infty} u_n X^n = \frac{r(X)}{f^-(X)},$$

where r is a polynomial of degree less than d determined by the initial values of \mathbf{u} .

Matrix notation for a linear recurrence sequence

The linear recurrence sequence

$$(\star) \quad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n \quad \text{for } n \geq 0$$

can be written

$$\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+d} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}.$$

Matrix notation for a linear recurrence sequence

$$U_{n+1} = AU_n$$

with

$$U_n = \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix}.$$

The determinant of $I_d X - A$ (the characteristic polynomial of A) is nothing but

$$f(X) = X^d - a_1 X^{d-1} - \cdots - a_d,$$

the characteristic polynomial of the linear recurrence sequence. By induction

$$U_n = A^n U_0.$$

Powers of matrices

Let $A = (a_{ij})_{1 \leq i, j \leq d} \in \text{GL}_{d \times d}(\mathbb{K})$ be a $d \times d$ matrix with coefficients in \mathbb{K} and nonzero determinant. For $n \geq 0$, define

$$A^n = (a_{ij}^{(n)})_{1 \leq i, j \leq d}.$$

Then each of the d^2 sequences $(a_{ij}^{(n)})_{n \geq 0}$, $(1 \leq i, j \leq d)$ is a linear recurrence sequence. The roots of the characteristic polynomial of these linear recurrences are the eigenvalues of A .

In particular the sequence $(\text{Tr}(A^n))_{n \geq 0}$ satisfies the linear recurrence, the characteristic polynomial of which is the characteristic polynomial of the matrix A .

Conversely :

Given a linear recurrence sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$, there exist an integer $d \geq 1$ and a matrix $A \in \text{GL}_d(\mathbb{K})$ such that, for each $n \geq 0$,

$$u_n = a_{11}^{(n)}.$$

The characteristic polynomial of A is the characteristic polynomial of the linear recurrence sequence.

EVEREST G., VAN DER POORTEN A., SHPARLINSKI I., WARD T. – *Recurrence sequences*, Mathematical Surveys and Monographs (AMS, 2003), volume 104.

Linear recurrence sequences : simple roots

A basis of E_a over \mathbb{K} is obtained by attributing to the initial values u_0, \dots, u_{d-1} the values given by the canonical basis of \mathbb{K}^d .

Given γ in \mathbb{K}^\times , a necessary and sufficient condition for a sequence $(\gamma^n)_{n \geq 0}$ to satisfy (\star) is that γ is a root of the characteristic polynomial

$$f(X) = X^d - a_1 X^{d-1} - \dots - a_d.$$

If this polynomial has d distinct roots $\gamma_1, \dots, \gamma_d$ in \mathbb{K} ,

$$f(X) = (X - \gamma_1) \cdots (X - \gamma_d), \quad \gamma_i \neq \gamma_j,$$

then a basis of E_a over \mathbb{K} is given by the d sequences $(\gamma_i^n)_{n \geq 0}$, $i = 1, \dots, d$.

Linear recurrence sequences : double roots

The characteristic polynomial of the linear recurrence $u_n = 2\gamma u_{n-1} - \gamma^2 u_{n-2}$ is $X^2 - 2\gamma X + \gamma^2 = (X - \gamma)^2$ with a double root γ .

The sequence $(n\gamma^n)_{n \geq 0}$ satisfies

$$n\gamma^n = 2\gamma(n-1)n\gamma^{n-1} - \gamma^2(n-2)\gamma^{n-2}.$$

A basis of E_a for $a_1 = 2\gamma$, $a_2 = -\gamma^2$ is given by the two sequences $(\gamma^n)_{n \geq 0}$, $(n\gamma^n)_{n \geq 0}$.

Given $\gamma \in \mathbb{K}^\times$, a necessary and sufficient condition for the sequence $n\gamma^n$ to satisfy the linear recurrence relation (\star) is that γ is a root of multiplicity ≥ 2 of $f(X)$.

Linear recurrence sequences : multiple roots

In general, when the characteristic polynomial splits as

$$X^d - a_1X^{d-1} - \dots - a_d = \prod_{i=1}^{\ell} (X - \gamma_i)^{t_i},$$

a basis of E_a is given by the d sequences

$$(n^k \gamma_i^n)_{n \geq 0}, \quad 0 \leq k \leq t_i - 1, \quad 1 \leq i \leq \ell.$$

Polynomial combinations of powers

The sum and the product of any two linear recurrence sequences are linear recurrence sequences.

The set $\cup_a E_a$ of all linear recurrence sequences with coefficients in \mathbb{K} is a sub- \mathbb{K} -algebra of $\mathbb{K}^{\mathbb{N}}$.

Given polynomials p_1, \dots, p_ℓ in $\mathbb{K}[X]$ and elements $\gamma_1, \dots, \gamma_\ell$ in \mathbb{K}^\times , the sequence

$$(p_1(n)\gamma_1^n + \dots + p_\ell(n)\gamma_\ell^n)_{n \geq 0}$$

is a linear recurrence sequence.

Fact : any linear recurrence sequence is of this form.

Sum of polynomial combinations of powers

If \mathbf{u}_1 and \mathbf{u}_2 are two linear recurrence sequences of characteristic polynomials f_1 and f_2 respectively, then $\mathbf{u}_1 + \mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\frac{f_1 f_2}{\gcd(f_1, f_2)}.$$

Product of polynomial combinations of powers

If the characteristic polynomials of the two linear recurrence sequences \mathbf{u}_1 and \mathbf{u}_2 are respectively

$$f_1(T) = \prod_{j=1}^{\ell} (T - \gamma_j)^{t_j} \quad \text{and} \quad f_2(T) = \prod_{k=1}^{\ell'} (T - \gamma'_k)^{t'_k},$$

then $\mathbf{u}_1 \mathbf{u}_2$ satisfies the linear recurrence, the characteristic polynomial of which is

$$\prod_{j=1}^{\ell} \prod_{k=1}^{\ell'} (T - \gamma_j \gamma'_k)^{t_j + t'_k - 1}.$$

Linear recurrence sequences and Brahmagupta–Pell–Fermat Equation

Let d be a positive integer, not a square. The solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of the Brahmagupta–Pell–Fermat Equation

$$x^2 - dy^2 = \pm 1$$

form a sequence $(x_n, y_n)_{n \in \mathbb{Z}}$ defined by

$$x_n + \sqrt{d}y_n = (x_1 + \sqrt{d}y_1)^n.$$

From

$$2x_n = (x_1 + \sqrt{d}y_1)^n + (x_1 - \sqrt{d}y_1)^n$$

we deduce that $(x_n)_{n \geq 0}$ is a linear recurrence sequence. Same for y_n , and also for $n \geq 0$.

Doubly infinite linear recurrence sequences

A sequence $(u_n)_{n \in \mathbb{Z}}$ indexed by \mathbb{Z} is a linear recurrence sequence if it satisfies

$$(*) \quad u_{n+d} = a_1 u_{n+d-1} + \cdots + a_d u_n.$$

for all $n \in \mathbb{Z}$.

Recall $a_d \neq 0$.

Such a sequence is determined by d consecutive values.

Discrete version of linear differential equations

A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ can be viewed as a linear map $\mathbb{N} \rightarrow \mathbb{K}$. Define the discrete derivative \mathcal{D} by

$$\begin{aligned} \mathcal{D}\mathbf{u} : \mathbb{N} &\rightarrow \mathbb{K} \\ n &\mapsto u_{n+1} - u_n. \end{aligned}$$

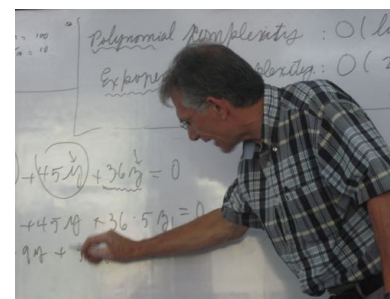
A sequence $\mathbf{u} \in \mathbb{K}^{\mathbb{N}}$ is a linear recurrence sequence if and only if there exists $Q \in \mathbb{K}[T]$ with $Q(1) \neq 1$ such that

$$Q(\mathcal{D})\mathbf{u} = 0.$$

Linear recurrence sequences are a discrete version of linear differential equations with constant coefficients.

The condition $Q(1) \neq 0$ reflects $a_d \neq 0$ – otherwise one gets *ultimately* recurrent sequences.

Joint work with Claude Levesque



Linear recurrence sequences and twisted binary forms.
 Proceedings of the International Conference on Pure and Applied Mathematics ICPAM-GOROKA 2014. South Pacific Journal of Pure and Applied Mathematics.

<http://webusers.imj-prg.fr/~michel.waldschmidt//articles/pdf/ProcConfPNG2014.pdf>

Families of binary forms

Consider a binary form $F_0(X, Y) \in \mathbb{C}[X, Y]$ which satisfies $F_0(1, 0) = 1$. We write it as

$$F_0(X, Y) = X^d + a_1 X^{d-1} Y + \dots + a_d Y^d = \prod_{i=1}^d (X - \alpha_i Y).$$

Let $\epsilon_1, \dots, \epsilon_d$ be d nonzero complex numbers not necessarily distinct. Twisting F_0 by the powers $\epsilon_1^n, \dots, \epsilon_d^n$ ($n \in \mathbb{Z}$) boils down to considering the family of binary forms

$$F_n(X, Y) = \prod_{i=1}^d (X - \alpha_i \epsilon_i^n Y),$$

which we write as

$$X^d - U_1(n) X^{d-1} Y + \dots + (-1)^d U_d(n) Y^d.$$

Therefore

$$U_h(0) = (-1)^h a_h \quad (1 \leq h \leq d).$$

Families of Diophantine equations

Each of the sequences $(U_h(n))_{n \in \mathbb{Z}}$ coming from the coefficients of the relation

$$F_n(X, Y) = X^d - U_1(n) X^{d-1} Y + \dots + (-1)^d U_d(n) Y^d$$

is a linear recurrence sequence.

For example, for $n \in \mathbb{Z}$,

$$U_1(n) = \sum_{i=1}^d \alpha_i \epsilon_i^n, \quad U_d(n) = \prod_{i=1}^d \alpha_i \epsilon_i^n.$$

For $1 \leq h \leq d$, the sequence $(U_h(n))_{n \in \mathbb{Z}}$ is a linear combination of the sequences

$$((\epsilon_{i_1} \cdots \epsilon_{i_h})^n)_{n \in \mathbb{Z}}, \quad (1 \leq i_1 < \dots < i_h \leq d).$$

Families of Diophantine equations

With [Claude Levesque](#), we considered some families of diophantine equations

$$F_n(x, y) = m$$

obtained in the same way from a given irreducible form $F(X, Y)$ with coefficients in \mathbb{Z} , when $\epsilon_1, \dots, \epsilon_d$ are algebraic units and when the algebraic numbers $\alpha_1 \epsilon_1, \dots, \alpha_d \epsilon_d$ are [Galois](#) conjugates with $d \geq 3$.

Theorem. Let \mathbb{K} be a number field of degree $d \geq 3$, S a finite set of places of \mathbb{K} containing the places at infinity. Denote by \mathcal{O}_S the ring of S -integers of \mathbb{K} and by \mathcal{O}_S^\times the group of S -units of \mathbb{K} . Assume $\alpha_1, \dots, \alpha_d, \epsilon_1, \dots, \epsilon_d$ belong to \mathbb{K}^\times . Then there are only finitely many (x, y, n) in $\mathcal{O}_S \times \mathcal{O}_S \times \mathbb{Z}$ satisfying

$$F_n(x, y) \in \mathcal{O}_S^\times, \quad xy \neq 0 \quad \text{and} \quad \text{Card}\{\alpha_1 \epsilon_1^n, \dots, \alpha_d \epsilon_d^n\} \geq 3.$$

Some units of Bernstein and Hasse

Let t and s be two positive integers, D an integer ≥ 1 , and $c \in \{-1, +1\}$. Let $\omega > 1$ satisfy

$$\omega^{st} = D^{st} + c,$$

where it is assumed that $\mathbb{Q}(\omega)$ is of degree st .

Consider

$$\alpha = D - \omega, \quad \epsilon = D^t - \omega^t.$$

[L. Bernstein](#) and [H. Hasse](#) noticed that α and ϵ are units of degree st and s respectively, and showed that these units can be obtained from the [Jacobi–Perron](#) algorithm. [H.-J. Stender](#) proved that for $s = t = 2$, $\{\alpha, \epsilon\}$ is a fundamental system of units of the quartic field $\mathbb{Q}(\omega)$.

Helmut Hasse (1898-1979)

$$D > 0, s \geq 1, t \geq 1,$$

$$c \in \{-1, +1\}, \omega > 0,$$

$$\omega^{st} = D^{st} + c,$$

$$\alpha = D - \omega,$$

$$\epsilon = D^t - \omega^t.$$



$$(\alpha - D)^{st} = (-1)^{st}(D^{st} + c).$$

Diophantine equations associated with some units of Bernstein and Hasse

The irreducible polynomial of α is $F_0(X, 1)$, with

$$F_0(X, Y) = (X - DY)^{st} - (-1)^{st}(D^{st} + c)Y^{st}.$$

For $n \in \mathbb{Z}$, the binary form $F_n(X, Y)$, obtained by twisting $F_0(X, Y)$ with the powers ϵ^n of ϵ , is the homogeneous version of the irreducible polynomial $F_n(X, 1)$ of $\alpha\epsilon^n$. So F_n depends of the parameters n, D, s, t and c .

Theorem (LW). *Suppose $st \geq 3$. There exists an effectively computable constant κ , depending only on D, s and t , with the following property. Let m, a, x, y be rational integers satisfying $m \geq 2, xy \neq 0, [\mathbb{Q}(\alpha\epsilon^a) : \mathbb{Q}] = st$ and*

$$|F_n(x, y)| \leq m.$$

Then

$$\max\{\log |x|, \log |y|, |n|\} \leq \kappa \log m.$$

19th International Mathematics Conference Bangladesh Mathematical Society

Linear recurrence sequences: an introduction

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