Third lecture: April 12 2011

4 Hilbert–Poincaré series

We shall work with commutative algebras, namely polynomial algebras in variables having each a weight, a conjectural (recall the open problem 2) example being the algebra $\mathfrak{Z}$ of MZV which is a subalgebra of the real numbers. This algebra $\mathfrak{Z}$ is the image by specialization of an algebra of polynomials in infinitely many variables, with one variable of weight 2 (corresponding to $\zeta(2)$), one of weight 3 (corresponding to $\zeta(3)$), one of weight 5 (corresponding to $\zeta(3,2)$) and so on. According to Conjecture 12, the number of variables of weight p should be the number $N(p)$ of Lyndon words on the alphabet $\{2,3\}\}$ with 2 having weight 2 and 3 weight 3. In general, we shall consider countably many variables with $N(p)$ variables of weight p for $p \geq 0$, with $N(0) = 0$. But later we shall also consider non–commutative variables; for instance the free algebra on the words $\{2,3\}$ will play a role.

5 Graduated algebras and Hilbert–Poincaré series

5.1 Graduations

We introduce basic definitions from algebra. A *graduation* on a ring A is a decomposition into a direct sum of additive subgroups

$$A = \bigoplus_{k\geq 0} A_k,$$

such that the multiplication $A \times A \to A$ which maps (a,b) onto the product ab maps $A_k \times A_h$ into A_{k+h} for all pairs (k,h) of non–negative integers. For us here it will be sufficient to take for indices the non–negative integers, but we could more generally take a commutative additive monoid (see [10] Chap. X § 5). The elements in A_k are *homogeneous of weight (or degree) k.* Notice that A_0 is a subring of A and that each A_k is a A_0–module[1].

Given a graduated ring A, a *graduation* on a A–module E is a decomposition into a direct sum of additive subgroups

$$E = \bigoplus_{k\geq 0} E_k,$$

such that $A_k E_n \subset E_{k+n}$. In particular each E_n is a A_0–module. The elements of E_n are *homogeneous of weight (or degree) n.*

[1] According to this definition, 0 is homogeneous of weight k for all $k \geq 0$

A *graduated K–algebra* is a K–algebra A with a graduation as a ring $A = \bigoplus_{k\geq 0} A_k$ such that $KA_k \subset A_k$ for all $k \geq 0$ and $A_0 = K$ (see [10] Chap. XVI, § 6). If the dimension d_k of each A_k as a K–vector space is finite with $d_0 = 1$, the *Hilbert–Poincaré series* of the graduated algebra A is

$$\mathcal{H}_A(t) = \sum_{p\geq 0} d_p t^p.$$

If the K–algebra A is the tensor product $A' \otimes A''$ of two graded algebras A' and A'' over the field K, then A is graded with the generators of A_p as K–vector space being the elements $x' \otimes x''$, where x' runs over the generators of the homogeneous part A'_k of A' and where x'' runs over the generators of the homogeneous part A''_ℓ of A'', with $k + \ell = p$. Hence the dimensions d_p, d'_k, d''_ℓ of the homogeneous subspaces of A, A' and A'' satisfy

$$d_p = \sum_{k+\ell=p} d'_k d''_\ell,$$

which means that the Hilbert–Poincaré series of A is the product of the Hilbert–Poincaré series of A' and A'':

$$\mathcal{H}_{A'\otimes A''}(t) = \mathcal{H}_{A'}(t)\mathcal{H}_{A''}(t).$$

5.2 Commutative polynomials algebras

Let

$$\big(N(1), N(2), \dots, N(p), \dots\big)$$

be a sequence of non–negative integers and let A denote the commutative K–algebra of polynomials with coefficients in K in the variables Z_{np} ($p \geq 1$, $1 \leq n \leq N(p)$). We endow the K-algebra A with the graduation for which each Z_{np} is homogeneous of weight p. We denote by d_p the dimension of the homogeneous space A_p over K.

Lemma 13. *The Hilbert–Poincaré series of A is*

$$\mathcal{H}_A(t) = \prod_{p\geq 1} \frac{1}{(1-t^p)^{N(p)}}.$$

Proof. For $p \geq 1$, the K–vector space A_p of homogeneous elements of weight p has a basis consisting of monomials

$$\prod_{k=1}^{\infty} \prod_{n=1}^{N(k)} Z_{nk}^{h_{nk}},$$

where $\underline{h} = (h_{nk})_{\substack{k\geq 1\\ 1\leq n\leq N(k)}}$ runs over the set of tuples of non–negative integers satisfying

$$\sum_{k=1}^{\infty} \sum_{n=1}^{N(k)} k h_{nk} = p. \tag{14}$$

Notice that these tuples $\underline{h}$ have a *support*

$$\{(n,k)\ ;\ k\geq 1,\ 1\leq n\leq N(k),\ h_{nk}\neq 0\}$$

which is finite, since $h_{n,k}=0$ for $k>p$. The dimension d_p of the K–vector space A_p is the number of these tuples $\underline{h}$ (with $d_0=1$), and by definition we have

$$\mathcal{H}_A(t)=\sum_{p\geq 0} d_p t^p.$$

In the identity

$$\frac{1}{(1-z)^N}=\sum_{h_1\geq 0}\cdots\sum_{h_N\geq 0}\prod_{n=1}^{N} z^{h_n}$$

we replace z by t^k and N by $N(k)$. We deduce

$$\prod_{k\geq 1}\frac{1}{(1-t^k)^{N(k)}}=\sum_{\underline{h}}\prod_{k=1}^{\infty}\prod_{n=1}^{N(k)} t^{kh_{nk}}.$$

The coefficient of t^p in the right hand side is the number of tuples $\underline{h}=(h_{nk})_{\substack{k\geq 1\\ 1\leq n\leq N(k)}}$ with $h_{nk}\geq 0$ satisfying (14); hence it is nothing else than d_p. □

5.3 Examples

Example 15. In Lemma 13, take $N(p)=0$ for $p\geq 2$ and write N instead of $N(1)$. Then A is the ring of polynomials $K[Z_1,\ldots,Z_N]$ with the standard graduation of the total degree (each variable Z_i, $i=1,\ldots,N$, has weight 1). The Hilbert–Poincaré series is

$$\frac{1}{(1-t)^N}=\sum_{\ell\geq 0}\binom{N+\ell-1}{\ell}t^{\ell}.$$

If each variable Z_i has a weight other than 1 but all the same, say p, it suffices to replace t by t^p. For instance the Hilbert–Poincaré series of the algebra of polynomials $K[Z]$ in one variable Z having weight 2 is $(1-t^2)^{-1}$.

Example 16. *More generally, if there are only finitely many variables, which means that there exists an integer $p_0\geq 1$ such that $N(j)=0$ for $j>p_0$, the same proof yields*

$$d_\ell=\prod_{\ell_1+2\ell_2+\cdots+j_0\ell_{k_0}=\ell}\ \prod_{j=1}^{k_0}\binom{N(j)+\ell_j-1}{\ell_j}.$$

Example 17. Denote by μ the Möbius function (see [9]— § 16.3):

$$\begin{cases}\mu(1)=1,\\ \mu(p_1\cdots p_r)=(-1)^r \text{ if } p_1,\ldots,p_r \text{ are distinct prime numbers distincts,}\\ \mu(n)=0 \text{ if } n \text{ has a square factor} >1.\end{cases}$$

Given a positive integer c, under the assumptions of Lemma 13, the following conditions are equivalent:

(i) *The Hilbert–Poincaré series of A is*

$$\mathcal{H}_A(t) = \frac{1}{1-ct}.$$

(ii) *For $p \geq 0$, we have $d_p = c^p$.*

(iii) *For $p \geq 1$ we have*

$$c^p = \sum_{n|p} nN(n) \quad \textit{for all } p \geq 1.$$

(iv) *For $k \geq 1$ we have*

$$N(k) = \frac{1}{k}\sum_{n|k} \mu(k/n)c^n.$$

Proof. The equivalence between (i) and (ii) follows from the definition of $\mathcal{H}_A$ and the power series expansion

$$\frac{1}{1-ct} = \sum_{p\geq 0} c^p t^p.$$

The equivalence between (iii) and (iv) follows from Möbius inversion formula (see [10] Chap. II Ex. 12.c and Chap. V, Ex. 21; [9] § 16.4).

It remains to check the equivalence between (i) and (iii).The constant term of each of the developments of

$$\frac{1}{1-ct} \quad \text{and} \quad \prod_{k\geq 1}\frac{1}{(1-t^k)^{N(k)}}$$

into power series is 1; hence the two series are the same if and only if their logarithmic derivatives are the same. The logarithmic derivative of $1/(1-ct)$ is

$$\frac{c}{1-ct} = \sum_{p\geq 1} c^p t^{p-1}.$$

The logarithmic derivative of $\prod_{k\geq 1}(1-t^k)^{-N(k)}$ is

$$\sum_{k\geq 1}\frac{kN(k)t^{k-1}}{1-t^k} = \sum_{p\geq 1}\left(\sum_{n|p} nN(n)\right)t^{p-1}. \tag{18}$$

□

Example 19. Let a and c be two positive integers. Define two sequences of integers $(\delta_p)_{p\geq 1}$ and $(P_\ell)_{\ell\geq 1}$ by

$$\begin{cases} \delta_p = 0 & \text{if } a \text{ does not divide } p,\\ \delta_p = c^{p/a} & \text{if } a \text{ divides } p \end{cases}$$

and

$$\begin{cases} P_\ell = 0 & a \text{ does not divide } \ell,\\ P_\ell = ac^{\ell/a} & \text{if } a \text{ divides } \ell. \end{cases}$$

Under the hypotheses of Lemma 13, the following properties are equivalent:

(i) The Hilbert–Poincaré series of A is

$$\mathcal{H}_A(t) = \frac{1}{1-ct^a}\cdot$$

(ii) For any $p \geq 1$, we have $d_p = \delta_p$.

(iii) For any $\ell \geq 1$, we have

$$\sum_{n|\ell} nN(n) = P_\ell.$$

(iv) For any $k \geq 1$, we have

$$N(k) = \frac{1}{k}\sum_{\ell|k} \mu(k/\ell)P_\ell.$$

Proof. The definition of the numbers δ_p means

$$\frac{1}{1-ct^a} = \sum_{p\geq 0} \delta_p t^p,$$

while the definition of P_ℓ can be written

$$\sum_{\ell\geq 1} P_\ell t^{\ell-1} = \frac{cat^{a-1}}{1-ct^a},$$

where the right hand side is the logarithmic derivative of $1/(1-ct^a)$. Recall that the logarithmic derivative of $\prod_{k\geq 1}(1-t^k)^{-N(k)}$ is given by (18). This completes the proof. □

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