Sixth lecture: April 20, 2011

6.2 The free algebra $\mathfrak{H}$ and its two subalgebras $\mathfrak{H}^1$ and $\mathfrak{H}^0$

Our first example of an alphabet was the trivial one with a single letter, the free monoid on a set with a single element 1 is just $\mathbf{N}$. The next example is when $X = \{x_0, x_1\}$ has two elements; in this case the algebra $K\langle x_0, x_1\rangle$ will be denoted by $\mathfrak{H}$. Each word w in X^* can be written $x_{\epsilon_1} \cdots x_{\epsilon_p}$ where each ϵ_i is either 0 or 1 and the integer p is the weight of w. The number of $i \in \{1, \dots p\}$ with $\epsilon_i = 1$ is called the *length* (or the *depth*) of w.

We shall denote by X^*x_1 the set of word which end with x_1, and by $x_0X^*x_1$ the set of words which start with x_0 and end with x_1.

Consider a word w in X^*x_1. We write it $w = x_{\epsilon_1} \cdots x_{\epsilon_p}$ with $\epsilon_p = 1$. Let k be the number of occurrences of the letter x_1 in w. We have $p \geq 1$ and $k \geq 1$. We can write $w = x_0^{s_1-1}x_1x_0^{s_2-1}x_1 \cdots x_0^{s_k-1}x_1$ by defining $s_1 - 1$ as the number of occurrences of the letter x_0 before the first x_1 and, for $2 \leq j \geq k$, by defining $s_j - 1$ as the number of occurrences of the letter x_0 between the $(j-1)$-th and the j–th occurrence of x_1. This produces a sequence of non–negative integers $(s_1, \dots, s_k) \in \mathbf{N}^k$. Such a sequence $\underline{s} = (s_1, \dots, s_k)$ of positive integers with $k \geq 1$ is a called *composition* (the set of compositions is the union over $k \geq 1$ of these k–tuples $(s_1, \dots, s_k)$).

For $s \geq 1$, define $y_s = x_0^{s-1}x_1$:

$$y_1 = x_1, \ \ y_2 = x_0x_1, \ \ y_3 = x_0^2x_1, \ \ y_4 = x_0^3x_1, \ \ \dots$$

and let $Y = \{y_1, \ y_2, \ y_3, \dots\}$. For $\underline{s} = (s_1, \dots, s_k) \in \mathbf{N}^k$ with $s_j \geq 1$ $(1 \leq j \leq k)$, set $y_{\underline{s}} = y_{s_1} \cdots y_{s_k}$, so that

$$y_{\underline{s}} = x_0^{s_1-1}x_1 \cdots x_0^{s_k-1}x_1.$$

Lemma 27.

a) The set X^*x_1 is the same as the set of $y_{s_1} \cdots y_{s_k}$, where $(s_1, \dots, s_k)$ ranges over the finite sequences of positive integers with $k \geq 1$ and $s_j \geq 1$ for $1 \leq j \leq k$.

b) The free monoid Y^* on the set Y is $\{e\} \cup X^*x_1$.

c) The set $x_0X^*x_1$ is the set $\{y_{\underline{s}}\}$ where $\underline{s}$ is a composition having $s_1 \geq 2$.

The subalgebra of $\mathfrak{H}$ spanned by X^*x_1 is

$$\mathfrak{H}^1 = Ke + \mathfrak{H}x_1$$

and $\mathfrak{H}x_1$ is a left ideal of $\mathfrak{H}$. The algebra $\mathfrak{H}^1$ is the free algebra $K\langle Y\rangle$ on the set Y. We observe an interesting phenomenon, which does not occur in the commutative case, is that the free algebra $K\langle x_0, x_1\rangle$ on a set with only two elements contains as a subalgebra the free algebra $K\langle y_1, y_2, \dots\rangle$ on a set with countably many elements. Notice that for each $n \geq 1$ this last algebra also contains as a subalgebra the free algebra on a set with n elements, namely

$K\langle y_1, y_2, \dots, y_n\rangle$. From this point of view it suffices to deal with only two variables! Any finite message can be encoded with an alphabet with only two letters. Also, we see that a naive definition of a dimension for such spaces, where $K\langle y_1, \dots, y_n\rangle$ would have dimension n, could be misleading.

A word in $x_0X^*x_1$ is called *convergent*. The reason is that one defines a map on the convergent words

$$\widehat{\zeta} : x_0X^*x_1 \to \mathbf{R}$$

by setting $\widehat{\zeta}(y_{\underline{s}}) = \zeta(s)$. Also the subalgebra of $\mathfrak{H}^1$ spanned by $x_0X^*x_1$ is

$$\mathfrak{H}^0 = Ke + x_0\mathfrak{H}x_1.$$

One extends the map $\widehat{\zeta} : x_0X^*x_1 \to \mathbf{R}$ by K–linearity and obtain a map $\widehat{\zeta} : \mathfrak{H}^0 \to \mathbf{R}$ such that $\widehat{\zeta}(e) = 1$.

On $\mathfrak{H}^0$ there is a structure of non–commutative algebra, given by the concatenation – however, the map $\widehat{\zeta}$ has no good property for this structure. The concatenation of $y_2 = x_1x_0$ and $y_3 = x_0^2x_1$ is $y_2y_3 = y_{(2,3)} = x_0x_1x_0^2x_1$, and $\zeta(2)\zeta(3) \neq \zeta(2,3)$ and $\zeta(2)\zeta(3) \neq \zeta(2,3)$; indeed according to [1] we have $\zeta(2) = 1.644\dots$, $\zeta(3) = 1.202\dots$

$$\zeta(2)\zeta(3) = 1.977\dots \quad \zeta(2,3) = 0.711\dots \quad \zeta(3,2) = 0.228\dots$$

so

$$\widehat{\zeta}(y_2)\widehat{\zeta}(y_3) \neq \widehat{\zeta}(y_2y_3).$$

As we have seen in §3, it is expected that $\zeta(2)$, $\zeta(3)$ and $\zeta(2,3)$ are algebraically independent.

However there are other structures of algebras on $\mathfrak{H}^0$, in particular two commutative algebra structures ш (*shuffle*) and $\star$ (*stuffle* or *harmonic law*) for which $\widehat{\zeta}$ will become an algebra homomorphism: for w and w' in $\mathfrak{H}^0$,

$$\widehat{\zeta}(w ш w') = \zeta(w)\zeta(w') \quad \text{and} \quad \widehat{\zeta}(w \star w') = \zeta(w)\zeta(w').$$

7 MZV as integrals and the Shuffle Product

7.1 Zeta values as integrals

We first check

$$\zeta(2) = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}.$$

For t_2 in the interval $0 < t_2 < 1$ we expand $1/(1-t_2)$ in power series; next we integrate over the interval $[0, t_1]$ where t_1 is in the interval $0 < t_1 < 1$, so that the integration terms by terms is licit:

$$\frac{1}{1-t_2} = \sum_{n\geq 1} t_2^{n-1}, \qquad \int_0^{t_1} \frac{dt_2}{1-t_2} = \sum_{n\geq 1} \int_0^{t_1} t_2^{n-1} dt_2 = \sum_{n\geq 1} \frac{t_1^n}{n}.$$

Hence, the integral is

$$\int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} = \sum_{n\geq 1} \frac{1}{n} \int_0^1 t_1^{n-1} dt_1 = \sum_{n\geq 1} \frac{1}{n^2} = \zeta(2).$$

In the same way one checks

$$\zeta(3) = \int_{1>t_1>t_2 t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{t_2} \cdot \frac{dt_3}{1-t_3} \cdot$$

We continue by induction and give an integral formula for $\zeta(s)$ when $s \geq 2$ is an integer, and more generally for $\zeta(\underline{s})$ when $\underline{s} = (s_1, \ldots, s_k)$ is a composition. To state the result, it will be convenient to introduce a definition: we set

$$\omega_0(t) = \frac{dt}{t} \quad \text{and} \quad \omega_1(t) = \frac{dt}{1-t} \cdot$$

For $p \geq 1$ and $\epsilon_1, \ldots, \epsilon_p$ in $\{0, 1\}$, we define

$$\omega_{\epsilon_1} \cdots \omega_{\epsilon_p} = \omega_{\epsilon_1}(t_1) \cdots \omega_{\epsilon_p}(t_p).$$

We shall integrate this differential form on the simplex

$$\Delta_p = \{(t_1, \ldots, t_p) \in \mathbf{R}^p \ ; \ 1 > t_1 > \cdots > t_p > 0\}$$

The two previous formulae are

$$\zeta(2) = \int_{\Delta_2} \omega_0 \omega_1 \quad \text{and} \quad \zeta(3) = \int_{\Delta_3} \omega_0^2 \omega_1.$$

The integral formula for zeta values that follows by induction is

$$\zeta(s) = \int_{\Delta_s} \omega_0^{s-1} \omega_1 \quad \text{for} \quad s \geq 2.$$

We extend this formula to multiple zeta values as follows. Firstly, for $s \geq 1$ we define $\omega_s = \omega_0^{s-1}\omega_1$, which matches the previous definition when $s = 1$ and produces, for $s \geq 2$,

$$\zeta(s) = \int_{\Delta_s} \omega_s.$$

Next, for $\underline{s} = (s_1, \ldots, s_k)$, we define $\omega_{\underline{s}} = \omega_{s_1} \cdots \omega_{s_k}$.

Proposition 28. *Assume $s_1 \geq 2$. Let $p = s_1 + \cdots + s_k$. Then*

$$\zeta(\underline{s}) = \widehat{\zeta}(y_{\underline{s}}) = \int_{\Delta_p} \omega_{\underline{s}}.$$

The proof is by induction on p. For this induction it is convenient to introduce the *multiple polylogarithm functions in one variable*:

$$\mathrm{Li}_{\underline{s}}(z) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}},$$

for $\underline{s} = (s_1, \ldots, s_k)$ with $s_j \geq 1$ and for $z \in \mathbf{C}$ with $|z| < 1$. Notice that $\mathrm{L}i(z)$ is defined also at $z = 1$ when $s_1 \geq 2$ where it takes the value $\mathrm{L}i_{\underline{s}}(z) = 1$. One checks, by induction on the weight p, for $0 < z < 1$,

$$\mathrm{Li}_{\underline{s}}(z) = \int_{\Delta_p(z)} \omega_{\underline{s}},$$

where $\Delta_p(z)$ is the simplex

$$\Delta_p(z) = \{(t_1, \ldots, t_p) \in \mathbf{R}^p \; ; \; z > t_1 > \cdots > t_p > 0\}.$$

We now consider products of such integrals. Consider $\zeta(2)^2$ as product of two integrals

$$\zeta(2)^2 = \int_{\substack{1 > t_1 > t_2 > 0 \\ 1 > u_1 > u_2 > 0}} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1 - t_2} \cdot \frac{du_1}{u_1} \cdot \frac{du_2}{1 - u_2}.$$

We decompose the domain

$$1 > t_1 > t_2 > 0, \quad 1 > u_1 > u_2 > 0$$

into six disjoint domains (and further subsets of zero dimension) obtained by "shuffling" (t_1, t_2) with (u_1, u_2):

$$\begin{aligned}
&1 > t_1 > t_2 > u_1 > u_2 > 0, \quad 1 > t_1 > u_1 > t_2 > u_2 > 0,\\
&1 > u_1 > t_1 > t_2 > u_2 > 0, \quad 1 > t_1 > u_1 > u_2 > t_2 > 0,\\
&1 > u_1 > t_1 > u_2 > t_2 > 0, \quad 1 > u_1 > u_2 > t_1 > t_2 > 0,
\end{aligned}$$

Each of the six simplices have either t_1 or u_1 as the largest variable (corresponding to $\omega_0(t) = 1/t$) and u_2 or t_2 as the lowest (corresponding to $\omega_1(t) = dt/(1-t)$). The integrals of $\omega_0^2\omega_1^2$ produce $\zeta(3, 1)$, there are 4 of them, the integrals of $\omega_0\omega_1\omega_0\omega_1$ produce $\zeta(2, 2)$, and there are 2 of them. From Proposition 28, we deduce

$$\zeta(2)^2 = 4\zeta(3, 1) + 2\zeta(2, 2).$$

This is a typical example of a "shuffle relation":

$$\omega_0\omega_1 ш \omega_0\omega_1 = 4\omega_0^2\omega_1^2 + 2\omega_0\omega_1\omega_0\omega_1.$$