Algebraic Values of Analytic Functions

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Given an analytic function of one complex variable f, we investigate the arithmetic nature of the values of f at algebraic points. A typical question is whether $f(\alpha)$ is a transcendental number for each algebraic number α. Since there exist transcendental entire functions f such that

$$f^{(s)}(\alpha) \in \mathbb{Q}[\alpha]$$

for any $s \geq 0$ and any algebraic number α, one needs to restrict the situation by adding hypotheses, either on the functions, or on the points, or else on the set of values.

Hermite-Lindemann:
The entire function e^z takes an algebraic value at an algebraic point α only for $\alpha = 0$.

Weierstraß (1886):
There exists a transcendental entire function f such that

$$f(p/q) \in \mathbb{Q} \quad \textit{for any} \quad p/q \in \mathbb{Q}.$$

In a letter to Straus he suggests:
There exists a transcendental entire function f such that

$$f(\alpha) \in \overline{\mathbb{Q}} \quad \textit{for any} \quad \alpha \in \overline{\mathbb{Q}}.$$

Here, $\overline{\mathbb{Q}}$ denotes the set of algebraic numbers (algebraic closure of $\mathbb{Q}$ into $\mathbb{C}$)

Strauss *There exists an analytic function f on $|z| < 1$, not rational, such that*

$$f(\alpha) \in \overline{\mathbb{Q}} \quad \text{for any } \alpha \in \overline{\mathbb{Q}} \text{ with } |\alpha| < 1.$$

Stäckel (using Hilbert's irreducibility Theorem)
This function f is transcendental.

Moreover,

> *If Σ is a countable subset of $\mathbb{C}$ and T a dense subset of $\mathbb{C}$, then there exists a transcendental entire function such that $f(\Sigma) \subset T$.*

For a transcendental entire function f, define

$$S_f = \{\alpha \in \overline{\mathbb{Q}}\,;\; f(\alpha) \in \overline{\mathbb{Q}}\}.$$

Examples.

For $f(z) = e^z$, $S_f = \{0\}$

For $f(z) = e^{P(z)}$ with $P \in \mathbb{C}[z]$ any non constant polynomial, S_f is the set of zeroes of P.

For $f(z) = e^{2i\pi z}$, $S_f = \mathbb{Q}$
 using Gel'fond-Schneider's Theorem.

For $f(z) = \sin(\pi z)e^z$, $S_f = \mathbb{Z}$
 assuming Schanuel's Conjecture.

There exists f with $S_f = \overline{\mathbb{Q}}$
 Follows from Stäckel's Theorem with
 $\Sigma = \overline{\mathbb{Q}}$ and $T = \overline{\mathbb{Q}}$.

There exists f with $S_f = \emptyset$
 Follows from Stäckel's Theorem with
 $\Sigma = \overline{\mathbb{Q}}$ and $T = \mathbb{C} \setminus \overline{\mathbb{Q}}$.

Proposition. *For any subset Σ of $\overline{\mathbb{Q}}$, there exists a transcendental entire function f such that $S_f = \Sigma$.*

For the proof, extend Stäckel's result as follows:

> *For any disjoint countable subsets Σ_1 and Σ_2 of $\mathbb{C}$, and any dense subsets T_1 and T_2 of $\mathbb{C}$, there exists a transcendental entire function f such that $f(\Sigma_1) \subset T_1$ and $f(\Sigma_2) \subset T_2$.*

Moreover one can construct such a f of low growth order: if we set

$$|f|_r = \max_{|z|=r} |f(z)|$$

for $r \geq 0$, and if ψ is any non polynomial entire function with $\psi(0) \neq 0$, one can construct f such that $|f|_r \leq |\psi|_r$ for any $r \geq 0$

Derivatives can be included:

$$f^{(s)} = (d/dz)^s f, \qquad s \geq 0.$$

Stäckel:

There exists a transcendental entire function f such that
$f^{(s)}(\alpha) \in \overline{\mathbb{Q}}$
for any $\alpha \in \overline{\mathbb{Q}}$ and any $s \geq 0$.

A.J. Van der Poorten:

There exists a transcendental entire function f such that
$f^{(s)}(\alpha) \in \mathbb{Q}(\alpha)$
for any $\alpha \in \overline{\mathbb{Q}}$ and any $s \geq 0$.

F. Gramain:

If Σ is a countable subset of $\mathbb{R}$ and T a dense subset of $\mathbb{R}$, then there exists a transcendental entire function such that $f^{(s)}(\Sigma) \subset T$ for any $s \geq 0$.

Proposition. *Denote by K either $\mathbb{R}$ or else $\mathbb{C}$. Let $(\zeta_n)_{n\geq 1}$ be a sequence of pairwise distinct elements of K. For each $n \geq 1$ and $s \geq 0$, let T_{ns} be a dense subset of K. Let ψ be a transcendental entire function with $\psi(0) \neq 0$. Then there exists a transcendental entire function f satisfying*

$$f^{(s)}(\zeta_n) \in T_{ns} \quad \text{for any} \quad n \geq 1 \quad \text{and} \quad s \geq 0$$

and

$$|f|_r \leq |\psi|_r \quad \text{for any} \quad r \geq 0.$$

Proof.

Order the set

$$\{(\zeta_n, s)\,;\, n \geq 1,\ s \geq 0\} \subset \mathbb{C} \times \mathbb{N}$$

by the usual diagonal process

$$\begin{aligned}&\{(w_0, \sigma_0),\ (w_1, \sigma_1), \ldots\} = \\ &\quad \{(\zeta_1, 0),\ (\zeta_2, 0),\ (\zeta_1, 1),\ (\zeta_3, 0), \ldots, \\ &\qquad\qquad (\zeta_n, 0),\ (\zeta_{n-1}, 1), \ldots, (\zeta_1, n),\ (\zeta_{n+1}, 0), \ldots\}.\end{aligned}$$

For $k \geq 0$, if n_k is the positive integer such that

$$\frac{n_k(n_k - 1)}{2} \leq k < \frac{n_k(n_k + 1)}{2}$$

then

$$\sigma_k = k - \frac{n_k(n_k - 1)}{2},$$

and

$$w_k = \zeta_{n_k - \sigma_k}.$$

The polynomial

$$P_k(z) = \prod_{j=0}^{k-1} (z - w_j)$$

for $k \geq 0$ (with $P_0 = 1$) has a zero of multiplicity σ_k at w_k, while for any $\ell > k$ the polynomial P_ℓ has a zero of multiplicity $> \sigma_k$ at w_k.

For $r > 0$, we have

$$|P|_r \leq (r + r_k)^k$$

with

$$r_k = \max_{0 \leq j < k} |w_j|.$$

We construct f as

$$\sum_{k\geq 0} a_k P_k(z)$$

where the coefficients $a_k \in K$ are selected by induction on k as follows. For $k = 0$, one selects $a_0 \in T_{10}$ with

$$0 < |a_0| < \frac{1}{2}|\psi(0)|.$$

Once $a_0, a_1, \ldots, a_{k-1}$ are known, one chooses $a_k \in K$, $a_k \neq 0$, such that

$$a_k P_k^{(\sigma_k)}(w_k) + \sum_{j=0}^{k-1} a_j P_j^{(\sigma_k)}(w_k) \in T_{n_k,\sigma_k}$$

and

$$|a_k| \leq 2^{-k} \inf_{r>0} (r + r_k)^{-k} |\psi|_r.$$

How often can a transcendental function take algebraic values?

For $p/q \in \mathbb{Q}$ with $\gcd(p,q) = 1$ and $q > 0$, define

$$\mathrm{h}(p/q) = \log \max\{|p|,\ q\}.$$

N. Elkies: *For any $\epsilon > 0$, there exists a positive constant A_ϵ such that, for any transcendental analytic function f in $|z| < 1$,*

$$\mathrm{Card}\{p/q \in \mathbb{Q},\ |p| < q,\ f(p/q) \in \mathbb{Q},\ \mathrm{h}(p/q) \le N,\ \mathrm{h}\bigl(f(p/q)\bigr) \le N\} \le A_\epsilon e^{\epsilon N}$$

for any $N \geq 1$.

Question: *Is this optimal?*

Answer by A. Surroca:

One cannot replace ϵN by a function $o(N)$.

Define the *absolute logarithmic height* of an algebraic number α by

$$\mathrm{h}(\alpha) = \frac{1}{d}\log|a_0| + \frac{1}{d}\sum_{j=1}^{d}\log\max\{1, |\alpha_j|\}$$

for $\alpha \in \overline{\mathbb{Q}}$ with minimal polynomial

$$a_0X^d + \cdots + a_d = a_0\prod_{j=1}^{d}(X - \alpha_j) \in \mathbb{C}[X].$$

Let $E_{D,N}$ be the set of $\alpha \in \overline{\mathbb{Q}}$ with degree $\leq D$ and height $\mathrm{h}(\alpha) \leq N$.

T. Loher : $\quad \mathrm{Card}E_{D,N} \geq c(D)e^{D(D+1)N}.$

S.J. Chern, J.D. Vaaler : $\quad \mathrm{Card}E_{D,N} \leq c'(D)e^{D(D+1)N}.$

For an analytic function f in the unit disk of $\mathbb{C}$, define

$$\Sigma_{D,N}(f) = \{\alpha \in \overline{\mathbb{Q}}\,;\ |\alpha| < 1,\ f(\alpha) \in \overline{\mathbb{Q}},\\ [\mathbb{Q}\big(\alpha, f(\alpha)\big) : \mathbb{Q}] \le D,\ \mathrm{h}(\alpha) \le N,\ \mathrm{h}\big(f(\alpha)\big) \le N\}.$$

Theorem 1 (A. Surroca)**.** *Let ϕ be a real valued function satisfying $\phi(x)/x \to 0$ as $x \to \infty$. Then there exists a transcendental entire function f such that*

$$f^{(s)}(\alpha) \in \mathbb{Q}(\alpha) \quad \textit{for any } \alpha \in \overline{\mathbb{Q}} \textit{ and any } s \ge 0$$

and such that, for any $D \ge 1$, there exist infinitely may $N \ge 1$ for which

$$\mathrm{Card}\Sigma_{D,N}(f) > e^{D(D+1)\phi(N)}.$$

Theorem 2 (A. Surroca). *Let f be a transcendental function f which is analytic in the unit disc $|z| < 1$. There exists a positive constant c such that, for any $D \geq 1$, there exist infinitely many $N \geq 1$ for which*

$$\mathrm{Card}\Sigma_{D,N}(f) < cD^2N^3.$$

Sketch of proof. Assume there exist $D \geq 1$ and a sufficiently large $c > 0$ such that

$$\mathrm{Card}\Sigma_{D,N}(f) > cD^2N^3$$

for any $N \geq N_0$. For each $N \geq N_0$ let E_N be a subset of $\Sigma_{D,N}(f)$ with cD^2N^3 elements. Using Dirichlet's box principle (Thue-Siegel's Lemma - a refined version is required), construct a non zero auxiliary function

$$F(z) = P\big(z, f(z)\big)$$

with a zero at each $\alpha \in E_{N_0}$. By induction show that F vanishes at each $\alpha \in E_N$, and conclude $F = 0$, hence the contradiction.