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The four exponentials conjecture, the six exponentials theorem and related statements.

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Transcendental Number Theory

Liouville (1844). Transcendental numbers exist.

Hermite (1873). The number e is transcendental.

Lindemann (1882). The number π is transcendental.

Corollary: Squaring the circle using rule and compass only is not possible.

Theorem (Hermite-Lindemann). Let β be a non zero complex number. Set $\alpha = e^{\beta}$. Then one at least of the two numbers α, β is transcendental.

Corollary 1. If β is a non zero algebraic number, then e^{β} is transcendental.

Example. The numbers e and $e^{\sqrt{2}}$ are transcendental.

Corollary 2. If α is a non zero algebraic number and $\log \alpha$ a non zero logarithm of α, then $\log \alpha$ is transcendental.

Example. The numbers $\log 2$ and π are transcendental.

Recall: the set $\overline{\mathbf{Q}}$ of algebraic numbers is a subfield of $\mathbf{C}$ - it is the algebraic closure of $\mathbf{Q}$ into $\mathbf{C}$.

The exponential map

$$\begin{array}{rccc} \exp : & \mathbf{C} & \longrightarrow & \mathbf{C}^\times \\ & z & \longmapsto & e^z \end{array}$$

is a morphism from the additive group of $\mathbf{C}$ onto the multiplicative group of $\mathbf{C}^\times$.

Hermite-Lindemann: $\overline{\mathbf{Q}}^\times \cap \exp(\overline{\mathbf{Q}}) = \{1\}$.

Also, if we define $\mathcal{L} = \exp^{-1}(\overline{\mathbf{Q}}^\times)$, then $\overline{\mathbf{Q}} \cap \mathcal{L} = \{0\}$.

Theorem (Gel'fond-Schneider).

(1) Let λ_1 and λ_2 be two elements in $\mathcal{L}$ which are linearly independent over $\mathbf{Q}$. Then λ_1 and λ_2 are linearly independent over $\overline{\mathbf{Q}}$.

(2) Let $\lambda \in \mathbf{C}$, $\lambda \neq 0$ and let $\beta \in \mathbf{C} \setminus \mathbf{Q}$. Then one at least of the three numbers

$$e^{\lambda},\ \beta,\ e^{\beta\lambda}$$

is transcendental.

Remark $(1) \iff (2)$ follows from

$$(\lambda_1, \lambda_2) \longleftrightarrow (\lambda, \beta\lambda) \qquad \text{and} \qquad (\lambda_1, \lambda_2/\lambda_1) \longleftrightarrow (\lambda, \beta).$$

Remarks.

(1) means:

$$\frac{\log \alpha_1}{\log \alpha_2} \not\in \overline{\mathbf{Q}} \setminus \mathbf{Q}.$$

(2) means: for $\alpha \in \mathbf{C} \setminus \{0\}$, $\beta \in \mathbf{C} \setminus \mathbf{Q}$ and any $\log \alpha \neq 0$, one at least of the three numbers

$$\alpha, \quad \beta \quad \text{and} \quad \alpha^\beta = e^{\beta \log \alpha}$$

is transcendental.

Corollaries. Transcendence of

$$\frac{\log 2}{\log 3}, \quad 2^{\sqrt{2}}, \quad e^{\pi}.$$

Proof. Take respectively

$$\lambda_1 = \log 2, \quad \lambda_2 = \log 3,$$

$$\lambda_1 = \log 2, \quad \lambda_2 = \sqrt{2}\log 2,$$

and

$$\lambda_1 = i\pi, \quad \lambda_2 = \pi.$$

Theorem (A. Baker). Let $\lambda_1, \ldots, \lambda_n$ be elements in $\mathcal{L}$ which are linearly independent over $\mathbf{Q}$. Then $1, \lambda_1, \ldots, \lambda_n$ are linearly independent over $\overline{\mathbf{Q}}$.

Corollary. Transcendence of numbers like

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

and

$$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}.$$

Corollary 1. (Hermite-Lindemann) Transcendence of e^β.

Proof. Take $n = 1$ in Baker's Theorem.

Corollary 2. (Gel'fond-Schneider) Transcendence of α^β.

Proof. Take $n = 2$ in Baker's Theorem.

Baker's Theorem means: The injection of $\mathbf{Q} + \mathcal{L}$ into $\mathbf{C}$ extends to an injection of $(\mathbf{Q} + \mathcal{L}) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ into $\mathbf{C}$. The image is the $\overline{\mathbf{Q}}$-vector space $\widetilde{\mathcal{L}} \subset \mathbf{C}$ spanned by 1 and $\mathcal{L}$:

$$(\mathbf{Q} + \mathcal{L}) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} \simeq \widetilde{\mathcal{L}}.$$

Algebraic independence of logarithms of algebraic numbers

Conjecture *Let* $\alpha_1, \ldots, \alpha_n$ *be non zero algebraic numbers. For* $1 \le j \le n$ *let* $\lambda_j \in \mathbf{C}$ *satisfy* $e^{\lambda_j} = \alpha_j$. *Assume* $\lambda_1, \ldots, \lambda_n$ *are linearly independent over* $\mathbf{Q}$. *Then* $\lambda_1, \ldots, \lambda_n$ *are algebraically independent.*

Write $\lambda_j = \log \alpha_j$.

If $\log \alpha_1, \ldots, \log \alpha_n$ *are* $\mathbf{Q}$*-linearly independent then they are algebraically independent.*

Recall that $\mathcal{L}$ is the $\mathbf{Q}$ vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \{\lambda \in \mathbf{C}\,;\; e^\lambda \in \overline{\mathbf{Q}}\} = \{\log \alpha\,;\; \alpha \in \overline{\mathbf{Q}}^\times\} = \exp^{-1}(\overline{\mathbf{Q}}^\times).$$

The conjecture on algebraic independence of logarithms of algebraic numbers can be stated:

The injection of $\mathcal{L}$ into $\mathbf{C}$ extends to an injection of the symmetric algebra $\mathrm{Sym}_{\mathbf{Q}}(\mathcal{L})$ on $\mathcal{L}$ into $\mathbf{C}$.

Four Exponentials Conjecture

History.

A. Selberg (50's).

Th. Schneider(1957) - first problem.

S Lang (60's).

K. Ramachandra (1968).

Leopoldt's Conjecture on the p-adic rank of the units of an algebraic number field (non vanishing of the p-adic regulator).

Quadratic relations between logarithms of algebraic numbers

Homogeneous:

Four Exponentials Conjecture *For* $i = 1, 2$ *and* $j = 1, 2$, *let* α_{ij} *be a non zero algebraic number and* λ_{ij} *a complex number satisfying* $e^{\lambda_{ij}} = \alpha_{ij}$. *Assume* $\lambda_{11}, \lambda_{12}$ *are linearly independent over* $\mathbf{Q}$ *and also* $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$. *Then*

$$\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}.$$

Quadratic relations between logarithms of algebraic numbers

Example. Transcendence of $2^{(\log 2)/\log 3}$:

$$(\log 2)^2 = (\log 3)(\log \alpha)?$$

Other open problem: Transcendence of $2^{\log 2}$.

Non homogeneous quadratic relations $(\log \alpha)(\log \beta) = \log \gamma$.

$$(\log 2)^2 = \log \gamma?$$

Quadratic relations between logarithms of algebraic numbers

Non homogeneous:

Three exponentials Conjecture *Let λ_1, λ_2, λ_3 be three elements in $\mathcal{L}$ satisfying $\lambda_1\lambda_2 = \lambda_3$. Then $\lambda_3 = 0$.*

Example. Special case of the open question on the transcendence of $\alpha^{\log \alpha}$: transcendence of e^{π^2}?

Four Exponentials Conjecture *For* $i = 1, 2$ *and* $j = 1, 2$, *let* α_{ij} *be a non zero algebraic number and* λ_{ij} *a complex number satisfying* $e^{\lambda_{ij}} = \alpha_{ij}$. *Assume* $\lambda_{11}, \lambda_{12}$ *are linearly independent over* $\mathbf{Q}$ *and also* $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$. *Then*

$$\lambda_{11}\lambda_{22} \neq \lambda_{12}\lambda_{21}.$$

Notice:

$$\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = \det \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{vmatrix}.$$

Four Exponentials Conjecture *(again)* *Let* x_1, x_2 *be two* $\mathbf{Q}$*-linearly independent complex numbers and* y_1, y_2 *also two* $\mathbf{Q}$*-linearly independent complex numbers. Then one at least of the four numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2}$$

is transcendental.

Four Exponentials Conjecture *(again)* *Let* x_1, x_2 *be two* $\mathbf{Q}$-*linearly independent complex numbers and* y_1, y_2 *also two* $\mathbf{Q}$-*linearly independent complex numbers. Then one at least of the four numbers*

$$e^{x_1 y_1},\ e^{x_1 y_2},\ e^{x_2 y_1},\ e^{x_2 y_2}$$

is transcendental.

Hint: Set

$$x_i y_j = \lambda_{ij}\ (i = 1, 2;\ j = 1, 2).$$

A rank one matrix is a matrix of the form

$$\begin{pmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \end{pmatrix}.$$

Sharp Four Exponentials Conjecture. *If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2*, are two complex numbers which are* $\mathbf{Q}$*-linearly independent and if* $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ *are four algebraic numbers such that the four numbers*

$$e^{x_1y_1-\beta_{11}},\ e^{x_1y_2-\beta_{12}},\ e^{x_2y_1-\beta_{21}},\ e^{x_2y_2-\beta_{22}}$$

are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2$ *and* $j = 1, 2$*.*

Sharp Four Exponentials Conjecture. *If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2, *are two complex numbers which are* $\mathbf{Q}$*-linearly independent and if* $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ *are four algebraic numbers such that the four numbers*

$$e^{x_1y_1-\beta_{11}}, \; e^{x_1y_2-\beta_{12}}, \; e^{x_2y_1-\beta_{21}}, \; e^{x_2y_2-\beta_{22}}$$

are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2$ *and* $j = 1, 2$.

If $x_iy_j = \lambda_{ij} + \beta_{ij}$ then

$$x_1x_2y_1y_2 = (\lambda_{11} + \beta_{11})(\lambda_{22} + \beta_{22}) = (\lambda_{12} + \beta_{12})(\lambda_{21} + \beta_{21}).$$

Recall that $\widetilde{\mathcal{L}}$ is the $\overline{\mathbf{Q}}$-vector space spanned by 1 and $\mathcal{L}$ (linear combinations of logarithms of algebraic numbers with algebraic coefficients):

$$\widetilde{\mathcal{L}} = \left\{ \beta_0 + \sum_{h=1}^{\ell} \beta_h \log \alpha_h \; ; \; \ell \geq 0, \; \alpha\text{'s in } \overline{\mathbf{Q}}^{\times}, \; \beta\text{'s in } \overline{\mathbf{Q}} \right\}$$

Strong Four Exponentials Conjecture. *If* x_1, x_2 *are* $\overline{\mathbf{Q}}$-*linearly independent and if* y_1, y_2, *are* $\overline{\mathbf{Q}}$-*linearly independent, then one at least of the four numbers*

$$x_1y_1, \; x_1y_2, \; x_2y_1, \; x_2y_2$$

does not belong to $\widetilde{\mathcal{L}}$.

Six Exponentials Theorem. *If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2, y_3 *are three complex numbers which are* $\mathbf{Q}$*-linearly independent, then one at least of the six numbers*

$$e^{x_i y_j} \quad (i = 1, 2,\ j = 1, 2, 3)$$

is transcendental.

Sharp Six Exponentials Theorem. *If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2, y_3 *are three complex numbers which are* $\mathbf{Q}$*-linearly independent and if* β_{ij} *are six algebraic numbers such that*

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \textit{for} \quad i = 1, 2,\ j = 1, 2, 3,$$

then $x_i y_j = \beta_{ij}$ *for* $i = 1, 2$ *and* $j = 1, 2, 3.$

Strong Six Exponentials Theorem (*D. Roy*). *If* x_1, x_2 *are* $\overline{\mathbf{Q}}$*-linearly independent and if* y_1, y_2, y_3 *are* $\overline{\mathbf{Q}}$*-linearly independent, then one at least of the six numbers*

$$x_i y_j \quad (i = 1, 2,\ j = 1, 2, 3)$$

does not belong to $\widetilde{\mathcal{L}}$.

Five Exponentials Theorem. *If* x_1, x_2 *are* $\mathbf{Q}$*-linearly independent,* y_1, y_2 *are* $\mathbf{Q}$*-linearly independent and* γ *is a non zero algebraic number, then one at least of the five numbers*

$$e^{x_1 y_1},\ e^{x_1 y_2},\ e^{x_2 y_1},\ e^{x_2 y_2},\ e^{\gamma x_2/x_1}$$

is transcendental.

This is a consequence of the sharp six exponentials Theorem: set $y_3 = \gamma/x_1$ and use Baker's Theorem for checking that y_1, y_2, y_3 are linearly independent over $\mathbf{Q}$.

Sharp Five Exponentials Conjecture. *If* x_1, x_2 *are* $\mathbf{Q}$-*linearly independent, if* y_1, y_2 *are* $\mathbf{Q}$-*linearly independent and if* $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ *are six algebraic numbers with* $\gamma \neq 0$ *such that*

$$e^{x_1y_1-\beta_{11}},\ e^{x_1y_2-\beta_{12}},\ e^{x_2y_1-\beta_{21}},\ e^{x_2y_2-\beta_{22}},\ e^{(\gamma x_2/x_1)-\alpha}$$

are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2,\ j = 1, 2$ *and also* $\gamma x_2 = \alpha x_1$.

Sharp Five Exponentials Conjecture. *If* x_1, x_2 *are* $\mathbf{Q}$-*linearly independent, if* y_1, y_2 *are* $\mathbf{Q}$-*linearly independent and if* $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ *are six algebraic numbers with* $\gamma \neq 0$ *such that*

$$e^{x_1 y_1 - \beta_{11}}, \; e^{x_1 y_2 - \beta_{12}}, \; e^{x_2 y_1 - \beta_{21}}, \; e^{x_2 y_2 - \beta_{22}}, \; e^{(\gamma x_2 / x_1) - \alpha}$$

are algebraic, then $x_i y_j = \beta_{ij}$ *for* $i = 1, 2$, $j = 1, 2$ *and also* $\gamma x_2 = \alpha x_1$.

Difficult case: when $y_1, y_2, \gamma/x_1$ are $\mathbf{Q}$-linearly dependent.

Example: $x_1 = y_1 = \gamma = 1$.

Sharp Five Exponentials Conjecture. *If* x_1, x_2 *are* $\mathbf{Q}$-*linearly independent, if* y_1, y_2 *are* $\mathbf{Q}$-*linearly independent and if* $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ *are six algebraic numbers with* $\gamma \neq 0$ *such that*

$$e^{x_1y_1-\beta_{11}},\; e^{x_1y_2-\beta_{12}},\; e^{x_2y_1-\beta_{21}},\; e^{x_2y_2-\beta_{22}},\; e^{(\gamma x_2/x_1)-\alpha}$$

are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2,$ $j = 1, 2$ *and also* $\gamma x_2 = \alpha x_1$.

Consequence: *Transcendence of the number* e^{π^2}.

Proof. Set $x_1 = y_1 = 1$, $x_2 = y_2 = i\pi$, $\gamma = 1$, $\alpha = 0$, $\beta_{11} = 1$, $\beta_{ij} = 0$ for $(i,j) \neq (1,1)$.

Known: *One* **at least** *of the two statements is true.*

- e^{π^2} *is transcendental.*

- *The two numbers* e *and* π *are algebraically independent.*

Other consequence of the sharp five exponentials conjecture: *Transcendence of the number* $e^{\lambda^2} = \alpha^{\log \alpha}$ *for* $\lambda \in \mathcal{L}$, $e^\lambda = \alpha \in \overline{\mathbf{Q}}^\times$.

Proof. Set $x_1 = y_1 = 1$, $x_2 = y_2 = \lambda$, $\gamma = 1$, $\alpha = 0$, $\beta_{11} = 1$, $\beta_{ij} = 0$ for $(i,j) \neq (1,1)$.

Other consequence of the sharp five exponentials conjecture: *Transcendence of the number* $e^{\lambda^2}=\alpha^{\log\alpha}$ *for* $\lambda\in\mathcal{L}$, $e^\lambda=\alpha\in\overline{\mathbf{Q}}^\times$.

Proof. Set $x_1=y_1=1,\ x_2=y_2=\lambda,\ \gamma=1,\ \alpha=0,\ \beta_{11}=1$, $\beta_{ij}=0$ for $(i,j)\neq(1,1)$.

Known: *One at least of the two numbers*

$$e^{\lambda^2}=\alpha^{\log\alpha},\ e^{\lambda^3}=\alpha^{(\log\alpha)^2}$$

is transcendental.

Also a consequence of the sharp six exponentials Theorem!

Strong Five Exponentials Conjecture. *Let* x_1, x_2 *be* $\mathbf{Q}$*-linearly independent and* y_1, y_2 *be* $\mathbf{Q}$*-linearly independent. Then one at least of the five numbers*

$$x_1y_1,\ x_1y_2,\ x_2y_1,\ x_2y_2,\ x_1/x_2$$

does not belong to $\widetilde{\mathcal{L}}$.

9 statements

Four exponentials

sharp Five exponentials

strong Six exponentials

9 statements

	Four exponentials	
		Conjecture
sharp	Five exponentials	
		Theorem
strong	Six exponentials	

9 statements

	Four exponentials	
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Four exponentials: three conjectures

Six exponentials: three theorems

Five exponentials: two conjectures (for sharp and strong)
one theorem

$$
\begin{array}{ccccc}
\text{Alg. indep. C} & & & & \\
\Downarrow & & & & \\
\text{Strong 4 exp C} & \Rightarrow & \text{Strong 5 exp C} & \Rightarrow & \text{Strong 6 exp T} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{Sharp 4 exp C} & \Rightarrow & \text{Sharp 5 exp C} & \Rightarrow & \text{Sharp 6 exp T} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{4 exp C} & \Rightarrow & \text{5 exp T} & \Rightarrow & \text{6 exp T}
\end{array}
$$

Remark. The sharp 6 exponentials Theorem implies the 5 exponentials Theorem.

Consequence of the sharp 4 exponentials Conjecture

Let λ_{ij} $(i = 1, 2$, $j = 1, 2)$ be four non zero logarithms of algebraic numbers.

Assume

$$\lambda_{11} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{22}} \in \overline{\mathbf{Q}}.$$

Then

$$\lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Proof. Assume

$$\lambda_{11} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{22}} = \beta \in \overline{\mathbf{Q}}.$$

Use the sharp four exponentials conjecture with

$$(\lambda_{11} - \beta)\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Consequence of the strong 4 exponentials Conjecture

Let λ_{ij} $(i = 1, 2$, $j = 1, 2)$ be four non zero logarithms of algebraic numbers.

Assume

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \in \overline{\mathbf{Q}}.$$

Then

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} \in \mathbf{Q}.$$

Proof: Assume

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}\lambda_{21}} = \beta \in \overline{\mathbf{Q}}.$$

Use the strong four exponentials conjecture with

$$\lambda_{11}\lambda_{22} = \beta\lambda_{12}\lambda_{21}.$$

Consequence of the strong 4 exponentials Conjecture

Let λ_{ij} $(i = 1, 2,\ j = 1, 2)$ be four non zero logarithms of algebraic numbers.

Assume

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \in \overline{\mathbf{Q}}.$$

Then

- either $\lambda_{11}/\lambda_{12} \in \mathbf{Q}$ and $\lambda_{21}/\lambda_{22} \in \mathbf{Q}$
- or $\lambda_{12}/\lambda_{22} \in \mathbf{Q}$ and

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} \in \mathbf{Q}.$$

Remark:

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{b\lambda_{11} - a\lambda_{12}}{b\lambda_{12}} = \frac{a}{b}.$$

Proof: Assume

$$\frac{\lambda_{11}}{\lambda_{12}} - \frac{\lambda_{21}}{\lambda_{22}} = \beta \in \overline{\mathbf{Q}}.$$

Use the strong four exponentials conjecture with

$$\lambda_{12}(\beta\lambda_{22} + \lambda_{21}) = \lambda_{11}\lambda_{22}.$$

Question: Let λ_{ij} ($i=1,2$, $j=1,2$) be four non zero logarithms of algebraic numbers. Assume

$$\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21}\in\overline{\mathbf{Q}}.$$

Deduce

$$\lambda_{11}\lambda_{22}=\lambda_{12}\lambda_{21}.$$

Question: Let λ_{ij} ($i = 1, 2$, $j = 1, 2$) be four non zero logarithms of algebraic numbers. Assume

$$\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \in \overline{\mathbf{Q}}.$$

Deduce

$$\lambda_{11}\lambda_{22} = \lambda_{12}\lambda_{21}.$$

Answer: *This is a consequence of the Conjecture on algebraic independence of logarithms of algebraic numbers.*

Consequences of the strong 6 exponentials Theorem

Let λ_{ij} $(i = 1, 2,\ \ j = 1, 2, 3)$ be six non zero logarithms of algebraic numbers. Assume

• $\lambda_{11}, \lambda_{21}$ are linearly independent over $\mathbf{Q}$

and

• $\lambda_{11}, \lambda_{12}, \lambda_{13}$ are linearly independent over $\mathbf{Q}$.

- *One at least of the two numbers*

$$\lambda_{12} - \frac{\lambda_{11}\lambda_{22}}{\lambda_{21}}, \quad \lambda_{13} - \frac{\lambda_{11}\lambda_{23}}{\lambda_{21}}$$

is transcendental.

- *One at least of the two numbers*

$$\frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}, \quad \frac{\lambda_{13}\lambda_{21}}{\lambda_{11}\lambda_{23}}$$

is transcendental.

• *One at least of the two numbers*

$$\frac{\lambda_{12}}{\lambda_{11}} - \frac{\lambda_{22}}{\lambda_{21}}, \quad \frac{\lambda_{13}}{\lambda_{11}} - \frac{\lambda_{23}}{\lambda_{21}}$$

is transcendental.

• *Also one at least of the two numbers*

$$\frac{\lambda_{21}}{\lambda_{11}} - \frac{\lambda_{22}}{\lambda_{12}}, \quad \frac{\lambda_{21}}{\lambda_{11}} - \frac{\lambda_{23}}{\lambda_{13}}$$

is transcendental.

Replacing λ_{21} by 1.

- *One at least of the two numbers*

$$\lambda_{12}-\lambda_{11}\lambda_{22},\quad \lambda_{13}-\lambda_{11}\lambda_{23}$$

is transcendental.

- *The same holds for*

$$\frac{\lambda_{12}}{\lambda_{11}}-\lambda_{22},\quad \frac{\lambda_{13}}{\lambda_{11}}-\lambda_{23}.$$

is transcendental.

• *Finally one at least of the two numbers*

$$\frac{\lambda_{11}\lambda_{22}}{\lambda_{12}}, \quad \frac{\lambda_{11}\lambda_{23}}{\lambda_{13}}$$

is transcendental, and also one at least of the two numbers

$$\frac{1}{\lambda_{11}} - \frac{\lambda_{22}}{\lambda_{12}}, \quad \frac{1}{\lambda_{11}} - \frac{\lambda_{23}}{\lambda_{13}}.$$

is transcendental.

Theorem 1. *Let* λ_{ij} *(*$i = 1, 2,\ j = 1, 2, 3, 4, 5$*) be ten non zero logarithms of algebraic numbers. Assume*

- $\lambda_{11}, \lambda_{21}$ *are linearly independent over* $\mathbf{Q}$

and

- $\lambda_{11}, \ldots, \lambda_{15}$ *are linearly independent over* $\mathbf{Q}$.

Then one at least of the four numbers

$$\lambda_{1j}\lambda_{21} - \lambda_{2j}\lambda_{11}, \quad (j = 2, 3, 4, 5)$$

is transcendental.

Elliptic Analogue

Theorem 2. *Let $\wp$ and $\wp^*$ be two non isogeneous Weierstraß elliptic functions with algebraic invariants g_2, g_3 and g_2^*, g_3^* respectively. For $1 \le j \le 9$ let u_j (resp. u_j^*) be a non zero logarithm of an algebraic point of $\wp$ (resp. $\wp^*$). Assume $u_1, \ldots, u_9$ are $\mathbf{Q}$-linearly independent. Then one at least of the eight numbers*

$$u_j u_1^* - u_j^* u_1 \quad (j = 2, \ldots, 9)$$

is transcendental

Motivation: periods of $K3$ surfaces.

Sketch of Proofs

Theorem 3. *Let* $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \times G_2$ *be a commutative algebraic group over* $\overline{\mathbf{Q}}$ *of dimension* $d = d_0 + d_1 + d_2$, V *a hyperplane of the tangent space* $T_e(G)$, Y *a finitely generated subgroup of* V *of rank* ℓ_1 *such that* $\exp_G(Y) \subset G(\overline{\mathbf{Q}})$ *with*

$$\ell_1 > (d-1)(d_1 + 2d_2).$$

Then V *contains a non zero algebraic Lie subalgebra of* $T_e(G)$ *defined over* $\overline{\mathbf{Q}}$.

Sketch of Proof of Theorem 1

Assume

$$\lambda_{1j}\lambda_{21}-\lambda_{2j}\lambda_{11}=\gamma_j \quad \text{for} \quad 1\le j\le \ell_1.$$

Take $G=\mathbf{G}_a\times\mathbf{G}_m^2$, $d_0=1$, $d_1=2$, $G_2=\{0\}$, V is the hyperplane

$$z_0=\lambda_{21}z_1-\lambda_{11}z_2$$

and $Y=\mathbf{Z}y_1+\cdots+\mathbf{Z}y_{\ell_1}$ with

$$y_j=(\gamma_j,\lambda_{1j},\lambda_{2j}), \quad (1\le j\le \ell_1).$$

Since $(d-1)(d_1+2d_2)=4$, we need $\ell_1\ge 5$.

Sketch of Proof of Theorem 2

Assume

$$u_j u_1^* - u_j^* u_1 = \gamma_j \quad \text{for} \quad 1 \le j \le \ell_1.$$

Take $G = \mathbf{G}_a \times \mathcal{E} \times \mathcal{E}^*$, $d_0 = 1$, $d_1 = 0$, $d_2 = 2$, V is the hyperplane

$$z_0 = u_1^* z_1 - u_1 z_2$$

and $Y = \mathbf{Z}y_1 + \cdots + \mathbf{Z}y_{\ell_1}$ with

$$y_j = (\gamma_j, u_j, u_j^*), \quad (1 \le j \le \ell_1).$$

Since $(d-1)(d_1 + 2d_2) = 8$, we need $\ell_1 \ge 9$.

In both cases we need to check that V does not contain a non zero Lie subalgebra of $T_e(G)$. For Theorem 1 this follows from the assumption

$$\lambda_{11},\ \lambda_{21} \quad \text{are } \mathbf{Q}\text{-linearly independent},$$

while for Theorem 2 this follows from the assumption

$$\mathcal{E},\ \mathcal{E}^* \quad \text{are not isogeneous.}$$

Further developments

- Abelian varieties in place of product of elliptic curves.
- Semi abelian varieties. Commutative algebraic groups.
- Taking periods into account.
- Conjectures (A. Grothendieck, Y. André, C. Bertolin.)
- Quadratic relations among logarithms of algebraic numbers.