Special year in number theory at IMSc

Lectures on Multiple Zeta Values IMSC 2011

by

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1 Introduction to the results of F. Brown on Zagier's Conjecture

1.1 Zeta values: main Diophantine conjecture

Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

has been considered before Riemann by Euler for integer values of the variable s, both positive and negative ones. Among the many results he proved are

$$\zeta(2) = \frac{\pi^2}{6} \quad \text{and} \quad \frac{\zeta(2n)}{\zeta(2)^n} \in \mathbf{Q}$$

for any integer $n \geq 1$.

A quite ambitious goal is to determine the algebraic relations among the numbers

$$\pi,\ \zeta(3),\ \zeta(5), \dots, \zeta(2n+1), \dots$$

The expected answer is disappointingly simple: it is widely believed that there are no relations, which means that these numbers should be algebraically independent:

Conjecture 1. *For any $n \geq 0$ and any nonzero polynomial $P \in \mathbf{Z}[T_0, \dots, T_n]$,*

$$P\bigl(\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+1)\bigr) \neq 0.$$

If true, this property would mean that there is no interesting algebraic structure.

There are very few results on the arithmetic nature of these numbers, even less on their independence: it is known that π is a transcendental numbers, hence so are all $\zeta(2n)$, $n \geq 1$. It is also known that $\zeta(3)$ is irrational (Apéry, 1978), and that infinitely many $\zeta(2n+1)$ are irrational (further sharper results have been achieved by T. Rivoal and others – see [4] But so far it has not been disproved that all these numbers lie in the ring $\mathbf{Q}[\pi^2]$ (see the Open Problem 2).

1.2 Multizeta values: Zagier's conjecture

The situation changes drastically if we enlarge our set so as to include the so-called Multiple Zeta Values (MZV, also called *Polyzeta values, Euler-Zagier numbers* or *multiple harmonic series*):

$$\zeta(s_1,\ldots,s_k)=\sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

which are defined for k, $s_1,\ldots,s_k$ positive integers with $s_1\geq 2$. There are plenty of relations between them, providing a rich algebraic structure.

We shall call k the *length* of the tuple $\underline{s}=(s_1,\ldots,s_k)$ and $|s|:=s_1+\cdots+s_k|$ the *weight* of this tuple. There are 2^{p-2} tuples $\underline{s}$ of weight p with $(s_1\geq 2$ and $s_j\geq 1$ for $2\leq j\leq k$. The length k and the weight p are related by $k+1\leq p$.

One easily gets quadratic relations between MZV when one multiplies two such series: it is easy to express the product as a linear combination of MZV. We shall study this phenomenon in detail, but we just give one easy example. Splitting the set of (n,m) with $n\geq 1$ and $m\geq 1$ into three disjoint subsets with respectively $n>m$, $m>n$ and $n=m$, we deduce, for $s\geq 2$ and $s'\geq 2$,

$$\sum_{n\geq 1}n^{-s}\sum_{m\geq 1}m^{-s'}=\sum_{n>m\geq 1}n^{-s}m^{-s'}+\sum_{m>n\geq 1}m^{-s'}n^{-s}+\sum_{n\geq 1}n^{-s-s'},$$

which is the so–called *Nielsen Reflexion Formula*:

$$\zeta(s)\zeta(s')=\zeta(s,s')+\zeta(s',s)+\zeta(s+s')$$

for $s\geq 2$ and $s'\geq 2$. For instance,

$$\zeta(2)^2=2\zeta(2,2)+\zeta(4).$$

Such expressions of the product of two zeta values as a linear combination of zeta values, arising from the product of two series, will be called "stuffle relations".

They show that the $\mathbf{Q}$–vector space spanned by the multiple zeta values is in fact an algebra: a product of linear combinations of numbers of the form $\zeta(\underline{s})$ is again a linear combination of such numbers. Again, we stress that it has not been disproved that this algebra is $\mathbf{Q}[\pi^2]$. One should keep in mind that the following problem is still open when one is looking at the result on the formal symbols representing the MZV: many results on these symbols are known, but almost nothing is known on the kernel of the corresponding specialization which maps a symbol onto the corresponding real number.

We denote by $\mathfrak{Z}$ the $\mathbf{Q}$–vector space spanned by the numbers $\zeta(\underline{s})$, for $p\geq 2$ we denote by $\mathfrak{Z}_p$ the $\mathbf{Q}$-subspace of $\mathfrak{Z}$ spanned by the numbers $\zeta(\underline{s})$ with $|\underline{s}|=p$, for $k\geq 1$ we denote by $\mathcal{F}^k\mathfrak{Z}$ the $\mathbf{Q}$-subspace of $\mathfrak{Z}$ spanned by the numbers $\zeta(\underline{s})$ with $\underline{s}$ of length $\leq k$ and finally for $p\geq k+1\geq 2$ we denote by $\mathcal{F}^k\mathfrak{Z}_p$ the $\mathbf{Q}$-subspace of $\mathfrak{Z}$ spanned by the numbers $\zeta(\underline{s})$ with $|\underline{s}|$ of weight p and length $\leq k$. The inclusion $\mathcal{F}^k\mathfrak{Z}_p\subset\mathcal{F}^k\mathfrak{Z}\cap\mathfrak{Z}_p$ is plain, that there is equality is only a conjecture. It is also conjectured but not proved that the weight defines a graduation on $\mathfrak{Z}$. It is a fact that the subspaces $\mathcal{F}^k\mathfrak{Z}$ define an increasing filtration of the algebra $\mathfrak{Z}$ (see §5.4), but this filtration may be trivial: for instance, it could happen that

$$\mathcal{F}^k\mathfrak{Z}_p = \mathfrak{Z}_p = \mathfrak{Z} = \mathbf{Q}[\pi^2]$$

for all $k \geq 1$ and $p \geq 2$. For $p \geq 2$ the space $\mathcal{F}^1\mathfrak{Z}_p$ has dimension 1, it is spanned by $\zeta(p)$. From Rivoal's result we know that $\mathfrak{Z}_p \neq \mathbf{Q}$ for infinitely many odd p.

Open Problem 2. *Is–it true that* $\mathfrak{Z} \neq \mathbf{Q}[\pi^2]$*?*

All know linear relations that express a multizeta $\zeta(\underline{s})$ as a linear combination of such numbers are homogeneous for the weight. The next conjecture (which is still open) is that any linear relations among these numbers splits into homogeneous linear relations.

Conjecture 3. *The* $\mathbf{Q}$*–subspaces* $\mathfrak{Z}_p$ *of* $\mathbf{R}$ *are in direct sum:*

$$\bigoplus_{p\geq 2} \mathfrak{Z}_p \subset \mathbf{R}.$$

This is equivalent to saying that the weight defines a graduation (see §5.1) on the algebra $\mathfrak{Z}$. A very special case of Conjecture 3 which is open is $\mathfrak{Z}_2 \cap \mathfrak{Z}_3 = \{0\}$, which means that the number $\zeta(3)/\pi^2$ should be irrational.

For $p \geq 1$ we denote by d_p the dimension of $\mathfrak{Z}_p$ with $d_1 = 0$; we also set $d_0 = 1$. It is clear that $d_1 = 0$ and that $d_p \geq 1$ because $\zeta(p)$ is not zero for $p \geq 2$. For $p \geq 1$ and $k \geq 1$, we denote by $d_{p,k}$ the dimension of $\mathcal{F}^k\mathfrak{Z}_p/\mathcal{F}^{k-1}\mathfrak{Z}_p$ with $d_{p,1} = 1$ for $p \geq 1$. We also set $d_{0,0} = 1$ and $d_{0,k} = 0$ for $k \geq 1$, $d_{p,0} = 0$ for $p \geq 1$. We have for all $p \geq 0$

$$d_p = \sum_{k\geq 0} d_{p,k} \tag{4}$$

and $d_{p,k} = 0$ for $k \geq p \geq 1$.

We have $d_2 = 1$, since $\mathfrak{Z}_2$ is spanned by $\zeta(2)$. The relation $\zeta(2,1) = \zeta(3)$, which is again due to Euler, shows that $d_3 = 1$. Also the relations, essentially going back to Euler,

$$\zeta(3,1) = \frac{1}{4}\zeta(4), \quad \zeta(2,2) = \frac{3}{4}\zeta(4), \quad \zeta(2,1,1) = \zeta(4) = \frac{2}{5}\zeta(2)^2$$

show that $d_4 = 1$. These are the only values of d_p which are known. It is not yet proved that there exists a $p \geq 5$ with $d_p \geq 2$. The upper bound $\zeta(5) \leq 2$ follows from the fact that there are 6 independent linear relations among the 8 numbers

$$\zeta(5),\ \zeta(4,1),\ \zeta(3,2),\ \zeta(3,1,1),\ \zeta(2,3),\ \zeta(2,2,1),\ \zeta(2,1,2),\ \zeta(2,1,1,1),$$

and $\mathfrak{Z}_5$ is the $\mathbf{Q}$-vector subspace of $\mathbf{R}$ spanned by $\zeta(2,3)$ and $\zeta(3,2)$:

$$\zeta(5) = 4\zeta(3,2) + 6\zeta(2,3) = \zeta(2,1,1,1),$$

$$\begin{aligned}
\zeta(4,1) &= -\frac{1}{5}\zeta(3,2) + \frac{1}{5}\zeta(2,3) = \zeta(3,1,1),\\
\zeta(2,2,1) &= \zeta(3,2)\\
\zeta(2,1,2) &= \zeta(2,3).
\end{aligned}$$

The dimension of $\mathfrak{Z}_5$ is 2 if $\zeta(2,3)/\zeta(3,2)$ is irrational (which is conjectured, but not yet proved), and 1 otherwise.

Similarly the $\mathbf{Q}$–space $\mathfrak{Z}_6$ has dimension ≤ 2, as it is spanned by $\zeta(2,2,2)$ and $\zeta(3,3)$:

$$\begin{aligned}
\zeta(6) &= \frac{16}{3}\zeta(2,2,2) = \zeta(2,1,1,1,1),\\
\zeta(5,1) &= \frac{4}{3}\zeta(2,2,2) - \zeta(3,3) = \zeta(3,1,1,1),\\
\zeta(4,2) &= -\frac{16}{9}\zeta(2,2,2) + 2\zeta(3,3) = \zeta(2,2,1,1),\\
\zeta(4,1,1) &= \frac{7}{3}\zeta(2,2,2) - 2\zeta(3,3),\\
\zeta(3,2,1) &= -\frac{59}{9}\zeta(2,2,2) + 6\zeta(3,3),\\
\zeta(2,4) &= \frac{59}{9}\zeta(2,2,2) - 2\zeta(3,3) = \zeta(2,1,2,1),\\
\zeta(2,3,1) &= \frac{34}{9}\zeta(2,2,2) - 3\zeta(3,3) = \zeta(3,1,2),\\
\zeta(2,1,2,1) &= \zeta(3,3).
\end{aligned}$$

Here is Zagier's conjecture on the dimension d_p of the $\mathbf{Q}$-vector space $\mathfrak{Z}_p$.

Conjecture 5. *For $p \ge 3$ we have*

$$d_p = d_{p-2} + d_{p-3}.$$

Since $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$, this conjecture can be written

$$\sum_{p\ge 0} d_p X^p = \frac{1}{1 - X^2 - X^3}.$$

It has been proved independently by Goncharov and Terasoma that the numbers defined by the recurrence relation of Zagier's Conjecture 5 with initial values $d_0 = 1$, $d_1 = 0$ provide upper bounds for the actual dimension d_p. This shows that there are plenty of linear relations among the numbers $\zeta(\underline{s})$. For each p from 2 to 11, we display the number of tuples $\underline{s}$ of length p which is 2^{p-2}, the number d_p given by Zagier's Conjecture 5 and the difference which is (a lower bound for) the number of linear relations among these numbers.

p	2	3	4	5	6	7	8	9	10	11
2^{p-2}	1	2	4	8	16	32	64	128	256	512
d_p	1	1	1	2	2	3	4	5	7	9
$2^{p-2} - d_p$	0	1	3	6	14	29	60	123	249	503

Since d_p grows like a constant multiple of r^p where $r = 1.324\,717\,957\,244\,7\dots$ is the real root of $x^3 - x - 1$, the difference is asymptotic to 2^{p-2}.

According to Zagier's Conjecture 5, a basis for $\mathfrak{Z}_p$ should be given as follows:

$p = 2$, $d_2 = 1$, $\zeta(2)$;
$p = 3$, $d_3 = 1$, $\zeta(3)$;
$p = 4$, $d_4 = 1$, $\zeta(2,2)$;
$p = 5$, $d_5 = 2$, $\zeta(2,3)$, $\zeta(3,2)$;
$p = 6$, $d_6 = 2$, $\zeta(2,2,2)$, $\zeta(3,3)$;
$p = 7$, $d_7 = 3$, $\zeta(2,2,3)$, $\zeta(2,3,2)$, $\zeta(3,2,2)$;
$p = 8$, $d_8 = 4$, $\zeta(2,2,2,3)$, $\zeta(2,3,3)$, $\zeta(3,2,3)$, $\zeta(3,3,2)$;
$p = 9$, $d_8 = 5$, $\zeta(2,2,2,3)$, $\zeta(2,2,3,2)$, $\zeta(2,3,2,2)$, $\zeta(3,2,2,2)\zeta(3,3,3)$.

For these small values of p, the dimension $d_{p,k}$ of $\mathcal{F}^k\mathfrak{Z}_p/\mathcal{F}^{k-1}\mathfrak{Z}_p$ is conjecturally given by the number of elements in the box (p,k) of the next figure, where conjectural generators of $\mathcal{F}^k\mathfrak{Z}_p/\mathcal{F}^{k-1}\mathfrak{Z}_p$ should be given by the classes of the following MZV:

k \ p	2	3	4	5	6	7	8	9
1	$\zeta(2)$	$\zeta(3)$	$\zeta(4)$	$\zeta(5)$	$\zeta(6)$	$\zeta(7)$	$\zeta(8)$	$\zeta(9)$
2				$\zeta(4,1)$	$\zeta(5,1)$	$\zeta(6,1)$ $\zeta(5,2)$	$\zeta(7,1)$ $\zeta(6,2)$	$\zeta(8,1)$ $\zeta(7,2)$ $\zeta(6,3)$
3							$\zeta(6,1,1)$	$\zeta(6,2,1)$
d_p	1	1	1	2	2	3	4	5

The displayed elements should all be linearly independent over $\mathbf{Q}$. The numerical computations have been performed online thanks to the computer program EZface [1].

1.3 Known results

A conjecture by M. Hoffman is that a basis of $\mathfrak{Z}_p$ over $\mathbf{Q}$ is given by the numbers $\zeta(s_1,\dots,s_k)$, $s_1+\cdots+s_k = p$, where each s_i is either 2 or 3: the dimension agrees with Conjecture 5.

As a side product of his main recent results, F. Brown [5, 6] obtains:

Theorem 6. *The numbers* $\zeta(s_1,\dots,s_k)$ *with* $k\ge 1$ *and* $s_j\in\{2,3\}$ *span the* $\mathbf{Q}$*–space* $\mathfrak{Z}$ *of multizeta.*

One of the auxiliary result which was need by Brown is a formula which he conjectured and which has been established by D. Zagier (see §2).

1.4 Occurrences of powers of π^2 in the set of generators

The space $\mathcal{F}^1\mathfrak{Z}_{2n}$ is spanned by π^{2n} over $\mathbf{Q}$. We now show that the numbers π^{2n} occur in the set of generators $\zeta(\underline{s})$ with $s_j\in\{2,3\}$ by taking all s_j equal to 2.

For $s\ge 2$ and $n\ge 1$, we use the notation $\{s\}_a$ for a string with n elements all equal to s, that is $\{s\}_n=(s_1,\dots,s_n)$ with $s_1=\cdots=s_n=s$.

Proposition 7. *For* $s\ge 2$*,*

$$\sum_{n\ge 0}\zeta\big(\{s\}_n\big)x^n=\prod_{j\ge 1}\left(1+\frac{x}{j^s}\right)=\exp\left(\sum_{k\ge 1}\frac{(-1)^{k-1}x^k\zeta(sk)}{k}\right).$$

The proof will involve the infinite product

$$F_s(x)=\prod_{j\ge 1}\left(1+\frac{x}{j^s}\right).$$

Proof. Expanding $F_s(x)$ as a series:

$$\begin{aligned}F_s(x)&=1+x\sum_{j\ge 1}\frac{1}{j^s}+x^2\sum_{j_1>j_2\ge 1}\frac{1}{(j_1j_2)^s}+\cdots\\&=1+x\zeta(s)+x^2\zeta(s,s)+\cdots\end{aligned}$$

yields the first equality in Proposition 7. For the second one, consider the logarithmic derivative of $F_s(x)$:

$$\begin{aligned}\frac{F_s'(x)}{F_s(x)}&=\sum_{j\ge 1}\frac{1}{j^s+x}=\sum_{j\ge 1}\frac{1}{j^s}\sum_{k\ge 1}(-1)^{k-1}\frac{x^{k-1}}{j^{s(k-1)}}\\&=\sum_{k\ge 1}(-1)^{k-1}x^{k-1}\sum_{j\ge 1}\frac{1}{j^{sk}}=\sum_{k\ge 1}(-1)^{k-1}x^{k-1}\zeta(sk).\end{aligned}$$

Since $F_s(0)=\zeta\big(\{s\}_0\big)=1$, Proposition 7 follows by integration. □

Lemma 8. *We have*

$$F_2(-z^2)=\frac{\sin(\pi z)}{\pi z}\cdot$$

Proof. Lemma 8 follows from the product expansion of the sine function

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

□

From Proposition 7 and Lemma 8 we deduce:

Corollary 9. *For any $n \geq 0$,*

$$\zeta\bigl(\{2\}_n\bigr) = \frac{\pi^{2n}}{(2n+1)!}.$$

Proof. From the Taylor expansion of the sine function

$$\sin z = \sum_{k \geq 0} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

and from Lemma 8 we infer

$$F_2(-z^2) = \frac{\sin(\pi z)}{\pi z} = \sum_{k \geq 0} (-1)^k \frac{\pi^{2k} z^{2k}}{(2k+1)!}.$$

Corollary 9 now follows from Proposition 7. □