LIOUVILLE NUMBERS, LIOUVILLE SETS
AND LIOUVILLE FIELDS

K. SENTHIL KUMAR, R. THANGADURAI AND M. WALDSCHMIDT

Abstract. Following earlier work by É. Maillet 100 years ago, we introduce the definition of a Liouville set, which extends the definition of a Liouville number. We also define a Liouville field, which is a field generated by a Liouville set. Any Liouville number belongs to a Liouville set $S$ having the power of continuum and such that $\mathbb{Q} \cup S$ is a Liouville field.

Update: May 25, 2013

1. Introduction

For any integer $q$ and any real number $x \in \mathbb{R}$, we denote by

$$\|qx\| = \min_{m \in \mathbb{Z}} |qx - m|$$

the distance of $qx$ to the nearest integer. Following É. Maillet [3, 4], an irrational real number $\xi$ is said to be a Liouville number if, for each integer $n \geq 1$, there exists an integer $q_n \geq 2$ such that the sequence $(u_n(\xi))_{n \geq 1}$ of real numbers defined by

$$u_n(\xi) = -\log \|q_n \xi\| \log q_n$$

satisfies $\lim_{n \to \infty} u_n(\xi) = \infty$. If $p_n$ is the integer such that $\|q_n \xi\| = |q_n \xi - p_n|$, then the definition of $u_n(\xi)$ can be written

$$|q_n \xi - p_n| = \frac{1}{q_n u_n(\xi)}.$$

An equivalent definition is to saying that a Liouville number is a real number $\xi$ such that, for each integer $n \geq 1$, there exists a rational number $p_n/q_n$ with $q_n \geq 2$ such that

$$0 < |\xi - \frac{p_n}{q_n}| \leq \frac{1}{q_n^2}.$$

We denote by $\mathbb{L}$ the set of Liouville numbers. Following [4], any Liouville number is transcendental.

We introduce the notions of a Liouville set and of a Liouville field. They extend what was done by É. Maillet in Chap. III of [3].

Definition. A Liouville set is a subset $S$ of $\mathbb{L}$ for which there exists an increasing sequence $(q_n)_{n \geq 1}$ of positive integers having the following property: for any $\xi \in S$, there exists a sequence $(b_n)_{n \geq 1}$ of positive rational integers and there exist two positive constants $\kappa_1$ and $\kappa_2$ such that, for any sufficiently large $n$, (1)

$$1 \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n \xi\| \leq \frac{1}{q_n^{\kappa_2}}.$$
It would not make a difference if we were requesting these inequalities to hold for any \( n \geq 1 \): it suffices to change the constants \( \kappa_1 \) and \( \kappa_2 \).

**Definition.** A Liouville field is a field of the form \( \mathbb{Q}(S) \) where \( S \) is a Liouville set.

From the definitions, it follows that, for a real number \( \xi \), the following conditions are equivalent:

(i) \( \xi \) is a Liouville number.

(ii) \( \xi \) belongs to some Liouville set.

(iii) The set \( \{ \xi \} \) is a Liouville set.

(iv) The field \( \mathbb{Q}(\xi) \) is a Liouville field.

If we agree that the empty set is a Liouville set and that \( \mathbb{Q} \) is a Liouville field, then any subset of a Liouville set is a Liouville set, and also (see Theorem 1) any subfield of a Liouville field is a Liouville field.

**Definition.** Let \( q = (q_n)_{n \geq 1} \) be an increasing sequence of positive integers and let \( u = (u_n)_{n \geq 1} \) be a sequence of positive real numbers such that \( u_n \to \infty \) as \( n \to \infty \). Denote by \( S_{q,u} \) the set of \( \xi \in \mathbb{L} \) such that there exist two positive constants \( \kappa_1 \) and \( \kappa_2 \) and there exists a sequence \( (b_n)_{n \geq 1} \) of positive rational integers with

\[
1 \leq b_n \leq q_{\kappa_1} \quad \text{and} \quad \|b_n \xi\| \leq \frac{1}{q_{\kappa_2} u_n}.
\]

Denote by \( u \) the sequence \( u = (u_n)_{n \geq 1} := (1, 2, 3, \ldots) \) with \( u_n = n \) \( (n \geq 1) \). For any increasing sequence \( q = (q_n)_{n \geq 1} \) of positive integers, we denote by \( S_q \) the set \( S_{q,u} \).

Hence, by definition, a Liouville set is a subset of some \( S_q \). In section 2 we prove the following lemma:

**Lemma 1.** For any increasing sequence \( q \) of positive integers and any sequence \( u \) of positive real numbers which tends to infinity, the set \( S_{q,u} \) is a Liouville set.

Notice that if \( (m_n)_{n \geq 1} \) is an increasing sequence of positive integers, then for the subsequence \( q' = (q_{m_n})_{n \geq 1} \) of the sequence \( q \), we have \( S_{q',u} \supset S_{q,u} \).

**Example.** Let \( u = (u_n)_{n \geq 1} \) be a sequence of positive real numbers which tends to infinity. Define \( f : \mathbb{N} \to \mathbb{R}_{>0} \) by \( f(1) = 1 \) and

\[
f(n) = u_1 u_2 \cdots u_{n-1} \quad (n \geq 2),
\]

so that \( f(n+1)/f(n) = u_n \) for \( n \geq 1 \). Define the sequence \( q = (q_n)_{n \geq 1} \) by \( q_n = \lfloor 2^{f(n)} \rfloor \). Then, for any real number \( t > 1 \), the number

\[
\xi_t = \sum_{n \geq 1} \frac{1}{\lfloor t^{f(n)} \rfloor}
\]

belongs to \( S_{q,u} \). The set \( \{ \xi_t \mid t > 1 \} \) has the power of continuum, since \( \xi_{t_1} < \xi_{t_2} \) for \( t_1 > t_2 > 1 \).

The sets \( S_{q,u} \) have the following property (compare with Theorem I_3 in [3]):

**Theorem 1.** For any increasing sequence \( q \) of positive integers and any sequence \( u \) of positive real numbers which tends to infinity, the set \( \mathbb{Q} \cup S_{q,u} \) is a field.

We denote this field by \( K_{q,u} \), and by \( K_q \) for the sequence \( u = n \). From Theorem 1, it follows that a field is a Liouville field if and only if it is a subfield of some \( K_q \).

Another consequence is that, if \( S \) is a Liouville set, then \( \mathbb{Q}(S) \setminus \mathbb{Q} \) is a Liouville set.
It is easily checked that if
\[ \liminf_{n \to \infty} \frac{u_n}{u'_n} > 0, \]
then \( K_{q,u} \) is a subfield of \( K_{q,u'} \). In particular if
\[ \liminf_{n \to \infty} \frac{u_n}{n} > 0, \]
then \( K_{q,u} \) is a subfield of \( K_q \), while if
\[ \limsup_{n \to \infty} \frac{u_n}{n} < +\infty \]
then \( K_q \) is a subfield of \( K_{q,u} \).

If \( R \in \mathbb{Q}(X_1, \ldots, X_\ell) \) is a rational fraction and if \( \xi_1, \ldots, \xi_\ell \) are elements of a Liouville set \( S \) such that \( \eta = R(\xi_1, \ldots, \xi_\ell) \) is defined, then Theorem 1 implies that \( \eta \) is either a rational number or a Liouville number, and in the second case \( S \cup \{ \eta \} \) is a Liouville set. For instance, if, in addition, \( R \) is not constant and \( \xi_1, \ldots, \xi_\ell \) are algebraically independent over \( \mathbb{Q} \), then \( \eta \) is a Liouville number and \( S \cup \{ \eta \} \) is a Liouville set. For \( \ell = 1 \), this yields:

**Corollary 1.** Let \( R \in \mathbb{Q}(X) \) be a rational fraction and let \( \xi \) be a Liouville number. Then \( R(\xi) \) is a Liouville number and \( \{ \xi, R(\xi) \} \) is a Liouville set.

We now show that \( S_{q,u} \) is either empty or else uncountable and we characterize such sets.

**Theorem 2.** Let \( q \) be an increasing sequence of positive integers and \( u = (u_n)_{n \geq 1} \) be an increasing sequence of positive real numbers such that \( u_{n+1} \geq u_n + 1 \). Then the Liouville set \( S_{q,u} \) is non-empty if and only if
\[ \limsup_{n \to \infty} \frac{\log q_{n+1}}{u_n \log q_n} > 0. \]
Moreover, if the set \( S_{q,u} \) is non empty, then it has the power of continuum.

Let \( t \) be an irrational real number which is not a Liouville number. By a result due to P. Erdős [1], we can write \( t = \xi + \eta \) with two Liouville numbers \( \xi \) and \( \eta \). Let \( q \) be an increasing sequence of positive integers and \( u \) be an increasing sequence of real numbers such that \( \xi \in S_{q,u} \). Since any irrational number in the field \( K_{q,u} \) is in \( S_{q,u} \), it follows that the Liouville number \( \eta = t - \xi \) does not belong to \( S_{q,u} \).

One defines a reflexive and symmetric relation \( R \) between two Liouville numbers by \( \xi R \eta \) if \( \{ \xi, \eta \} \) is a Liouville set. The equivalence relation which is induced by \( R \) is trivial, as shown by the next result, which is a consequence of Theorem 2.

**Corollary 2.** Let \( \xi \) and \( \eta \) be Liouville numbers. Then there exists a subset \( \vartheta \) of \( L \) having the power of continuum such that, for each such \( \varrho \in \vartheta \), both sets \( \{ \xi, \varrho \} \) and \( \{ \eta, \varrho \} \) are Liouville sets.

In [3], É Mailliet introduces the definition of Liouville numbers corresponding to a given Liouville number. However this definition depends on the choice of a given sequence \( q \) giving the rational approximations. This is why we start with a sequence \( q \) instead of starting with a given Liouville number.

The intersection of two nonempty Liouville sets maybe empty. More generally, we show that there are uncountably many Liouville sets \( S_q \) with pairwise empty intersections.
Proposition 1. For $0 < \tau < 1$, define $q^{(\tau)}_n$ as the sequence $(q^{(\tau)}_n)_{n \geq 1}$ with
\[ q^{(\tau)}_n = 2^{n! \lfloor n \tau \rfloor} \quad (n \geq 1). \]
Then the sets $S_{q^{(\tau)}}$, $0 < \tau < 1$, are nonempty (hence uncountable) and pairwise disjoint.

To prove that a real number is not a Liouville number is most often difficult. But to prove that a given real number does not belong to some Liouville set $S_{q^{(\tau)}}$ is easier. If $q'$ is a subsequence of a sequence $q$, one may expect that $S_{q'}$ may often contain strictly $S_q$. Here is an example.

Proposition 2. Define the sequences $q$, $q'$ and $q''$ by
\[ q_n = 2^{n!}, \quad q'_n = q_{2n} = 2^{(2n)!} \quad \text{and} \quad q''_n = q_{2n+1} = 2^{(2n+1)!} \quad (n \geq 1), \]
so that $q$ is the increasing sequence deduced from the union of $q'$ and $q''$. Let $\lambda_n$ be a sequence of positive integers such that
\[ \lim_{n \to \infty} \lambda_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\lambda_n}{n} = 0. \]
Then the number
\[ \xi := \sum_{n \geq 1} \frac{1}{q^{(2n-1)!}\lambda_n} \]
belongs to $S_{q'}$ but not to $S_{q''}$. Moreover
\[ S_q = S_{q'} \cap S_{q''}. \]

When $q$ is the increasing sequence deduced form the union of $q'$ and $q''$, we always have $S_q \subset S_{q'} \cap S_{q''}$: Proposition 1 gives an example where $S_{q'} \neq \emptyset$ and $S_{q''} \neq \emptyset$, while $S_q$ is the empty set. In the example from Proposition 2 the set $S_q$ coincides with $S_{q'} \cap S_{q''}$. This is not always the case.

Proposition 3. There exists two increasing sequences $q'$ and $q''$ of positive integers with union $q$ such that $S_q$ is a strict nonempty subset of $S_{q'} \cap S_{q''}$.

Also, we prove that given any increasing sequence $q$, there exists a subsequence $q'$ of $q$ such that $S_q$ is a strict subset of $S_{q'}$. More generally, we prove

Proposition 4. Let $u = (u_n)_{n \geq 1}$ be a sequence of positive real numbers such that for every $n \geq 1$, we have $\sqrt{u_{n+1}} \leq u_n + 1 \leq u_{n+1}$. Then any increasing sequence $q$ of positive integers has a subsequence $q'$ for which $S_{q', u}$ strictly contains $S_{q, u}$. In particular, for any increasing sequence $q$ of positive integers has a subsequence $q'$ for which $S_{q'}$ is strictly contains $S_q$.

Proposition 5. The sets $S_{2, u}$ are not $G_\delta$ subsets of $\mathbb{R}$. If they are non empty, then they are dense in $\mathbb{R}$.

The proof of lemma 1 is given in section 2, the proof of Theorem 1 in section 3, the proof of Theorem 2 in section 4, the proof of Corollary 2 in section 5. The proofs of Propositions 1, 2, 3 and 4 are given in section 6 and the proof of Proposition 5 is given in section 7.
2. Proof of Lemma

Proof of Lemma 1. Given \( q \) and \( u \), define inductively a sequence of positive integers \((m_n)_{n \geq 1}\) as follows. Let \( m_1 \) be the least integer \( m \geq 1 \) such that \( u_m > 1 \). Once \( m_1, \ldots, m_{n-1} \) are known, define \( m_n \) as the least integer \( m > m_{n-1} \) for which \( u_m > n \). Consider the subsequence \( q' \) of \( q \) defined by \( q'_n = q_{m_n} \). Then \( S_{q', u} \subset S_{q, u} \), hence \( S_{q, u} \) is a Liouville set.

Remark 1. In the definition of a Liouville set, if assumption (1) is satisfied for some \( \kappa_1 \), then it is also satisfied with \( \kappa_1 \) replaced by any \( \kappa'_1 > \kappa_1 \). Hence there is no loss of generality to assume \( \kappa_1 > 1 \). Then, in this definition, one could add to (1) the condition \( q_n \leq b_n \leq q_{\kappa_1 n} \). Indeed, if, for some \( n \), we have \( b_n < q_n \), then we set \( b'_n = \lfloor q_n b_n \rfloor b_n \), so that \( b_n \leq b'_n \leq q_n + b_n \leq 2q_n \).

Denote by \( a_n \) the nearest integer to \( b_n \xi \) and set \( a'_n = \lceil q_n b_n \rceil a_n \).

Then, for \( \kappa'_2 < \kappa_2 \) and, for sufficiently large \( n \), we have

\[
|b_n' \xi - a'_n| = \left\lfloor \frac{q_n}{b_n} \right\rfloor |b_n \xi - a_n| \leq \frac{q_n}{q_{\kappa_2 n}} \leq \frac{1}{(q_n)^{\kappa'_2 n}}.
\]

Hence condition (1) can be replaced by

\[
q_n \leq b_n \leq q_{\kappa_1 n} \quad \text{and} \quad \|b_n \xi\| \leq \frac{1}{q_{\kappa_2 n}}.
\]

Also, one deduces from Theorem 2 that the sequence \((b_n)_{n \geq 1}\) is increasing for sufficiently large \( n \). Note also that same way we can assume that

\[
q_n \leq b_n \leq q_{\kappa_1 n} \quad \text{and} \quad \|b_n \xi\| \leq \frac{1}{q_{\kappa_2 n}}.
\]

3. Proof of Theorem

We first prove the following:

Lemma 2. Let \( q \) be an increasing sequence of positive integers and \( u = (u_n)_{n \geq 1} \) be an increasing sequence of real numbers. Let \( \xi \in S_{q', u} \). Then \( 1/\xi \in S_{q, u} \).

As a consequence, if \( S \) is a Liouville set, then, for any \( \xi \in S \), the set \( S \cup \{1/\xi\} \) is a Liouville set.

Proof of Lemma 2. Let \( q = (q_n)_{n \geq 1} \) be an increasing sequence of positive integers such that, for sufficiently large \( n \),

\[
\|b_n \xi\| \leq q_{-u_n}^{-1},
\]

where \( b_n \leq q_{\kappa_1 n} \). Write \( \|b_n \xi\| = |b_n \xi - a_n| \) with \( a_n \in \mathbb{Z} \). Since \( \xi \notin \mathbb{Q} \), the sequence \((|a_n|)_{n \geq 1}\) tends to infinity; in particular, for sufficiently large \( n \), we have \( a_n \neq 0 \). Writing

\[
\frac{1}{\xi} - \frac{b_n}{a_n} = -\frac{b_n}{\xi a_n} \left( \xi - \frac{a_n}{b_n} \right),
\]

so that
one easily checks that, for sufficiently large $n$,

$$\|a_n\xi^{-1}\| = |a_n|^{-u_n/2} \quad \text{and} \quad 1 \leq |a_n| < b_n^2 \leq q_n^{2\kappa_1}.$$  

\[ \text{square root} \]

\[ \text{square root} \]

**Proof of Theorem**

Let us check that for $\xi$ and $\xi'$ in $Q \cup S_{\mathbb{Q},u}$, we have $\xi - \xi' \in Q \cup S_{\mathbb{Q},u}$ and $\xi\xi' \in Q \cup S_{\mathbb{Q},u}$. Clearly, it suffices to check

1. For $\xi$ in $S_{\mathbb{Q},u}$ and $\xi'$ in $Q$, we have $\xi - \xi' \in S_{\mathbb{Q},u}$ and $\xi\xi' \in S_{\mathbb{Q},u}$.

2. For $\xi$ in $S_{\mathbb{Q},u}$ and $\xi'$ in $S_{\mathbb{Q},u}$ with $\xi - \xi' \notin Q$, we have $\xi - \xi' \in S_{\mathbb{Q},u}$.

3. For $\xi$ in $S_{\mathbb{Q},u}$ and $\xi'$ in $S_{\mathbb{Q},u}$ with $\xi\xi' \notin Q$, we have $\xi\xi' \in S_{\mathbb{Q},u}$.

The idea of the proof is as follows. When $\xi \in S_{\mathbb{Q},u}$ is approximated by $a_n/b_n$ and when $\xi' = r/s \in Q$, then $\xi - \xi'$ is approximated by $(sa_n - rb_n)/b_n$ and $\xi\xi'$ by $ra_n/b_n$. When $\xi \in S_{\mathbb{Q},u}$ is approximated by $a_n/b_n$ and $\xi' \in S_{\mathbb{Q},u}$ by $a'_n/b'_n$, then $\xi - \xi'$ is approximated by $(a_n b'_n - a'_n b_n)/b_n b'_n$ and $\xi\xi'$ by $a_n a'_n/b_n b'_n$. The proofs which follow amount to writing down carefully these simple observations.

Let $\xi'' = \xi - \xi'$ and $\xi^* = \xi\xi'$. Then the sequence $(a''_n)$ and $(b''_n)$ are corresponding to $\xi''$; similarly $(a'_n)$ and $(b'_n)$ correspond to $\xi^*$.

Here is the proof of (1). Let $\xi \in S_{\mathbb{Q},u}$ and $\xi' = r/s \in Q$, with $r$ and $s$ in $Z$, $s > 0$. There are two constants $\kappa_1$ and $\kappa_2$ and there are sequences of rational integers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ such that

$$1 \leq b_n \leq q_n^{\kappa_1} \quad \text{and} \quad 0 < |b_n \xi - a| \leq \frac{1}{q_n^{\kappa_2} u_n}.$$  

Let $\tilde{\kappa}_1 \geq \kappa_1$ and $\tilde{\kappa}_2 \geq \kappa_2$. Then,

$$b''_n = b''_n = s b_n,$$

$$a''_n = s a_n - r b_n,$$

$$a'_n = r a_n.$$

Then one easily checks that, for sufficiently large $n$, we have

$$0 < |b_n \xi'' - a''_n| = s |b_n \xi - a| \leq \frac{1}{q_n^{\kappa_2} u_n},$$

$$0 < |b'_n \xi^* - a'_n| = |r| |b_n \xi - a| \leq \frac{1}{q_n^{\kappa_2} u_n}.$$  

Here is the proof of (2) and (3). Let $\xi$ and $\xi'$ be in $S_{\mathbb{Q},u}$. There are constants $\kappa'_1$, $\kappa''_1$, $\kappa''_2$ and $\kappa''_2$ and there are sequences of rational integers $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$, $(a'_n)_{n \geq 1}$ and $(b'_n)_{n \geq 1}$ such that

$$1 \leq b_n \leq q_n^{\kappa'_1} \quad \text{and} \quad 0 < |b_n \xi - a| \leq \frac{1}{q_n^{\kappa''_2} u_n},$$

$$1 \leq b'_n \leq q_n^{\kappa''_1} \quad \text{and} \quad 0 < |b'_n \xi - a'| \leq \frac{1}{q_n^{\kappa''_2} u_n}.$$  

Define $\tilde{\kappa}_1 = \kappa'_1 + \kappa''_1$ and let $\tilde{\kappa}_2 > 0$ satisfy $\tilde{\kappa}_2 < \min\{\kappa'_1, \kappa''_1\}$. Set

$$b''_0 = b''_0 = b_n b_n,'$$

$$a''_0 = a_n b'_n - b_n a'_n,$$

$$a'_0 = a_n a'_n.$$
Then for sufficiently large $n$, we have
\[ b_n^\nu \xi^\nu - a_n^\nu = b_n^\nu (b_n \xi - a_n) - b_n (b_n^\nu \xi^\nu - a_n^\nu) \]
and
\[ b_n^\nu \xi^\nu - a_n^\nu = b_n \xi (b_n^\nu \xi^\nu - a_n^\nu) + a_n^\nu (b_n \xi - a_n), \]
hence
\[ |b_n^\nu \xi^\nu - a_n^\nu| \leq \frac{1}{q_n^{2a_n}} \]
and
\[ |b_n^\nu \xi^\nu - a_n^\nu| \leq \frac{1}{q_n^{3a_n}}. \]

Also we have
\[ 1 \leq b_n^\nu \leq q_n^{\kappa_1} \quad \text{and} \quad 1 \leq b_n^\nu \leq q_n^{\kappa_1}. \]
The assumption $\xi - \xi' \notin \mathbb{Q}$ (resp $\xi \xi' \notin \mathbb{Q}$) implies $b_n^\nu \xi^\nu \xi' \notin a_n^\nu$ (respectively, $b_n^\nu \xi^\nu \xi' \notin a_n^\nu$). Hence $\xi - \xi'$ and $\xi \xi'$ are in $\mathbb{S}_{\mathbb{Q}, \mathbb{K}}$. This completes the proof of (2) and (3).

It follows from (1), (2) and (3) that $\mathbb{Q} \cup \mathbb{S}_{\mathbb{Q}, \mathbb{K}}$ is a ring.

Finally, if $\xi \in \mathbb{Q} \cup \mathbb{S}_{\mathbb{Q}, \mathbb{K}}$ is not 0, then $1/\xi \in \mathbb{Q} \cup \mathbb{S}_{\mathbb{Q}, \mathbb{K}}$ by Lemma \[ \] This completes the proof of Theorem \[ \]

**Remark 2.** Since the field $\mathbb{K}_{\mathbb{Q}}$ does not contain irrational algebraic numbers, 2 is not a square in $\mathbb{K}_{\mathbb{Q}}$. For $\xi \in \mathbb{S}_{\mathbb{Q}, \mathbb{K}}$, it follows that $\eta = 2\xi^2$ is an element in $\mathbb{S}_{\mathbb{Q}, \mathbb{K}}$, which is not the square of an element in $\mathbb{S}_{\mathbb{Q}, \mathbb{K}}$. According to \[ \], we can write $\sqrt{2} = \xi_1 \xi_2$ with two Liouville numbers $\xi_1, \xi_2$; then the set $\{\xi_1, \xi_2\}$ is not a Liouville set.

Let $N$ be a positive integer such that $N$ cannot be written as a sum of two squares of an integer. Let us show that, for $\nu \in \mathbb{S}_{\mathbb{Q}, \mathbb{K}}$, the Liouville number $N \nu^2 \in \mathbb{S}_{\mathbb{Q}, \mathbb{K}}$ is not the sum of two squares of elements in $\mathbb{S}_{\mathbb{Q}, \mathbb{K}}$. Dividing by $\nu^2$, we are reduced to show that the equation $N = \xi^2 + (\xi')^2$ has no solution $(\xi, \xi')$ in $\mathbb{S}_{\mathbb{Q}, \mathbb{K}} \times \mathbb{S}_{\mathbb{Q}, \mathbb{K}}$. Otherwise, we would have, for suitable positive constants $\kappa_1$ and $\kappa_2$,
\[ \left| \xi - \frac{a_n}{b_n} \right| \leq \frac{1}{q_n^{2\kappa_1 a_n + 1}}, \quad 1 \leq b_n \leq q_n^{\kappa_1}, \]
\[ \left| \xi' - \frac{a_n'}{b_n} \right| \leq \frac{1}{q_n^{2\kappa_2 b_n + 1}}, \quad 1 \leq b_n' \leq q_n^{\kappa_2}, \]
hence
\[ \left| \xi^2 - \frac{a_n^2}{b_n^2} \right| \leq \frac{2(|\xi| + 1)}{q_n^{2\kappa_1 a_n + 1}}, \quad \left| (\xi')^2 - \frac{(a_n')^2}{(b_n')^2} \right| \leq \frac{2(|\xi'| + 1)}{q_n^{2\kappa_2 b_n + 1}} \]
and
\[ \left| \xi^2 + (\xi')^2 - \frac{(a_n b_n')^2 + (a_n' b_n)^2}{(b_n b_n')^2} \right| \leq \frac{2(|\xi| + |\xi'| + 1)}{q_n^{2\kappa a_n + 1}}. \]
Using $\xi^2 + (\xi')^2 = N$, we deduce
\[ |N(b_n b_n')^2 - (a_n b_n')^2 - (a_n' b_n)^2| < 1. \]
The left hand side is an integer, hence it is 0:
\[ N(b_n b_n')^2 = (a_n b_n')^2 + (a_n' b_n)^2. \]
This is impossible, since the equation $x^2 + y^2 = Nz^2$ has no solution in positive rational integers.
Therefore, if we write \( N = \xi^2 + (\xi')^2 \) with two Liouville numbers \( \xi, \xi' \), which is possible by the above mentioned result from P. Erdős [1], then the set \( \{ \xi, \xi' \} \) is not a Liouville set.

4. Proof of Theorem 2

We first prove the following lemma which will be required for the proof of part (ii) of Theorem 2.

**Lemma 3.** Let \( \xi \) be a real number, \( n, q \) and \( q' \) be positive integers. Assume that there exist rational integers \( p \) and \( p' \) such that \( \frac{p}{q}, \frac{p'}{q'} \not= \frac{p'}{q} \) and

\[
|q\xi - p| \leq \frac{1}{q^u_n}, \quad |q'\xi - p'| \leq \frac{1}{(q')^u_{n+1}}.
\]

Then we have either \( q' \geq q^{u_n} \) or \( q \geq (q')^{u_n} \).

**Proof of Lemma 3.** From the assumptions we deduce

\[
\frac{1}{qq'} \leq \frac{|pq' - p'q|}{qq'} \leq \left| \frac{p}{q} - \frac{p'}{q'} \right| \leq \frac{1}{q_{u_n+1}} + \frac{1}{(q')^{u_{n+1}}},
\]

hence

\[
q^{u_n}(q')^{u_{n+1}} \leq (q')^{u_{n+2}} + q^{u_{n+1}}.
\]

If \( q < q' \), we deduce

\[
q^{u_n} \leq q' + \left( \frac{q}{q'} \right)^{u_{n+1}} < q' + 1.
\]

Assume now \( q \geq q' \). Since the conclusion of Lemma 3 is trivial if \( u_n = 1 \) and also if \( q' = 1 \), we assume \( u_n > 1 \) and \( q' \geq 2 \). From

\[
q^{u_n}(q')^{u_{n+1}} \leq (q')^{u_{n+2}} + q^{u_{n+1}} \leq (q')^2 q^{u_n} + q^{u_{n+1}}
\]

we deduce

\[
(q')^{u_{n+1}} - (q')^2 \leq q.
\]

From \( (q')^{u_{n-1}} > (q')^{u_{n-2}} \) we deduce \( (q')^{u_{n-1}} \geq (q')^{u_{n-2}} + 1 \), which we write as

\[
(q')^{u_{n+1}} - (q')^2 \geq (q')^{u_n}.
\]

Finally

\[
(q')^{u_n} \leq (q')^{u_{n+1}} - (q')^2 \leq q.
\]

\[ \square \]

**Proof of Theorem 2.** Suppose \( \limsup_{n \to \infty} \frac{\log q_{n+1}}{u_n \log q_n} = 0 \). Then, we get,

\[
\lim_{n \to \infty} \frac{\log q_{n+1}}{u_n \log q_n} = 0.
\]

Suppose \( S_{2,u} \neq \emptyset \). Let \( \xi \in S_{2,u} \). From Remark 1 it follows that there exists a sequence \((b_n)_{n \geq 1}\) of positive integers and there exist two positive constants \( \kappa_1 \) and \( \kappa_2 \) such that, for any sufficiently large \( n \),

\[
q_n \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq q_n^{-\kappa_2 u_n}.
\]
Let $n_0$ be an integer $\geq \kappa_1$ such that these inequalities are valid for $n \geq n_0$ and such that, for $n \geq n_0$, $q_{n+1}^\kappa < q_n^a$ (by the assumption). Since the sequence $(q_n)_{n \geq 1}$ is increasing, we have $q_{n+1}^\kappa < q_n^a$ for $n \geq n_0$. From the choice of $n_0$ we deduce
\[ b_{n+1} \leq q_{n+1}^\kappa < q_n^a \leq b_n \]
and
\[ b_n \leq q_{n+1}^\kappa < q_n^a \leq b_{n+1} \]
for any $n \geq n_0$. Denote by $a_n$ (resp. $a_{n+1}$) the nearest integer to $\xi b_n$ (resp. to $\xi b_{n+1}$). Lemma 3 with $q$ replaced by $b_n$ and $q'$ by $b_{n+1}$ implies that for each $n \geq n_0$,
\[ \frac{a_n}{b_n} = \frac{a_{n+1}}{b_{n+1}}. \]
This contradicts the assumption that $\xi$ is irrational. This proves that $S_{q,a} = \emptyset$.

Conversely, assume
\[ \limsup_{n \to \infty} \frac{\log q_{n+1}}{a_n \log q_n} > 0. \]
Then there exists $\vartheta > 0$ and there exists a sequence $(N_{\ell})_{\ell \geq 1}$ of positive integers such that
\[ q_{N_{\ell}} > q_{N_{\ell+1}}^{\vartheta(\vartheta N_{\ell+1})} \]
for all $\ell \geq 1$. Define a sequence $(\varepsilon_{\ell})_{\ell \geq 1}$ of positive integers by
\[ 2^{\varepsilon_{\ell}} \leq q_{N_{\ell}} < 2^{\varepsilon_{\ell}+1}. \]
Let $\varepsilon = (\varepsilon_{\ell})_{\ell \geq 1}$ be a sequence of elements in $\{-1, 1\}$. Define
\[ \xi_{\varepsilon} = \sum_{\ell \geq 1} \varepsilon_{\ell} \frac{\varepsilon_{\ell}}{2^{\varepsilon_{\ell}}}. \]
It remains to check that $\xi_{\varepsilon} \in S_{q,a}$ and that distinct $\varepsilon$ produce distinct $\xi_{\varepsilon}$.

Let $\kappa_1 = 1$ and let $\kappa_2$ be in the interval $0 < \kappa_2 < \vartheta$. For sufficiently large $n$, let $\ell$ be the integer such that $N_{\ell-1} \leq n < N_{\ell}$. Set
\[ b_n = 2^{\varepsilon_{\ell}-1}, \quad a_n = \sum_{h=1}^{\ell-1} e_h 2^{\varepsilon_{\ell-1} - c_h}, \quad r_n = \frac{a_n}{b_n}. \]

We have
\[ \frac{1}{2^{\varepsilon_{\ell}}} < |\xi_{\varepsilon} - r_n| = \left| \xi_{\varepsilon} - \sum_{h \geq \ell} \frac{e_h}{2^{c_h}} \right| \leq \frac{2}{2^{\varepsilon_{\ell}}}. \]

Since $\kappa_2 < \vartheta$, $n$ is sufficiently large and $n \leq N_{\ell} - 1$, we have
\[ 4q_n^{\kappa_2 u_n} \leq 4q_{N_{\ell}-1}^{\kappa_2 u_{N_{\ell}-1}} \leq q_{N_{\ell}}, \]
hence
\[ \frac{2}{2^{\varepsilon_{\ell}}} < \frac{4}{q_{N_{\ell}}} < \frac{1}{q_n^{\kappa_2 u_n}} \]
for sufficiently large $n$. This proves $\xi_{\varepsilon} \in S_{q,a}$ and hence $S_{q,a}$ is not empty.

Finally, if $\varepsilon$ and $\varepsilon'$ are two elements of $\{-1, +1\}^N$ for which $e_h = e'_h$ for $1 \leq h < \ell$ and, say, $e_\ell = -1$, $e'_\ell = 1$, then
\[ \xi_{\varepsilon} < \sum_{h=1}^{\ell-1} \frac{e_h}{2^{c_h}} < \xi_{\varepsilon'}, \]
hence $\xi_\varrho \neq \xi_\varrho'$. This completes the proof of Theorem 2. 

5. PROOF OF COROLLARY 2

The proof of Corollary 2 as a consequence of Theorem 2 relies on the following elementary lemma.

Lemma 4. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be two increasing sequences of positive integers. Then there exists an increasing sequence of positive integers $(q_n)_{n \geq 1}$ satisfying the following properties:

(i) The sequence $(q_{2n})_{n \geq 1}$ is a subsequence of the sequence $(a_n)_{n \geq 1}$.

(ii) The sequence $(q_{2n+1})_{n \geq 0}$ is a subsequence of the sequence $(b_n)_{n \geq 1}$.

(iii) For $n \geq 1$, $q_{n+1} \geq q_n^2$.

Proof of Lemma 4. We construct the sequence $(q_n)_{n \geq 1}$ inductively, starting with $q_1 = b_1$ and with $q_2$ the least integer $a_i$ satisfying $a_i \geq b_1$. Once $q_n$ is known for some $n \geq 2$, we take for $q_{n+1}$ the least integer satisfying the following properties:

- $q_{n+1} \in \{a_1, a_2, \ldots\}$ if $n$ is odd, $q_{n+1} \in \{b_1, b_2, \ldots\}$ if $n$ is even.
- $q_{n+1} \geq q_n^2$.

Proof of Corollary 2. Let $\xi$ and $\eta$ be Liouville numbers. There exist two sequences of positive integers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$, which we may suppose to be increasing, such that

$$\|a_n\xi\| \leq a_n^{-n} \quad \text{and} \quad \|b_n\eta\| \leq b_n^{-n}$$

for sufficiently large $n$. Let $q = (q_n)_{n \geq 1}$ be an increasing sequence of positive integers satisfying the conclusion of Lemma 4. According to Theorem 2, the Liouville set $S_q$ is not empty. Let $q \in S_q$. Denote by $q'$ the subsequence $(q_2, q_4, \ldots, q_{2n}, \ldots)$ of $q$ and by $q''$ the subsequence $(q_1, q_3, \ldots, q_{2n+1}, \ldots)$. We have $q \in S_q = S_{q'} \cap S_{q''}$. Since the sequence $(a_n)_{n \geq 1}$ is increasing, we have $q_{2n} \geq a_n$, hence $\xi \in S_{q'}$. Also, since the sequence $(b_n)_{n \geq 1}$ is increasing, we have $q_{2n+1} \geq b_n$, hence $\eta \in S_{q''}$. Finally, $\xi$ and $\eta$ belong to the Liouville set $S_{q'}$, while $\eta$ and $\eta$ belong to the Liouville set $S_{q''}$. 

6. PROOFS OF PROPOSITIONS 1, 2, 3 AND 4

Proof of Proposition 1. The fact that for $0 < \tau < 1$ the set $S_{q^{(\tau)}}$ is not empty follows from Theorem 2 since

$$\lim_{n \to \infty} \frac{\log q_{n+1}^{(\tau)}}{n \log q_n^{(\tau)}} = 1.$$ 

In fact, if $(e_n)_{n \geq 1}$ is a bounded sequence of integers with infinitely many nonzero terms, then

$$\sum_{n \geq 1} \frac{e_n}{q_n^{(\tau)}} \in S_{q^{(\tau)}}.$$

Let $0 < \tau_1 < \tau_2 < 1$. For $n \geq 1$, define

$$q_{2n} = q_n^{(\tau_1)} = 2^n [n^{\tau_1}] \quad \text{and} \quad q_{2n+1} = q_n^{(\tau_2)} = 2^n [n^{\tau_2}].$$

One easily checks that $(q_n)_{n \geq 1}$ is an increasing sequence with

$$\frac{\log q_{2n+1}}{n \log q_{2n}} \to 0 \quad \text{and} \quad \frac{\log q_{2n+2}}{n \log q_{2n+1}} \to 0.$$
From Theorem 2 one deduces $S_{g(r_1)} \cap S_{g(r_2)} = \emptyset$. 

**Proof of Proposition 2.** For sufficiently large $n$, define 

$$a_n = \sum_{m=1}^{n} 2^{(2n)!-(2m-1)!}\lambda_m.$$ 

Then 

$$\frac{1}{q_{2n}^{(2n)!}\lambda_{n+1}} < \xi = \frac{a_n}{q_{2n}} = \sum_{m=n+1}^{\infty} \frac{1}{q_{2n}^{(2m-1)!}\lambda_m} \leq \frac{2}{q_{2n}^{(2m+1)!}\lambda_{n+1}}.$$ 

The right inequality with the lower bound $\lambda_{n+1} \geq 1$ proves that $\xi \in S_{g''}$. 

Let $\kappa_1$ and $\kappa_2$ be positive numbers, $n$ a sufficiently large integer, $s$ an integer in the interval $q_{2n+1} \leq s \leq q_{2n+1}$ and $r$ an integer. Since $\lambda_{n+1} < \kappa_2 n$ for sufficiently large $n$, we have 

$$q_{2n}^{(2n)!}\lambda_{n+1} < q_{2n}^{\kappa_2 n(2n)+1} = q_{2n+1}^{\kappa_2 n}.$$ 

Therefore, if $r/s = a_n/q_{2n}$, then 

$$\left| \xi - \frac{r}{s} \right| = \left| \xi - \frac{a_n}{q_{2n}} \right| > \frac{1}{q_{2n}^{(2n)!}\lambda_{n+1}} > \frac{1}{s^{\kappa_2 n}}.$$ 

On the other hand, for $r/s \neq a_n/q_{2n}$, we have 

$$\left| \xi - \frac{r}{s} \right| > \frac{2}{q_{2n}^{(2n)!}\lambda_{n+1}}.$$ 

Since $\lambda_n \to \infty$, for sufficiently large $n$ we have 

$$4q_{2n}s \leq 4q_{2n}q_{2n+1}^{\kappa_1} = 4q_{2n}^{1+n_1(2n+1)} \leq q_{2n}^{(2n)!}\lambda_{n+1}$$ 

hence 

$$\frac{2}{q_{2n}^{(2n)!}\lambda_{n+1}} < \frac{1}{2q_{2n}s}.$$ 

Further 

$$2q_{2n} < q_{2n+1} < q_{2n+1}^{\kappa_2 n-1} \leq s^{\kappa_2 n-1}.$$ 

Therefore 

$$\left| \xi - \frac{r}{s} \right| > \frac{2}{q_{2n}s} > \frac{1}{s^{\kappa_2 n-1}},$$ 

which shows that $\xi \notin S_{g''}$. 

**Proof of Proposition 3.** Let $(\lambda_s)_{s \geq 0}$ be a strictly increasing sequence of positive rational integers with $\lambda_0 = 1$. Define two sequences $(n'_k)_{k \geq 1}$ and $(n''_k)_{k \geq 1}$ of positive integers as follows. The sequence $(n'_k)_{k \geq 1}$ is the increasing sequence of the positive integers $n$ for which there exists $s \geq 0$ with $\lambda_{2s} \leq n < \lambda_{2s+1}$, while $(n''_k)_{k \geq 1}$ is the increasing sequence of the positive integers $n$ for which there exists $s \geq 0$ with $\lambda_{2s+1} \leq n < \lambda_{2s+2}$. 

For $s \geq 0$ and $\lambda_{2s} \leq n < \lambda_{2s+1}$, set 

$$k = n - \lambda_{2s} + \lambda_{2s-1} - \lambda_{2s-2} + \cdots + \lambda_1.$$ 

Then $n = n'_k$. 

For $s \geq 0$ and $\lambda_{2s+1} \leq n < \lambda_{2s+2}$, set 

$$h = n - \lambda_{2s+1} + \lambda_{2s} - \lambda_{2s-1} + \cdots - \lambda_1 + 1.$$
Then \( n = n_h'' \).

For instance, when \( \lambda_n = s + 1 \), the sequence \((n_k')_{k \geq 1}\) is the sequence \((1, 3, 5, \ldots)\) of odd positive integers, while \((n_h''')_{h \geq 1}\) is the sequence \((2, 4, 6, \ldots)\) of even positive integers. Another example is \( \lambda_n = \sqrt{n} \), which occurs in the paper [1] by Erdős.

In general, for \( n = \lambda_{2s} \), we write \( n = n_k'(s) \) where

\[
k(s) = \lambda_{2s} - \lambda_{2s-2} - \cdots - \lambda_1 < \lambda_{2s-1}.
\]

Notice that \( \lambda_{2s} - 1 = n''_h \) with \( h = \lambda_{2s} - k(s) \).

Next, define two increasing sequences \((d_n)_{n \geq 1}\) and \( q = (q_n)_{n \geq 1}\) of positive integers by induction, with \( d_1 = 2 \),

\[
d_{n+1} = \begin{cases} 
kd_n & \text{if } n = n_k', \\
hd_n & \text{if } n = n_h'' 
\end{cases}
\]

for \( n \geq 1 \) and \( q_n = 2^{d_n} \). Finally, let \( q' = (q_k')_{k \geq 1} \) and \( q'' = (q_h'')_{h \geq 1} \) be the two subsequences of \( q \) defined by

\[
q_k' = q_{n_k'}, \quad k \geq 1, \quad q_h'' = q_{n_h''}, \quad h \geq 1.
\]

Hence \( q \) is the union of these two subsequences. Now we check that the number

\[
\xi = \sum_{n \geq 1} \frac{1}{q_n}
\]

belongs to \( S_{q'} \cap S_{q''} \). Note that by Theorem 2 that \( S_{q'} \neq \emptyset \) as \( S_{q'} \neq \emptyset \) and \( S_{q''} \neq \emptyset \).

Define

\[
a_n = \sum_{m=1}^{n} 2^{d_n - d_m}.
\]

Then

\[
\frac{1}{q_{n+1}} < \xi - \frac{a_n}{q_n} = \sum_{m \geq n+1} \frac{1}{q_m} < \frac{2}{q_{n+1}}.
\]

If \( n = n_k' \), then

\[
\left| \xi - \frac{a_n'}{q_k'} \right| < \frac{2}{(q_k')^k}
\]

while if \( n = n_h'' \), then

\[
\left| \xi - \frac{a_n''}{q_h''} \right| < \frac{2}{(q_h'')^h}.
\]

This proves \( \xi \in S_{q'} \cap S_{q''} \).

Now, we choose \( \lambda_s = 2^{2^s} \) for \( s \geq 2 \) and we prove that \( \xi \) does not belong to \( S_{q} \).

Notice that \( \lambda_{2s-1} = \sqrt{\lambda_{2s}} \). Let \( n = \lambda_{2s} = n_k(s) \). We have \( k(s) < \sqrt{\lambda_{2s}} \) and

\[
\left| \xi - \frac{a_n}{q_n} \right| > \frac{1}{q_{n+1}} = \frac{1}{q_k(s)} > \frac{1}{q_n^{\sqrt{n}}}.
\]

Let \( \kappa_1 \) and \( \kappa_2 \) be two positive real numbers and assume \( s \) is sufficiently large. Further, let \( u/v \in \mathbb{Q} \) with \( v \leq q_n^{\kappa_2} \). If \( u/v = a_n/q_n \), then

\[
\left| \xi - \frac{u}{v} \right| = \left| \xi - \frac{a_n}{q_n} \right| > \frac{1}{q_n^{\sqrt{n}}} > \frac{1}{q_n^{\kappa_2}}.
\]
On the other hand, if \( u/v \neq a_n/q_n \), then
\[
\left| \xi - \frac{u}{v} \right| \geq \left| \frac{u}{v} - \frac{a_n}{q_n} \right| = \left| \frac{a_n}{q_n} \right|
\]
with
\[
\left| \frac{u}{v} - \frac{a_n}{q_n} \right| \geq \frac{1}{vq_n} \geq \frac{1}{q_n^{n+1}} > \frac{2}{q_n^{2n}}
\]
and
\[
\left| \xi - \frac{a_n}{q_n} \right| < \frac{1}{q_n^{2n}}.
\]
Hence
\[
\left| \xi - \frac{u}{v} \right| > \frac{1}{q_n^{n+1}} \geq \frac{1}{q_n^{2n}}
\]
This proves Proposition \( \square \)

*Proof of Proposition* \( \square \) Let \( u = (u_n)_{n \geq 1} \) be a sequence of positive real numbers such that \( \sqrt{u_{n+1}} \leq u_n + 1 \leq u_{n+1} \). We prove more precisely that for any sequence \( q \) such that \( q_{n+1} > q_n^n \) for all \( n \geq 1 \), the sequence \( q' = (q_{2m+1})_{m \geq 1} \) has \( S_{q',u} \neq S_{q,u} \). This implies the proposition, since any increasing sequence has a subsequence satisfying \( q_{n+1} > q_n^n \).

Assuming \( q_{n+1} > q_n^n \) for all \( n \geq 1 \), we define
\[
d_n = \begin{cases} 
q_n & \text{for even } n, \\
q_{n-1}^{\sqrt{n-1}} & \text{for odd } n.
\end{cases}
\]
We check that the number
\[
\xi = \sum_{n \geq 1} \frac{1}{d_n}
\]
satisfies \( \xi \in S_{q',u} \) and \( \xi \notin S_{q,u} \).

Set \( b_n = d_1 d_2 \cdots d_n \) and
\[
a_n = \sum_{m=1}^{n} \frac{b_n}{d_m} = \sum_{m=1}^{n} \prod_{1 \leq i \leq m, i \neq m} d_i,
\]
so that
\[
\xi - \frac{a_n}{b_n} = \sum_{m \geq n+1} \frac{1}{d_m}.
\]
It is easy to check from the definition of \( d_n \) and \( q_n \) that we have, for sufficiently large \( n \),
\[
b_n \leq q_1 \cdots q_n \leq q_n^{n-1} q_n \leq q_n^2
\]
and
\[
\frac{1}{d_{n+1}} \leq \xi - \frac{a_n}{b_n} \leq \frac{2}{d_{n+1}}.
\]
For odd \( n \), since \( d_{n+1} = q_{n+1} \geq q_n^n \), we deduce
\[
\left| \xi - \frac{a_n}{b_n} \right| \leq \frac{2}{q_n^n},
\]
hence \( \xi \in S_{q',u} \).
For even \( n \), we plainly have
\[
\left| \xi - \frac{a_n}{b_n} \right| > \frac{1}{d_{n+1}} = \frac{1}{q_n^{\lfloor \sqrt{n+1} \rfloor}}.
\]

Let \( \kappa_1 \) and \( \kappa_2 \) be two positive real numbers, and let \( n \) be sufficiently large. Let \( s \) be a positive integer with \( s \leq q_n^{\kappa_1} \) and let \( r \) be an integer. If \( r/s = a_n/b_n \), then
\[
\left| \xi - \frac{r}{s} \right| = \left| \xi - \frac{a_n}{b_n} \right| > \frac{1}{q_n^{\kappa_2 u_n}}.
\]

Assume now \( r/s \neq a_n/b_n \). From
\[
\left| \xi - \frac{r}{s} \right| \leq \frac{2}{q_n^{\lfloor \sqrt{n+1} \rfloor}} \leq \frac{1}{2q_n^{\kappa_1+2}},
\]
we deduce
\[
\frac{1}{q_n^{\kappa_1+2}} \leq \frac{1}{sb_n} \leq \left| \frac{r}{s} - \frac{a_n}{b_n} \right| \leq \left| \xi - \frac{r}{s} \right| + \left| \xi - \frac{a_n}{b_n} \right| \leq \left| \xi - \frac{r}{s} \right| + \frac{1}{2q_n^{\kappa_1+2}},
\]
hence
\[
\left| \xi - \frac{r}{s} \right| \geq \frac{1}{2q_n^{\kappa_1+2}} > \frac{1}{q_n^{\kappa_2 u_n}}.
\]
This completes the proof that \( \xi \notin S_{q,u} \).

\[ \square \]

7. Proof of Proposition 5

Proof of Proposition 5. If \( S_{q,u} \) is non empty, let \( \gamma \in S_{q,u} \). By Theorem \[1\] \( \gamma + \mathbb{Q} \) is contained in \( S_{q,u} \), hence \( S_{q,u} \) is dense in \( \mathbb{R} \).

Let \( t \) be an irrational real number which is not Liouville. Hence \( t \notin K_{q,u} \) and therefore, by Theorem \[1\] \( S_{q,u} \cap (t + S_{q,u}) = \emptyset \). This implies that \( S_{q,u} \) is not a \( G_{\delta} \) dense subset of \( \mathbb{R} \).

\[ \square \]

References