On the Markoff Equation

\[ x^2 + y^2 + z^2 = 3xyz \]

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Abstract

It is easy to check that the equation $x^2 + y^2 + z^2 = 3xyz$, where the three unknowns $x, y, z$ are positive integers, has infinitely many solutions. There is a simple algorithm which produces all of them. However, this does not answer to all questions on this equation: in particular, Frobenius asked whether it is true that for each integer $z > 0$, there is at most one pair $(x, y)$ such that $x < y < z$ and $(x, y, z)$ is a solution. This question is an active research topic nowadays.
Markoff’s equation occurred initially in the study of minima of quadratic forms at the end of the XIX–th century and the beginning of the XX–th century. It was investigated by many a mathematician, including Lagrange, Hermite, Korkine, Zolotarev, Markoff, Frobenius, Hurwitz, Cassels. The solutions are related with the Lagrange-Markoff spectrum, which consists of those quadratic numbers which are badly approximable by rational numbers. It occurs also in other parts of mathematics, in particular free groups, Fuchsian groups and hyperbolic Riemann surfaces (Ford, Lehner, Cohn, Rankin, Conway, Coxeter, Hirzebruch and Zagier…).

We discuss some aspects of this topic without trying to cover all of them.
The sequence of Markoff numbers

A *Markoff number* is a positive integer $z$ such that there exist two positive integers $x$ and $y$ satisfying

$$x^2 + y^2 + z^2 = 3xyz.$$ 

For instance, 1 is a Markoff number, since $(x, y, z) = (1, 1, 1)$ is a solution.

Photos:
http://www-history.mcs.st-andrews.ac.uk/history/
The sequence of Markoff numbers is available on the web

The On-Line Encyclopedia of Integer Sequences

http://oeis.org/A002559
Given a Markoff number $z$, there exist infinitely many pairs of positive integers $x$ and $y$ satisfying

$$x^2 + y^2 + z^2 = 3xyz.$$ 

This is a cubic equation in the 3 variables $(x, y, z)$, of which we know a solution $(1, 1, 1)$.

There is an algorithm producing all integer solutions.
Integer points on a surface

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Markoff’s cubic variety

The surface defined by Markoff’s equation

\[ x^2 + y^2 + z^2 = 3xyz. \]

is an algebraic variety with many automorphisms: permutations of the variables, changes of signs and

\[(x, y, z) \mapsto (3yz - x, y, z).\]
Algorithm producing all solutions

Let \((m, m_1, m_2)\) be a solution of Markoff’s equation:

\[m^2 + m_1^2 + m_2^2 = 3mm_1m_2.\]

Fix two coordinates of this solution, say \(m_1\) and \(m_2\). We get a quadratic equation in the third coordinate \(m\), of which we know a solution, hence, the equation

\[x^2 + m_1^2 + m_2^2 = 3xm_1m_2.\]

has two solutions, \(x = m\) and, say, \(x = m'\), with

\[m + m' = 3m_1m_2\] and \(mm' = m_1^2 + m_2^2\). This is the cord and tangente process.

Hence, another solution is \((m', m_1, m_2)\) with \(m' = 3m_1m_2 - m\).
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Three solutions derived from one

Starting with one solution \((m, m_1, m_2)\), we derive three new solutions:

\[
(m', m_1, m_2), \quad (m, m'_1, m_2), \quad (m, m_1, m'_2).
\]

If the solution we start with is \((1, 1, 1)\), we produce only one new solution, \((2, 1, 1)\) (up to permutation).

If we start from \((2, 1, 1)\), we produce only two new solutions, \((1, 1, 1)\) and \((5, 2, 1)\) (up to permutation).

A new solution means distinct from the one we start with.
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A new solution means distinct from the one we start with.
New solutions

We shall see that any solution different from \((1, 1, 1)\) and from \((2, 1, 1)\) yields three new different solutions – and we shall see also that, in each other solution, the three numbers \(m, m_1\) and \(m_2\) are pairwise distinct.

Two solutions are called *neighbors* if they share two components.

For instance

- \((1, 1, 1)\) has a single neighbor, namely \((2, 1, 1)\),
- \((2, 1, 1)\) has two neighbors : \((1, 1, 1)\) et \((5, 2, 1)\),
- any other solution has exactly three neighbors.
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Markoff’s tree

Assume we start with \((m, m_1, m_2)\) satisfying \(m > m_1 > m_2\). We shall check

\[ m'_2 > m'_1 > m > m'. \]

We order the solution according to the largest coordinate. Then two of the neighbors of \((m, m_1, m_2)\) are larger than the initial solution, the third one is smaller.

Hence, if we start from \((1, 1, 1)\), we produce infinitely many solutions, which we organize in a tree: this is Markoff’s tree.
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Hence, if we start from \((1, 1, 1)\), we produce infinitely many solutions, which we organize in a tree: this is Markoff’s tree.
This algorithm yields all the solutions

Conversely, starting from any solution other than \((1, 1, 1)\), the algorithm produces a smaller solution.

Hence, by induction, we get a sequence of smaller and smaller solutions, until we reach \((1, 1, 1)\).

Therefore the solution we started from was in Markoff’s tree.
First branches of Markoff’s tree
Markoff’s tree starting from $(2, 5, 29)$

Conversely, given a Markoff triple $(p, q, r)$ with $r > 1$, one checks easily that $3pq - r < r$; and from this it follows by induction that all Markoff triples occur, and occur only once, on this tree (for a fuller discussion of this and other properties of the Markoff tree, see [2]).

To prove the theorem we must analyze the asymptotic behavior of the Markoff tree. From the Markoff equation (1) we find that $3r^2 - 2 - 3pqr$ or $r^2 - pq$; if $p$ is large (which will happen for all but a small portion of the tree, contributing $O(\log x)$ to $M(x)$), then this implies that $r$ is much larger than $q$ and hence (1) gives $r^2 < 3pqr < r^2 + o(r^2)$ or $r - 3pq$. Multiplying both sides of this equation by 3 and taking logarithms gives

$$\log(3p) + \log(3q) = \log(3r) + o(1) \quad (p \text{ large})$$
Markoff’s tree up to 100 000

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\[3r^2 \geq 2p^2 + 3pqr\]

or

\[r^2 \geq pq;\]

if \(p\) is large (which will happen for all but a small portion of the tree, contributing \(O(\log x)\) to \(M(x)\)), then this implies that \(r\) is much larger than \(q\) and hence

\[r^2 < 3pqr < r^2 + o(r^2)\]

or \(r < 3pq\). Multiplying both sides of this equation by 3 and taking logarithms gives

\[\log(3p) + \log(3q) = \log(3r) + o(1)\]

\((p\) large)
Continued fractions and the Markoff tree

E. Bombieri,

*Continued fractions and the Markoff tree*,


no. 3, 187–213.
Markoff’s tree
Letter from Masanobu Kaneko to Yuri Manin  
26/06/2008

Markoff numbers, associated $\Theta_i$, and the period of its continued fraction expansion, and the value $\text{val}(\pm \Theta_i)$.
\[ a^2 + b^2 + c^2 = 3abc \]

\[ X^2 - 3abX + a^2 + b^2 = (X - c)(X - 3ab + c) \]
The Fibonacci sequence and the Markoff equation

The smallest Markoff number is 1. When we impose $z = 1$ in the Markoff equation $x^2 + y^2 + z^2 = 3xyz$, we obtain the equation

$$x^2 + y^2 + 1 = 3xy.$$  

Going along the Markoff’s tree starting from $(1, 1, 1)$, we obtain the subsequence of Markoff numbers

$$1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, \ldots$$

which is the sequence of Fibonacci numbers with odd indices

$$F_1 = 1, F_3 = 2, F_5 = 5, F_7 = 13, F_9 = 34, F_{11} = 89, \ldots$$
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The Fibonacci sequence \((F_n)_{n \geq 0}\): 

\begin{align*}
0, & \quad 1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \\
& \quad 34, \quad 55, \quad 89, \quad 144, \quad 233 \ldots
\end{align*}

is defined by

\begin{align*}
F_0 &= 0, \quad F_1 = 1, \\
F_n &= F_{n-1} + F_{n-2} \quad (n \geq 2).
\end{align*}
Encyclopedia of integer sequences (again)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, ...

The Fibonacci sequence is available online

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

http://oeis.org/A000045
Fibonacci numbers with odd indices are Markoff’s numbers:

\[ F_{m+3}F_{m-1} - F_{m+1}^2 = (-1)^m \quad \text{for} \quad m \geq 1 \]

and

\[ F_{m+3} + F_{m-1} = 3F_{m+1} \quad \text{for} \quad m \geq 1. \]

Set \( y = F_{m+1}, \) \( x = F_{m-1}, \) \( x' = F_{m+3}, \) so that, for even \( m, \)

\[ x + x' = 3y, \quad xx' = y^2 + 1 \]

and

\[ X^2 - 3yX + y^2 + 1 = (X - x)(X - x'). \]
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Order of the *new* solutions

Let \((m, m_1, m_2)\) be a solution of Markoff’s equation

\[ m^2 + m_1^2 + m_2^2 = 3mm_1m_2. \]

Denote by \(m'\) the other root of the quadratic polynomial

\[ X^2 - 3m_1m_2X + m_1^2 + m_2^2. \]

Hence,

\[ X^2 - 3m_1m_2X + m_1^2 + m_2^2 = (X - m)(X - m') \]

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$m_1 \neq m_2$

Let us check that if $m_1 = m_2$, then $m_1 = m_2 = 1$ : this holds only for the two exceptional solutions $(1, 1, 1)$, $(2, 1, 1)$.

Assume $m_1 = m_2$. We have

$$m^2 + 2m_1^2 = 3mm_1^2 \quad \text{hence} \quad m^2 = (3m - 2)m_1^2.$$  

Therefore $m_1$ divides $m$. Let $m = km_1$. We have

$$k^2 = 3km_1 - 2, \quad \text{hence} \quad k \text{ divise } 2.$$  

For $k = 1$ we get $m = m_1 = 1$.

For $k = 2$ we get $m_1 = 1, \quad m = 2$.

Consider now a solution distinct from $(1, 1, 1)$ or $(2, 1, 1)$ : hence $m_1 \neq m_2$. 
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For \( k = 1 \) we get \( m = m_1 = 1 \).

For \( k = 2 \) we get \( m_1 = 1, \ m = 2 \).

Consider now a solution distinct from \((1, 1, 1)\) or \((2, 1, 1)\) : hence \( m_1 \neq m_2 \).
Let us check that if \( m_1 = m_2 \), then \( m_1 = m_2 = 1 \): this holds only for the two exceptional solutions \((1, 1, 1)\), \((2, 1, 1)\).

Assume \( m_1 = m_2 \). We have

\[
m^2 + 2m_1^2 = 3mm_1 \quad \text{hence} \quad m^2 = (3m - 2)m_1^2.
\]

Therefore \( m_1 \) divides \( m \). Let \( m = km_1 \). We have

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Two larger, one smaller

Assume \( m_1 > m_2 \).

Question: Do we have \( m' > m_1 \) or else \( m' < m_1 \) ?

Consider the number \( a = (m_1 - m)(m_1 - m') \).

Since \( m + m' = 3m_1 m_2 \), and \( mm' = m_1^2 + m_2^2 \), we have

\[
a = m_1^2 - m_1(m + m') + mm' \\
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\]

However \( 2m_1^2 < 2m_1^2 m_2 \) and \( m_2^2 < m_1^2 m_2 \), hence \( a < 0 \).

This means that \( m_1 \) is in the interval defined by \( m \) and \( m' \).
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This means that $m_1$ is in the interval defined by $m$ and $m'$. 
Order of the solutions

If $m > m_1$, we have $m_1 > m'$ and the new solution $(m', m_1, m_2)$ is smaller than the initial solution $(m, m_1, m_2)$. If $m < m_1$, we have $m_1 < m'$ and the new solution $(m', m_1, m_2)$ is larger than the initial solution $(m, m_1, m_2)$. 

![Diagram](image)
Order of the solutions

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Prime factors

**Remark.** Let \( m \) be a Markoff number with

\[
m^2 + m_1^2 + m_2^2 = 3mm_1m_2.
\]

The same proof shows that the \( \gcd \) of \( m, m_1 \) and \( m_2 \) is 1: indeed, if \( p \) divides \( m_1, m_2 \) and \( m \), then \( p \) divides the new solutions which are produced by the preceding process – going down in the tree shows that \( p \) would divide 1.

The odd prime factors of \( m \) are all congruent to 1 modulo 4 (since they divide a sum of two relatively prime squares).

If \( m \) is even, then the numbers

\[
\frac{m}{2}, \quad \frac{3m - 2}{4}, \quad \frac{3m + 2}{8},
\]

are odd integers.
Prime factors

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Markoff’s Conjecture

The previous algorithm produces the sequence of Markoff numbers. Each Markoff number occurs infinitely often in the tree as one of the components of the solution.

According to the definition, for a Markoff number $m > 2$, there exist a pair $(m_1, m_2)$ of positive integers with $m > m_1 > m_2$ such that $m^2 + m_1^2 + m_2^2 = 3mm_1m_2$.

**Question**: Given $m$, is such a pair $(m_1, m_2)$ unique?

The answer is yes, as long as $m \leq 10^{105}$. 
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Frobenius’s work

*Markoff’s Conjecture* does not occur in Markoff’s 1879 and 1880 papers but in Frobenius’s one in 1913.

Ferdinand Georg Frobenius (1849–1917)
Special cases

The Conjecture has been proved for certain classes of Markoff numbers $m$ like $p^n$, $\frac{p^n \pm 2}{3}$ for $p$ prime.

A. Baragar (1996),
P. Schmutz (1996),
J.O. Button (1998),
M.L. Lang, S.P. Tan (2005),
Ying Zhang (2007).

http://www.nevada.edu/baragar/
Powers of a prime number

Anitha Srinivasan, 2007
A really simple proof of the Markoff conjecture for prime powers

Number Theory Web
Created and maintained by
Keith Matthews, Brisbane, Australia
www.numbertheory.org/pdfs/simpleproof.pdf
A triple \((a, b, c)\) of positive integers is called a Markoff triple iff it satisfies the diophantine equation \(a^2 + b^2 + c^2 = abc\). Recasting the Markoff tree, whose vertices are Markoff triples, in the framework of integral upper triangular \(3 \times 3\) matrices, it will be shown that the largest member of such a triple determines the other two uniquely. This answers a question which has been open for almost 100 years.
Markoff Equation and Nilpotent Matrices
arXiv:0709.1499 [math.NT]

From: Norbert Riedel
Submission history
[v2] Thu, 13 Sep 2007 18:45:29 GMT (11kb)
[v3] Tue, 4 Dec 2007 17:43:40 GMT (15kb)
[v7] Fri, 29 Mar 2013 12:36:56 GMT (0kb,I)

Comments: Most of the (correct) portion of this paper has been incorporated into the paper "On the Markoff equation" (arXiv:1208.4032)
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[v2] Mon, 1 Apr 2013 10 :21 :39 GMT (52kb)
[v4] Mon, 8 Jul 2013 17 :35 :21 GMT (53kb)
[v8] Sun, 12 Jan 2014 14 :59 :52 GMT (53kb)
[v10] Fri, 6 Feb 2015 20 :16 :28 GMT (59kb)
La théorie de Markoff et ses développements
Tessier et Ashpool, 2002.
Norbert Riedel

[v2] Thu, 13 Sep 2007 18:45:29 GMT (11kb)
[v3] Tue, 4 Dec 2007 17:43:40 GMT (15kb)

Comments by Serge Perrine (23/07/2009)
On the version 4 on March 10, 2009, 103 p.
Why the coefficient 3?

Let $n$ be a positive integer. If the equation $x^2 + y^2 + z^2 = nxyz$ has a solution in positive integers, then either $n = 3$ and $x$, $y$, $z$ are relatively prime, or $n = 1$ and the GCD of the numbers $x$, $y$, $z$ is 3.

Markoff type equations

*Bijection between the solutions for $n = 1$ and those for $n = 3$*

- if $x^2 + y^2 + z^2 = 3xyz$, then $(3x, 3y, 3z)$ is solution of $X^2 + Y^2 + Z^2 = XYZ$, since $(3x)^2 + (3y)^2 + (3z)^2 = (3x)(3y)(3z)$.

- if $X^2 + Y^2 + Z^2 = XYZ$, then $X$, $Y$, $Z$ are multiples of 3 and $(X/3)^2 + (Y/3)^2 + (Z/3)^2 = 3(X/3)(Y/3)(Z/3)$.

The squares modulo 3 are 0 and 1. If $X$, $Y$ and $Z$ are not multiples of 3, then $X^2 + Y^2 + Z^2$ is a multiple of 3.

If one or two (not three) integers among $X$, $Y$, $Z$ are multiples of 3, then $X^2 + Y^2 + Z^2$ is not a multiple of 3.
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- if \( x^2 + y^2 + z^2 = 3xyz \), then \((3x, 3y, 3z)\) is solution of \(X^2 + Y^2 + Z^2 = XYZ\), since
  \[(3x)^2 + (3y)^2 + (3z)^2 = (3x)(3y)(3z).\]

- if \(X^2 + Y^2 + Z^2 = XYZ\), then \(X, Y, Z\) are multiples of 3 and \((X/3)^2 + (Y/3)^2 + (Z/3)^2 = 3(X/3)(Y/3)(Z/3).\)

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Equations \( x^2 + ay^2 + bz^2 = (1 + a + b)xyz \)

If we insist that \((1, 1, 1)\) is a solution, then up to permutations there are only two more Diophantine equations of the type

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having infinitely many integer solutions, namely those with \((a, b) = (1, 2)\) and \((2, 3)\):

\[ x^2 + y^2 + 2z^2 = 4xyz \quad \text{and} \quad x^2 + 2y^2 + 3z^2 = 6xyz \]

- \(x^2 + y^2 + z^2\) : tessellation of the plane by equilateral triangles
- \(x^2 + y^2 + 2z^2 = 4xyz\) : tessellation of the plane by isosceles rectangle triangles
- \(x^2 + 2y^2 + 3z^2 = 6xyz\) : tessellation?
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Hurwitz’s equation (1907)

For each $n \geq 2$ the set $K_n$ of positive integers $k$ for which the equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 = kx_1 \cdots x_n$$

has a solution in positive integers is finite.

The largest value of $k$ in $K_n$ is $n$ — with the solution

$$(1, 1, \ldots, 1).$$

Examples:

$$K_3 = \{1, 3\},$$

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$$K_7 = \{1, 2, 3, 5, 7\}.$$
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When there is a solution in positive integers, there are infinitely many solutions, which can be organized in finitely many trees.

A. Baragar proved that there exists such equations which require an arbitrarily large number of trees:
J. Number Theory (1994), 49 No 1, 27-44.

The analog for the rank of elliptic curves over the rational number field is yet a conjecture.
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Growth of Markoff’s sequence

1978: order of magnitude of $m, m_1$ and $m_2$ for $m^2 + m_1^2 + m_2^2 = 3mm_1m_2$
with $m_1 < m_2 < m$,

$$\log(3m_1) + \log(3m_2) = \log(3m) + o(1).$$

To identify primitive words in a free group with two generators, H. Cohn used *Markoff forms*.

$$x \mapsto \log(3x) : (m_1, m_2, m) \mapsto (a, b, c) \text{ with } a + b \sim c.$$
Growth of Markoff’s sequence

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$x \mapsto \log(3x): (m_1, m_2, m) \mapsto (a, b, c)$ with $a + b \sim c$. 
Euclidean tree

Start with (0, 1, 1). From a triple \((a, b, c)\) satisfying \(a + b = c\) and \(a \leq b \leq c\), one produces two larger such triples \((a, c, a + c)\) and \((b, c, b + c)\) and a smaller one \((a, b - a, b)\) or \((b - a, a, b)\).
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Markoff and Euclidean trees

\( M \)

\( E \)

\( \psi \)

Figure 2. The Markoff tree and the Euclid tree

Tom Cusik & Mary Flahive,

*The Markoff and Lagrange spectra,*

Growth of Markoff’s sequence

Don Zagier (1982) : estimating the number of the Markoff triples bounded by $x$ :

$$c (\log x)^2 + O(\log x (\log \log x)^2),$$

$c = 0.1807170\ldots$

Conjecture : the $n$-th Markoff number $m_n$ is

$$m_n \sim A^{\sqrt{n}} \text{ with } A = 10.5101504\ldots$$

Remark : $c = 1/(\log A)^2$. 
Growth of Markoff’s sequence

Don Zagier (1982) : estimating the number of the Markoff triples bounded by $x$ :

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Markoff and Diophantine approximation

J.W.S. Cassels,
An introduction to
Diophantine approximation,
Cambridge Univ. Press (1957)
Connection with Hurwitz’s Theorem

Don Zagier,

*On the number of Markoff numbers below a given bound.*

Abstract. According to a famous theorem of Markoff, the indefinite quadratic forms with exceptionally large minima (greater than $\sqrt{\Delta}$ of the square root of the discriminant) are in 1:1 correspondence with the solutions of the Diophantine equation

$$p^2 + q^2 + r^2 = 3pqr,$$

By relating Markoff's algorithm for finding solutions of this equation to a problem of counting lattice points in triangles, it is shown that the number of solutions less than $x$ equals $C \log 23x + O(\log x \log \log 2x)$ with an explicitly computable constant $C = 0.18071704711507$. . . .

Numerical data up to is presented which suggests that the true error term is considerably smaller.

1. By a Markoff triple we mean a solution $(p, q, r)$ of the Markoff equation

$$p^2 + q^2 + r^2 = 3pqr (p, q, r \in \mathbb{Z}, \Delta \neq p \neq q \neq r);$$

a Markoff number is a member of such a triple. These numbers, of which the first few are

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, . . . ,$$

play a role in a famous theorem of Markoff [lo] (see also Frobenius [6], Cassels [21]): the $GL_2(\mathbb{Z})$-equivalence classes of real indefinite binary quadratic forms $Q$ of discriminant 1 for which the invariant $p(Q)$ is greater than $\frac{3}{4}$ are in one-to-one correspondence with the Markoff triples, the invariant $p(Q)$ for the form corresponding to $(p, q, r)$ being $(9 - 4r - 2 \sqrt{\Delta})/2$. Thus the part of the Markoff spectrum (the set of all $\mu(Q)$ lying above $\frac{1}{3}$ is described exactly by the Markoff numbers. An equivalent theorem is that, under the action of $SL_2(\mathbb{Z})$ on $\mathbb{R} \cup \{\infty\}$ given by $x \to (ax + b)/(cx + d)$, the $SL_2(\mathbb{Z})$-equivalence classes of real numbers $x$ for which the approximation measure

$$\mu(x) = \limsup_{q \to \infty} \left( q \cdot \min_{p \in \mathbb{Z}} |qx - p| \right)$$

is $> \frac{1}{3}$ are in 1:1 correspondence with the Markoff triples, the spectrum being the same as above (e.g. $\mu(x) = 5^{-1/2}$ for $x$ equivalent to the golden ratio and $\mu(x) \leq 8^{-1/2}$ for all other $x$). Thus the Markoff numbers are important both in the theory of quadratic forms and in the theory of Diophantine approximation. They have also M.W., *Open Diophantine Problems*, Moscow Mathematical Journal **4** N°1, 2004, 245–305.
Historical origin: rational approximation

**Hurwitz’s Theorem** (1891): For any real irrational number \( x \), there exist infinitely many rational numbers \( p/q \) such that

\[
| x - \frac{p}{q} | \leq \frac{1}{\sqrt{5}q^2}.
\]

Golden ratio
\[ \Phi = (1 + \sqrt{5})/2 = 1.6180339\ldots \]

Hurwitz’s result is optimal.

Adolf Hurwitz
(1859–1919)
Lagrange’s constant

For \( x \in \mathbb{R} \setminus \mathbb{Q} \) denote by \( \lambda(x) \in [\sqrt{5}, +\infty] \) the least upper bound of the numbers \( \lambda > 0 \) such that there exist infinitely many \( p/q \in \mathbb{Q} \) satisfying

\[
\left| x - \frac{p}{q} \right| \leq \frac{1}{\lambda q^2}.
\]

This means

\[
\frac{1}{\lambda(x)} = \lim \inf_{q \to \infty} (q \min_{p \in \mathbb{Z}} |qx - p|).
\]

Hurwitz : \( \lambda(x) \geq \sqrt{5} = 2.2360679 \cdots \) for any \( x \) and \( \lambda(\Phi) = \sqrt{5} \).
Lagrange’s constant

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Badly approximable numbers

An irrational real number $x$ is *badly approximable* by rational numbers if its Lagrange’s constant is finite. This means that there exists $\lambda > 0$ such that, for any $p/q \in \mathbb{Q}$,

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{\lambda q^2}.$$ 

For instance Liouville’s numbers have an infinite Lagrange’s constant.

A real number is badly approximable if and only if the sequence $(a_n)_{n \geq 0}$ of partial quotients in its continued fraction expansion

$$x = [a_0, a_1, a_2, \ldots, a_n, \ldots]$$

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Any quadratic irrational real number has a finite Lagrange’s constant (= is badly approximable).

It is not known whether there exist real algebraic numbers of degree $\geq 3$ which are badly approximable.

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One conjectures that any irrational real number which is not quadratic and which is badly approximable is transcendental. This means that one conjectures that no real algebraic number of degree $\geq 3$ is badly approximable.
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Lebesgue measure

The set of badly approximable real numbers has zero measure for Lebesgue’s measure.

Henri Léon Lebesgue (1875–1941)
Properties of the Lagrange’s constant

We have

\[ \lambda(x + 1) = \lambda(x) : \quad \left| x + 1 - \frac{p}{q} \right| = \left| x - \frac{p + q}{q} \right| \]

and

\[ \lambda(-x) = \lambda(x) : \quad \left| -x - \frac{p}{q} \right| = \left| x + \frac{p}{q} \right| , \]

Also \( \lambda(1/x) = \lambda(x) : \)

\[ p^2 \left| \frac{1}{x} - \frac{q}{p} \right| = q^2 \left| \frac{p}{qx} \right| \cdot \left| x - \frac{p}{q} \right| . \]
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The modular group

The multiplicative group generated by the three matrices
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
is the group \( \text{GL}_2(\mathbb{Z}) \) of \( 2 \times 2 \) matrices
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
with coefficients in \( \mathbb{Z} \) and determinant \( \pm 1 \).

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax + b}{cx + d}
\]

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x = x + 1 \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x = -x \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x = \frac{1}{x}
\]

\[
\lambda(x + 1) = \lambda(x) \quad \lambda(-x) = \lambda(x) \quad \lambda(1/x) = \lambda(x)
\]

**Consequence**: Let \( x \in \mathbb{R} \setminus \mathbb{Q} \) and let \( a, b, c, d \) be rational integers satisfying \( ad - bc = \pm 1 \). Set

\[
y = \frac{ax + b}{cx + d}.
\]

Then \( \lambda(x) = \lambda(y) \).
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\[
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Then \( \lambda(x) = \lambda(y) \).
The inequality $\lambda(x) \geq \sqrt{5}$ for all real irrational $x$ is optimal for the **Golden ratio** and for all the **noble** irrational numbers whose continued fraction expansion ends with an infinite sequence of 1’s – these numbers are roots of the quadratic polynomials having discriminant 5:

$$\Phi = [1, 1, 1, \ldots] = [1].$$

*Adolf Hurwitz, 1891*
Noble numbers

A noble number, whose continued fraction expansion ends with an infinite sequence of 1’s, is a number related to the Golden ratio $\Phi$ by a homography of determinant $\pm 1$:

$$\frac{a\Phi + b}{c\Phi + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}).$$
The first elements of the spectrum

For all the real numbers which are not noble numbers, a stronger inequality than Hurwitz’s inequality

$$\lambda(x) \geq \sqrt{5} = 2,236,067,977 \ldots$$

is valid, namely

$$\lambda(x) \geq 2\sqrt{2} = 2,828,427,125 \ldots$$

This is optimal for

$$\sqrt{2} = 1.414213562373095048801688724209698078 \ldots$$

whose continued fraction expansion is

$$[1; 2, 2, \ldots, 2, \ldots] = [1; \bar{2}].$$
The first elements of the spectrum

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\[ \sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\ldots \]

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\[ [1; 2, 2, \ldots, 2, \ldots] = [1; \bar{2}]. \]
The constant $\sqrt{5}$ is best possible in Hurwitz’s Theorem.

If $\alpha$ is not $\text{GL}(2, \mathbb{Z})$ equivalent to the Golden ratio, the constant $\sqrt{5}$ improves to $\sqrt{8}$.

If we also exclude the numbers which are $\text{GL}(2, \mathbb{Z})$ equivalent to $1 + \sqrt{2}$, then we can take $\sqrt{221}/5$ as the constant.

The sequence

$$\sqrt{5} = 2.236\ldots, \quad \sqrt{8} = 2.828\ldots, \quad \sqrt{221}/5 = 2.973\ldots$$

continues as follows.
Lagrange Spectrum

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There is an infinite increasing sequence of real numbers \((L_i)_{i \geq 1}\) with limit 3, and an associated sequence of quadratic irrationalities \((\theta_i)_{i \geq 1}\), such that for any \(i \geq 1\), if \(\alpha\) is not \(\text{GL}(2, \mathbb{Z})\) equivalent to any of \(\theta_1, \ldots, \theta_{i-1}\), then there is infinitely many rational numbers that satisfy

\[
\left| \alpha - \frac{p}{q} \right| < \frac{1}{L_i q^2},
\]

and this is best possible when \(\alpha\) is \(\text{GL}(2, \mathbb{Z})\) equivalent to \(\theta_i\).
Lagrange Spectrum

Denoting by $m_1, m_2, \ldots$ the sequence of Markoff numbers, one has

$$L_i = \sqrt{9 - \frac{4}{m_i^2}}$$

and

$$\theta_i = \frac{-3m_i + 2k_i + \sqrt{9m_i^2 - 4}}{2m_i},$$

where $k_i$ is an integer satisfying $a_ik_i \equiv b_i \pmod{m_i}$ and $(a_i, b_i, m_i)$ is a solution of Markoff’s equation with $m_i \geq \max\{a_i, b_i\}$.

Here one assumes Markoff’s Conjecture to get unicity of $(a_i, b_i)$. 
Lagrange Spectrum

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*Here one assumes Markoff’s Conjecture to get unicity of $(a_i, b_i)$.*
First values

\[ L_i = \sqrt{9 - \frac{4}{m_i^2}}, \quad \theta_i = \frac{-3m_i + 2k_i + \sqrt{9m_i^2 - 4}}{2m_i} \]

\[ F_i(X, Y) = m_iX^2 + (2k_i - 3m_i)XY + \frac{k_i^2 - 3k_i m_i + 1}{m_i} Y^2. \]

<table>
<thead>
<tr>
<th>( i )</th>
<th>( m_i )</th>
<th>( k_i )</th>
<th>( L_i )</th>
<th>( \theta_i )</th>
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<td>( 2(X^2 - 2XY - Y^2) )</td>
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<td>3</td>
<td>5</td>
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<td>( \sqrt{221/5} )</td>
<td>( -11 + \sqrt{221} )/10</td>
<td>( 5X^2 + 11XY - 5Y^2 )</td>
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Let \( f(X, Y) = aX^2 + bXY + cY^2 \) be a quadratic form with real coefficients. Denote by \( \Delta(f) \) its discriminant \( b^2 - 4ac \).

Consider the minimum \( m(f) \) of \( |f(x, y)| \) on \( \mathbb{Z}^2 \setminus \{(0, 0)\} \). Assume \( \Delta(f) \neq 0 \) and set

\[
C(f) = m(f) / \sqrt{|\Delta(f)|}.
\]

Let \( \alpha \) and \( \alpha' \) be the roots of \( f(X, 1) :\)

\[
f(X, Y) = a(X - \alpha Y)(X - \alpha' Y),
\]

\[
\{\alpha, \alpha'\} = \left\{ \frac{1}{2a} ( -b \pm \sqrt{\Delta(f)} ) \right\}.
\]
Minima of quadratic forms

Let $f(X, Y) = aX^2 + bXY + cY^2$ be a quadratic form with real coefficients. Denote by $\Delta(f)$ its discriminant $b^2 - 4ac$.

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Example with $\Delta < 0$

The form

$$f(X, Y) = X^2 + XY + Y^2$$

has discriminant $\Delta(f) = -3$ and minimum $m(f) = 1$, hence

$$C(f) = \frac{m(f)}{\sqrt{|\Delta(f)|}} = \frac{1}{\sqrt{3}}.$$

For $\Delta < 0$, the form

$$f(X, Y) = \sqrt{\frac{3}{\Delta}}(X^2 + XY + Y^2)$$

has discriminant $\Delta$ and minimum $\sqrt{\frac{|\Delta|}{3}}$. Again

$$C(f) = \frac{1}{\sqrt{3}}.$$
Example with $\Delta < 0$

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For $\Delta < 0$, the form

$$f(X, Y) = \sqrt{\frac{|\Delta|}{3}}(X^2 + XY + Y^2)$$

has discriminant $\Delta$ and minimum $\sqrt{|\Delta|/3}$. Again

$$C(f) = \frac{1}{\sqrt{3}}.$$
Definite quadratic forms ($\Delta < 0$)

If the discriminant is negative, J.L. Lagrange and Ch. Hermite (letter to Jacobi, August 6, 1845) proved $C(f) \leq 1/\sqrt{3}$ with equality for $f(X, Y) = X^2 + XY + Y^2$. For each $\varrho \in (0, 1/\sqrt{3}]$, there exists such a form $f$ with $C(f) = \varrho$.

Example with $\Delta > 0$

The form

$$f(X, Y) = X^2 - XY - Y^2$$

has discriminant $\Delta(f) = 5$ and minimum $m(f) = 1$, hence

$$C(f) = \frac{m(f)}{\sqrt{\Delta(f)}} = \frac{1}{\sqrt{5}}.$$ 

For $\Delta > 0$, the form

$$f(X, Y) = \sqrt{\frac{\Delta}{5}}(X^2 - XY - Y^2)$$

has discriminant $\Delta$ and minimum $\sqrt{\Delta / 5}$. Again

$$C(f) = \frac{1}{\sqrt{5}}.$$
**Example with \( \Delta > 0 \)**

The form

\[
f(X, Y) = X^2 - XY - Y^2
\]

has discriminant \( \Delta(f) = 5 \) and minimum \( m(f) = 1 \), hence

\[
C(f) = \frac{m(f)}{\sqrt{\Delta(f)}} = \frac{1}{\sqrt{5}}.
\]

For \( \Delta > 0 \), the form

\[
f(X, Y) = \sqrt{\frac{\Delta}{5}}(X^2 - XY - Y^2)
\]

has discriminant \( \Delta \) and minimum \( \sqrt{\Delta/5} \). Again

\[
C(f) = \frac{1}{\sqrt{5}}.
\]
Indefinite quadratic forms ($\Delta > 0$)

Assume $\Delta > 0$

A. Korkine and E.I. Zolotarev proved in 1873 $C(f) \leq 1/\sqrt{5}$ with equality for $f_0(X, Y) = X^2 - XY - Y^2$. For all forms which are not equivalent to $f_0$ under $GL(2, \mathbb{Z})$, they prove $C(f) \leq 1/\sqrt{8}$.

$1/\sqrt{5} = 0.447213595\ldots$

$1/\sqrt{8} = 0.353553391\ldots$

Egor Ivanovich Zolotarev (1847–1878)
Indefinite quadratic forms ($\Delta > 0$).

The works by Korkine and Zolotarev inspired Markoff who pursued the study of this question. He produced infinitely many values $C(f_i), \ i = 0, 1, \ldots$, between $1/\sqrt{5}$ and $1/3$, with the same property as $f_0$. These values form a sequence which converges to $1/3$. He constructed them by means of the tree of solutions of the Markoff equation.

A. Markoff, 1879 and 1880.
Indefinite quadratic forms ($\Delta > 0$)

Assume $f((X, Y) = aX^2 + bXY + cY^2 \in \mathbb{R}[X, Y]$ with $a > 0$ has discriminant $\Delta > 0$.

If $|f(x, y)|$ is small with $y \neq 0$, then $x/y$ is close to a root of $f(X, 1)$, say $\alpha$.

Then

$$|x - \alpha'y| \sim |y| \cdot |\alpha - \alpha'|$$

and $\alpha - \alpha' = \sqrt{\Delta}/a$.

Hence

$$|f(x, y)| = |a(x - \alpha y)(x - \alpha'y)| \sim y^2 \sqrt{\Delta} \left|\alpha - \frac{x}{y}\right|.$$
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Lagrange spectrum and Markoff spectrum

Markoff spectrum = set of values taken by

\[ \frac{1}{C(f)} = \sqrt{\Delta(f)/m(f)} \]

when \( f \) runs over the set of quadratic forms \( ax^2 + bxy + cy^2 \) with real coefficients of discriminant \( \Delta(f) = b^2 - 4ac > 0 \) and \( m(f) = \inf_{(x,y) \in \mathbb{Z}^2 \setminus \{0\}} |f(x, y)| \).

Lagrange spectrum = set of values taken by Lagrange’s constant.

\[ \lambda(x) = 1/\liminf_{q \to \infty} q(\min_{p \in \mathbb{Z}} |qx - p|) \]

when \( x \) runs over the set of real numbers.

The Markoff spectrum contains the Lagrange spectrum. The intersection with the interval \([\sqrt{5}, 3]\) is the same for both of them, and is a discrete sequence.
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Chronologie

J. Liouville, 1844
J-L. Lagrange et Ch. Hermite, 1845
A. Korkine et E.I. Zolotarev, 1873
A. Markoff, 1879
F. Frobenius, 1915
L. Ford, 1917, 1938
R. Remak, 1924
J.W.S. Cassels, 1949
J. Lehner, 1952, 1964
H. Cohn, 1954, 1980
R.A. Rankin, 1957
A. Schmidt, 1976
S. Perrine, 2002
Fraction continue et géométrie hyperbolique

Référence : Caroline Series
The Geometry of Markoff Numbers

Caroline Series,
The Geometry of Markoff Numbers,
Markoff’s tree can be seen as the dual of the triangulation of the hyperbolic upper half plane by the images of the fundamental domain of the modular invariant under the action of the modular group.

Lazarus Immanuel Fuchs (1833–1902)
Triangulation of polygons, metric properties of polytopes

Harold Scott MacDonald Coxeter (1907–2003)
Robert Alexander Rankin (1915–2001)
John Horton Conway
Fricke groups

The subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ generated by the two matrices

\[
\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
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is the free group with two generators.

The Riemann surface quotient of the Poincaré upper half plane by $\Gamma$ is a punctured torus. The minimal lengths of the closed geodesics are related to the $C(f)$, for $f$ indefinite quadratic form.
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\[(\text{tr}A)^2 + (\text{tr}B)^2 + (\text{tr}AB)^2 = (\text{tr}A)(\text{tr}B)(\text{tr}AB)\]

Harvey Cohn showed that quadratic forms with a Markoff constant $C(f) \in ]1/3, 1/\sqrt{5}]$ are equivalent to

\[cx^2 + (d - a)xy - by^2\]

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Fundamental domain of a punctured disc

Figure 13. Fundamental region for the punctured torus.
A simple curve on a punctured disc

Figure 1. A simple curve on the punctured torus.
Ford circles

The Ford circle associated to the irreducible fraction $p/q$ is tangent to the real axis at the point $p/q$ and has radius $1/2q^2$.

Ford circles associated to two consecutive elements in a Farey sequence are tangent.

Lester Randolph Ford (1886–1967)

Farey sequence of order 5

\[
\begin{align*}
0 & \quad 1 & \quad 1 & \quad 2 & \quad 1 & \quad 3 & \quad 2 & \quad 3 & \quad 4 & \quad 1 \\
1' & \quad 5' & \quad 4' & \quad 3' & \quad 5' & \quad 2' & \quad 5' & \quad 3' & \quad 4' & \quad 5' & \quad 1
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\]
Complex continued fraction

The third generation of Asmus Schmidt’s complex continued fraction method.

http://www.maa.org/editorial/mathgames/mathgames_03_15_04.html
Laurent’s phenomenon

Connection with Laurent polynomials.

If \( f, g, h \) are Laurent polynomials in two variables \( x \) and \( y \), i.e., polynomials in \( x, x^{-1}, y, y^{-1} \), in general

\[
h(f(x, y), g(x, y))
\]

is not a Laurent polynomial:

\[
f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x},
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\[
f(f(x)) = \frac{\left(x + \frac{1}{x}\right)^2 + 1}{x + \frac{1}{x}} = \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)}.
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Simultaneous rational approximation and Markoff spectrum

Relation between Markoff numbers and extremal numbers: simultaneous approximation of $x$ and $x^2$ by rational numbers with the same denominator.

Markoff–Lagrange spectrum and extremal numbers, arXiv.0906.0611 [math.NT] 2 June 2009

Damien Roy
Greatest prime factor of Markoff pairs

Pietro Corvaja and Umberto Zannier, 2006:
The greatest prime factor of the product $xy$ when $x, y, z$ is a solution of Markoff’s equation tends to infinity with $\max\{x, y, z\}$.

Equivalent statement:
If $S$ denotes a finite set of prime numbers, the equation

$$x^2 + y^2 + z^2 = 3xyz$$

has only finitely many solutions in positive integers $x, y, z$, such that $xy$ has no prime divisor outside $S$. (The integers $x$ and $y$ are called $S$–units.)
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On the Markoff Equation

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Michel Waldschmidt

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http://www.math.jussieu.fr/~miw/