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Transcendental Number Theory: Recent Results and Conjectures

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Abstract

In this lecture we survey some of the many recent results concerning the arithmetic nature (rational, algebraic irrational or else transcendental) of real or complex numbers. At the same time we present related open problems and conjectures.

For instance for each of the following numbers

$$\Gamma(1/5), \quad \text{Euler constant}, \quad \zeta(5), \quad e^{\pi^2}, \quad e+\pi$$

one conjectures that it is transcendental, but one does not know yet that it is irrational.

Periods

M. Kontevich and D. Zagier (2000) – *Periods.*

A period is a complex number whose real and imaginary part are values of absolutely convergent integrals of rational functions with rational coefficients over domains of $\mathbf{R}^n$ given by polynomials (in)equalities with rational coefficients.

Examples:

$$\sqrt{2} = \int_{2x^2 \le 1} dx,$$

$$\pi = \int_{x^2+y^2 \le 1} dxdy,$$

$$\log 2 = \int_{1<x<2} \frac{dx}{x},$$

$$\zeta(2) = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} = \frac{\pi^2}{6}.$$

Relations between periods

1 Additivity

$$\int_a^b \big(f(x)+g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2 Change of variables

$$\int_{\varphi(a)}^{\varphi(b)} f(t)dt = \int_a^b f\big(\varphi(u)\big)\varphi'(u)du.$$

3 Newton–Leibniz–Stokes

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Conjecture *(Kontsevich–Zagier). If a period has two representations, then one can pass from one formula to another using only rules* 1, 2 *and* 3 *in which all functions and domains of integrations are algebraic with algebraic coefficients.*

$$
\begin{aligned}
\pi &= \int_{x^2+y^2\le 1} dxdy \\
&= 2\int_{-1}^{1} \sqrt{1-x^2}\; dx \\
&= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \\
&= \int_{-\infty}^{\infty} \frac{dx}{1-x^2}.
\end{aligned}
$$

Transcendence of values of definite integrals

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Th. Schneider
(1934) – *Transzendenzuntersuchungen periodischer Funktionen.*
(1937) – *Arithmetische Untersuchungen elliptischer Integrale.*
Transcendence of elliptic integrals of first and second kind.

For real algebraic numbers a and b let $\mathcal{E}_{ab}$ be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let (x_0, y_0) and (x_1, y_1) be two points on $\mathcal{E}_{ab}$ with real algebraic coordinates. Set

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}}.$$

Then the arc length

$$\int_{x_0}^{x_1} \sqrt{\frac{a^2 - \epsilon^2 x^2}{a^2 - x^2}}\, dx$$

is either 0 *or else a transcendental number.*

For a a real algebraic numbers let $\mathcal{L}_a$ be the lemniscate

$$(x^2+y^2)^2 = 2a^2(x^2-y^2).$$

Set

$$t = \sqrt{\frac{x^2-y^2}{x^2+y^2}}.$$

Let (x_0, y_0) and (x_1, y_1) be two points on $\mathcal{L}_a$ with real algebraic coordinates. *Then the arc length*

$$\int_{t_0}^{t_1} \frac{a\sqrt{2}}{\sqrt{1-t^4}}\, dt$$

is either 0 *or else a transcendental number.*

Numerical examples:

$$\int_1^\infty \frac{dx}{\sqrt{4x^3-4x}} = \frac{1}{4}\mathrm{B}(1/4,1/2) = \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}}$$

and

$$\int_1^\infty \frac{dx}{\sqrt{4x^3-4}} = \frac{1}{6}\mathrm{B}(1/6,1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi}$$

are transcendental.

Th. Schneider (1940) – *Zur Theorie des Abelschen Funktionen und Integrale.*

For any rational numbers a *and* b *which are not integers and such that* $a+b$ *is not an integer, the number*

$$\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

is transcendental.

Remark. For any $p/q\in\mathbf{Q}$ with $p>0$ and $q>0$, $\Gamma(p/q)^q$ is a period.

A. Baker (1968) – *Linear forms in the logarithms of algebraic numbers*

If $\log\alpha_1,\ldots,\log\alpha_n$ *are* $\mathbf{Q}$*-linearly independent logarithms of algebraic numbers, then* $1,\log\alpha_1,\ldots,\log\alpha_n$ *are linearly independent over the field of algebraic numbers.*

Example:

The number

$$\int_0^1 \frac{dx}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right)$$

is transcendental.

A.J. van der Poorten (1970) – *On the arithmetic nature of definite integrals of rational functions.*

For P *and* Q *polynomials with algebraic coefficients satisfying* $\deg P < \deg Q$, *if* γ *is either a closed loop or else a path with endpoints which are either algebraic or infinite, if the integral*

$$\int_\gamma \frac{P(z)}{Q(z)} dz$$

exists, then it is either 0 *or transcendental.*

Abelian Integrals

Th. Schneider (1940), S. Lang (1960's), D.W. Masser (1980's).

G. Wüstholz (1989) – *Algebraische Punkte auf analytischen Untergruppen algebraischer Gruppen.*

Extension of Baker's Theorem to commutative algebraic groups.

Transcendence and linear independence over the field of algebraic numbers of abelian integrals of first, second and third kind.

Example: Gauss hypergeometric function

$$ {}_2F_1\left(a,\ b\ ;\ c \middle|\ z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} $$

where

$$ (a)_n = a(a+1)\cdots(a+n-1). $$

J. Wolfart: Transcendence of ${}_2F_1\left(a,\ b\ ;\ c \middle|\ z\right)$ for rational values of a, b, c, z.

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F. Beukers and J. Wolfart:

$$ {}_2F_1\left(\frac{1}{12}, \frac{5}{12}\ ;\ \frac{1}{2} \middle|\ \frac{1323}{1331}\right) = \frac{3}{4}\sqrt[4]{11} $$

G.V. Chudnovskij (1976) – *Algebraic independence of constants connected with exponential and elliptical functions*

Theorem *(G.V. Chudnovskij). Let $\wp$ be an elliptic function of Weierstrass with invariants g_2 and g_3. Let ω be a non zero period of $\wp$ and let η be the associated quasi-period of the Weierstrass zeta function ζ:*

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \qquad \zeta' = -\wp, \quad \zeta(z+\omega) = \zeta(z) + \eta.$$

Then two at least of the numbers

$$g_2,\ g_3,\ \omega/\pi,\ \eta/\pi$$

are algebraically independent.

Corollary. *With the elliptic curve* $y^2 = 4x^3 - 4x$ *one gets the algebraic independence of the two numbers*

$$\Gamma(1/4) \quad \textit{and} \quad \pi.$$

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Yu.V. Nesterenko (1996) – *Modular functions and transcendence questions*

$$\begin{aligned} P(q) &= E_2(q) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \\ Q(q) &= E_4(q) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n}, \\ R(q) &= E_6(q) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}. \end{aligned}$$

Theorem *(Yu.V. Nesterenko.) Let $q \in \mathbf{C}$ satisfy $0 < |q| < 1$. Then three at least of the four numbers*

$$q,\ P(q),\ Q(q),\ R(q)$$

are algebraically independent.

Connexion with elliptic functions

For a Weierstrass elliptic function $\wp$ with algebraic invariants g_2 and g_3 and fundamental periods ω_2, ω_1 set $\tau = \omega_1/\omega_2$, $q = e^{2i\pi\tau}$,

$$J(q) = j(\tau) = 1728 \frac{g_2^3}{g_2^3 - 27 g_3^2}.$$

Remark.

$$\Delta(q) = \frac{1}{1728}\bigl(Q(q)^3 - R(q)^2\bigr)$$

satisfies

$$J(q) = \frac{Q(q)^3}{\Delta(q)} \quad \text{and} \quad \Delta = q \prod_{n\geq 1}(1-q^n)^{24}.$$

LEMNISCATE $y^2 = 4x^3 - 4x$

$$g_2 = 4, \quad g_3 = 0, \quad j = 1728, \quad \tau = i, \quad q = e^{-2\pi}$$

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.6220575542\ldots$$

$$P(e^{-2\pi}) = \frac{3}{\pi}, \qquad Q(e^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4,$$

$$R(e^{-2\pi}) = 0, \qquad \Delta(e^{-2\pi}) = \frac{1}{2^6}\left(\frac{\omega_1}{\pi}\right)^{12}.$$

ANHARMONIQUE $\qquad y^2 = 4x^3 - 4$

$$g_2 = 0, \quad g_3 = 4, \quad j = 0, \quad \tau = \varrho, \quad q = -e^{-\pi\sqrt{3}}$$

$$\omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428650648\ldots$$

$$P(-e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi}, \qquad Q(-e^{-\pi\sqrt{3}}) = 0,$$

$$R(-e^{-\pi\sqrt{3}}) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6, \qquad \Delta(-e^{-\pi\sqrt{3}}) = -\frac{27}{256}\left(\frac{\omega_1}{\pi}\right)^{12}.$$

Corollary. *The three numbers*

$$\Gamma(1/4), \quad \pi \quad \textit{and} \quad e^{\pi}$$

are algebraically independent and the three numbers

$$\Gamma(1/3), \quad \pi \quad \textit{and} \quad e^{\pi\sqrt{3}}$$

are algebraically independent

Conjecture *(Nesterenko)*. *Let* $\tau \in \mathbf{C}$ *have positive imaginary part. Assume that* τ *is not quadratic. Set* $q = e^{2i\pi\tau}$. *Then* 4 *at least of the* 5 *numbers*

$$\tau,\ q,\ P(q),\ Q(q),\ R(q)$$

are algebraically independent.

Remark. Lindemann: $\Gamma(1/2) = \sqrt{\pi}$ *is transcendental.*

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are algebraically independent.

Standard Relations

$$\Gamma(a+1) = a\Gamma(a)$$

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)},$$

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

Define

$$G(z) = \frac{1}{\sqrt{2\pi}}\Gamma(z).$$

According to the multiplication theorem of Gauss and Legendre, for each positive integer N and each complex number z such that $Nz \not\equiv 0 \pmod{\mathbf{Z}}$,

$$\prod_{i=0}^{N-1} G\left(z + \frac{i}{N}\right) = N^{(1/2)-Nz} G(Nz).$$

The gamma function has no zero and defines a map from $\mathbf{C} \setminus \mathbf{Z}$ to $\mathbf{C}^\times$. Restrict to $\mathbf{Q} \setminus \mathbf{Z}$ and compose with the canonical map $\mathbf{C}^\times \rightarrow \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$. The composite map has period 1, and the resulting mapping

$$\overline{G} : \frac{\mathbf{Q}}{\mathbf{Z}} \setminus \{0\} \rightarrow \frac{\mathbf{C}^\times}{\overline{\mathbf{Q}}^\times}$$

is an odd *distribution* on $(\mathbf{Q}/\mathbf{Z}) \setminus \{0\}$:

$$\prod_{i=0}^{N-1} \overline{G}\left(a + \frac{i}{N}\right) = \overline{G}(Na) \quad \text{for} \quad a \in \frac{\mathbf{Q}}{\mathbf{Z}} \setminus \{0\}$$

and

$$\overline{G}(-a) = \overline{G}(a)^{-1}.$$

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Means: Any multiplicative relation

$$\pi^{b/2}\prod_{a\in\mathbf{Q}}\Gamma(a)^{m_a}\in\overline{\mathbf{Q}}$$

with b and m_a in $\mathbf{Z}$ can be derived for the standard relations.

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Example: (P. Das)

$$\frac{\Gamma(1/3)\Gamma(2/15)}{\Gamma(4/15)\Gamma(1/5)}\in\overline{\mathbf{Q}}.$$

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with b and m_a in $\mathbf{Z}$ can be derived for the standard relations.

This leads to the question whether the distribution relations, the oddness relation and the functional equations of the gamma function generate the ideal over $\overline{\mathbf{Q}}$ of all algebraic relations among the values of $G(a)$ for $a\in\mathbf{Q}$.

Euler's constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.5772157\ldots,$$

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Conjecture: γ is a transcendental number.

Conjecture: γ is not a period.

J. Sondow (2003) – *Criteria for Irrationality of Euler's Constant.*

A very sharp conjectured lower bound for infinitely many elements in a specific sequence

$$\left| e^{b_0} a_1^{b_1} \cdots a_m^{b_m} - 1 \right|$$

with b_0 arbitrary, and where all the exponents b_i have the same sign, would yield the irrationality of Euler's constant.

Analog of Euler's constant in finite characteristic

L. Carlitz (1935) – *On certain functions connected with polynomials in a Galois field.*

V.G. Drinfel'd (1974) – *Elliptic modules.*

I.I. Wade (1941), J.M. Geijsel (1978), P. Bundschuh (1978), Yu Jing (1980's), G.W. Anderson and D. Thakur (1990)...

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \qquad \gamma = \lim_{s \to 1} \left(\zeta(s) - \frac{1}{s-1} \right).$$

In finite characteristic p runs over the set of monic irreducible polynomials and $\zeta(1)$ converges.

An analog in dimension 2 of Euler constant

For $k \geq 2$, let A_k be the minimal area of a closed disk in $\mathbf{R}^2$ containing at least k points of $\mathbf{Z}^2$. For $n \geq 2$ define

$$\delta_n = -\log n + \sum_{k=2}^{n} \frac{1}{A_k} \quad \text{and} \quad \delta = \lim_{n\to\infty} \delta_n.$$

Gramain conjectures:

$$\delta = 1 + \frac{4}{\pi}\bigl(\gamma L'(1) + L(1)\bigr) = 1.82282524\ldots,$$

where

$$L(s) = \sum_{n\geq 0} (-1)^n (2n+1)^{-s}.$$

Best known estimates for δ (F. Gramain and M. Weber, 1985):

$$1.811\ldots < \delta < 1.897\ldots$$

Transcendence of sums of series

S.D. Adhikari, N. Saradha, T.N. Shorey, R. Tijdeman (2001) – *Transcendental infinite sums.*

N. Saradha, R. Tijdeman (2003) – *On the transcendence of infinite sums of values of rational functions.*

Question: What is the arithmetic nature of

$$\sum_{\substack{n\ge 0\\ Q(n)\ne 0}} \frac{P(n)}{Q(n)}\ ?$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1,$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

are rational numbers.

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2,$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}}$$

are transcendental numbers

Also

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}$$

is transcendental.

Question: *Arithmetic nature of*

$$\sum_{n\geq 1} \frac{1}{n^s}$$

for $s \geq 2$ *?*

Zeta Values – Euler Numbers

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \qquad \text{for } s \geq 2.$$

These are special values of the Riemann Zeta Function: $s \in \mathbf{C}$.

Euler: $\pi^{-2k}\zeta(2k) \in \mathbf{Q}$ *for* $k \geq 1$. (Bernoulli numbers).

Diophantine Question: *What are the algebraic relations among the numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7) \ldots ?$$

Conjecture. *There is no algebraic relation at all: these numbers*

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Known:

- Hermite-Lindemann: π *is transcendental, hence* $\zeta(2k)$ *also for* $k \geq 1$.

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• Rivoal (2000) + Ball, Zudilin. . . *Infinitely many* $\zeta(2k+1)$ *are irrational + lower bound for the dimension of the* $\mathbf{Q}$*-space they span.*

Let $\epsilon > 0$. For a be a sufficiently large odd integer the dimension of the $\mathbf{Q}$-space spanned by $1,\ \zeta(3),\ \zeta(5),\ \cdots, \zeta(a)$ is at least

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

W. Zudilin.

• *One at least of the four numbers*

$$\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$$

is irrational.

• *There is an odd integer* j *in the range* $[5, 69]$ *such that the three numbers* 1, $\zeta(3)$, $\zeta(j)$ *are linearly independent over* $\mathbf{Q}$.

Linearization of the problem *(Euler).* The product of two zeta values is not quite a zeta value, but something similar.

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From

$$\sum_{n_1\ge 1} n_1^{-s_1} \sum_{n_2\ge 1} n_2{}^{-s_2} = \sum_{n_1>n_2\ge 1} n_1^{-s_1} n_2{}^{-s_2} + \sum_{n_2>n_1\ge 1} n_2^{-s_2} n_1{}^{-s_1} + \sum_{n\ge 1} n^{-s_1-s_2}$$

one deduces, for $s_1 \ge 2$ and $s_2 \ge 2$,

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

with

$$\zeta(s_1, s_2) = \sum_{n_1>n_2\ge 1} n_1^{-s_1} n_2{}^{-s_2}.$$

For k, $s_1,\dots,s_k$ positive integers with $s_1 \geq 2$, define $\underline{s} = (s_1,\dots,s_k)$ and

$$\zeta(\underline{s}) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}.$$

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The product of two Multiple Zeta Values is a linear combination, with integer coefficients, of Multiple Zeta Values.

These numbers satisfy a quantity of linear relations with rational coefficients.

A complete description of these relations would in principle settle the problem of the algebraic independence of

$$\pi, \quad \zeta(3), \quad \zeta(5), \ldots, \quad \zeta(2k+1).$$

Goal: *Describe all linear relations among Multiple Zeta Values.*

Example of linear relation.

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$$\zeta(2,1) = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} \cdot \frac{dt_3}{1-t_3}.$$

$$\zeta(3) = \int_{1>t_1>t_2>t_3>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{t_2} \cdot \frac{dt_3}{1-t_3}.$$

Euler's result follows from $(t_1, t_2, t_3) \mapsto (1-t_3, 1-t_2, 1-t_1)$.

Denote by $\mathfrak{Z}_p$ the $\mathbf{Q}$-vector subspace of $\mathbf{R}$ spanned by the real numbers $\zeta(\underline{s})$ with $\underline{s}$ of weight $s_1+\cdots+s_k=p$, with $\mathfrak{Z}_0=\mathbf{Q}$ and $\mathfrak{Z}_1=\{0\}$.

Here is Zagier's conjecture on the dimension d_p of $\mathfrak{Z}_p$.

Conjecture *(Zagier). For $p\geq 3$ we have*

$$d_p=d_{p-2}+d_{p-3}.$$

$$(d_0,\ d_1,\ d_2,\ldots)=(1,0,1,1,1,2,2,\ldots).$$

This conjecture can be written

$$\sum_{p\geq 0} d_p X^p = \frac{1}{1-X^2-X^3}.$$

M. Hoffman conjectures: *a basis of* $\mathfrak{Z}_p$ *over* $\mathbf{Q}$ *is given by the numbers* $\zeta(s_1,\ldots,s_k)$, $s_1+\cdots+s_k=p$, *where each* s_i *is either* 2 *or* 3.

True for $p\leq 16$ (Hoang Ngoc Minh)

A.G. Goncharov (2000) – *Multiple ζ-values, Galois groups and Geometry of Modular Varieties.*

T. Terasoma (2002) – *Mixed Tate motives and multiple zeta values.*

The numbers defined by the recurrence relation of Zagier's Conjecture

$$d_p = d_{p-2} + d_{p-3}.$$

with initial values $d_0 = 1$, $d_1 = 0$ *provide upper bounds for the actual dimension of* $\mathfrak{Z}_p$.

Schanuel's Conjecture. *Let* $x_1, \ldots, x_n$ *be* $\mathbf{Q}$*-linearly independent complex numbers. Then at least* n *of the* $2n$ *numbers*

$$x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}$$

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Example: $n = 2$, $x_1 = 1$, $x_2 = 2i\pi$, $e^{x_1} = e$, $e^{x_2} = 1$. The two numbers e and π are algebraically independent. In particular $e + \pi$ and $e\pi$ are transcendental. **Unknown!**.

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Denote by $\mathcal{L}$ the set of logarithms of algebraic numbers.

Special case: Taking $x_i \in \mathcal{L}$ for $1 \leq i \leq n$ one gets the following conjecture of algebraic independence of logarithms of algebraic numbers.

Algebraic independence of logarithms of algebraic numbers

Conjecture. *Let* $\alpha_1, \ldots, \alpha_n$ *be non zero algebraic numbers. For* $1 \le j \le n$ *let* $\lambda_j \in \mathbf{C}$ *satisfy* $e^{\lambda_j} = \alpha_j$. *Assume* $\lambda_1, \ldots, \lambda_n$ *are linearly independent over* $\mathbf{Q}$. *Then* $\lambda_1, \ldots, \lambda_n$ *are algebraically independent.*

Write $\lambda_j = \log \alpha_j$.

If $\log \alpha_1, \ldots, \log \alpha_n$ *are* $\mathbf{Q}$*-linearly independent then they are algebraically independent.*

e^{π}

A.O. Gel'fond (1929) – *Sur les propriétés arithmétiques des fonctions entières.*

The number e^{π} *is transcendental.*

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Transcendence of α^β *for algebraic* α *and* β *with* $\alpha \neq 0$, $\log \alpha \neq 0$ *and* $\beta \notin \mathbf{Q}$.

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Transcendence of α^β *for algebraic* α *and* β *with* $\alpha \neq 0$, $\log \alpha \neq 0$ *and* $\beta \notin \mathbf{Q}$.

Example:

$$\alpha = 1, \;\; \log \alpha = 2i\pi, \;\; \beta = 1/2i, \;\; \alpha^\beta = \exp(\beta \log \alpha) = e^\pi.$$

$$\boxed{e^{\pi^2}}$$

Special case of the Conjecture of Algebraic Independence of Logarithms:

Conjecture. *If* λ_1, λ_2, λ_3 *are three non zero logarithms of algebraic numbers, then*

$$\lambda_1\lambda_2 \neq \lambda_3.$$

Example: Transcendence of $\alpha^{\log \beta}$, of $2^{\log 2}$ and of e^{π^2}.

W.D. Brownawell and M.W. (1970): *One at least of the two following properties is true:*

- *The two numbers* e *and* π *are algebraically independent.*
- *The number* e^{π^2} *is transcendental.*

Sharp Five Exponentials Conjecture. *If* x_1, x_2 *are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2 *are* $\mathbf{Q}$*-linearly independent and if* $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ *are six algebraic numbers with* $\gamma \neq 0$ *such that*

$$e^{x_1y_1-\beta_{11}},\ e^{x_1y_2-\beta_{12}},\ e^{x_2y_1-\beta_{21}},\ e^{x_2y_2-\beta_{22}},\ e^{(\gamma x_2/x_1)-\alpha}$$

are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2,\ j = 1, 2$ *and also* $\gamma x_2 = \alpha x_1$.

Sharp Five Exponentials Conjecture. *If* x_1, x_2 *are* $\mathbf{Q}$-*linearly independent, if* y_1, y_2 *are* $\mathbf{Q}$-*linearly independent and if* $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ *are six algebraic numbers with* $\gamma \neq 0$ *such that*

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are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2,$ $j = 1, 2$ *and also* $\gamma x_2 = \alpha x_1$.

Difficult case: when $y_1, y_2, \gamma/x_1$ are $\mathbf{Q}$-linearly dependent.

Example: $x_1 = y_1 = \gamma = 1$.

Sharp Five Exponentials Conjecture. *If* x_1, x_2 *are* $\mathbf{Q}$-*linearly independent, if* y_1, y_2 *are* $\mathbf{Q}$-*linearly independent and if* $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ *are six algebraic numbers with* $\gamma \neq 0$ *such that*

$$e^{x_1 y_1 - \beta_{11}}, \; e^{x_1 y_2 - \beta_{12}}, \; e^{x_2 y_1 - \beta_{21}}, \; e^{x_2 y_2 - \beta_{22}}, \; e^{(\gamma x_2 / x_1) - \alpha}$$

are algebraic, then $x_i y_j = \beta_{ij}$ *for* $i = 1, 2$, $j = 1, 2$ *and also* $\gamma x_2 = \alpha x_1$.

Consequence: *Transcendence of the number* e^{π^2}.

Proof. Set $x_1 = y_1 = 1$, $x_2 = y_2 = i\pi$, $\gamma = 1$, $\alpha = 0$, $\beta_{11} = 1$, $\beta_{ij} = 0$ for $(i, j) \neq (1, 1)$.

Five Exponentials Theorem *(exponential form). If x_1, x_2 are $\mathbf{Q}$-linearly independent, y_1, y_2 are $\mathbf{Q}$-linearly independent and γ is a non zero algebraic number, then one at least of the five numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2},\ e^{\gamma x_2/x_1}$$

is transcendental.

Five Exponentials Theorem *(exponential form).* *If* x_1, x_2 *are* $\mathbf{Q}$*-linearly independent,* y_1, y_2 *are* $\mathbf{Q}$*-linearly independent and* γ *is a non zero algebraic number, then one at least of the five numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2},\ e^{\gamma x_2/x_1}$$

is transcendental.

Five Exponentials Theorem *(logarithmic form).* *For* $i = 1, 2$ *and* $j = 1, 2$*, let* $\lambda_{ij} \in \mathcal{L}$*. Assume* $\lambda_{11}, \lambda_{12}$ *are linearly independent over* $\mathbf{Q}$*. Further let* $\gamma \in \overline{\mathbf{Q}}^{\times}$ *and* $\lambda \in \mathcal{L}$*. Then the matrix*

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \gamma \\ \lambda_{21} & \lambda_{22} & \lambda \end{pmatrix}$$

has rank 2*.*

Sharp Six Exponentials Theorem *(logarithmic form). For $i = 1, 2$ and $j = 1, 2, 3$, let $\lambda_{ij} \in \mathcal{L}$ and $\beta_{ij} \in \overline{\mathbf{Q}}$. Assume $\lambda_{11}, \lambda_{12}, \lambda_{13}$ are linearly independent over $\mathbf{Q}$ and also $\lambda_{11}, \lambda_{21}$ are linearly independent over $\mathbf{Q}$. Then the matrix*

$$\begin{pmatrix} \lambda_{11} + \beta_{11} & \lambda_{12} + \beta_{12} & \lambda_{13} + \beta_{13} \\ \lambda_{21} + \beta_{21} & \lambda_{22} + \beta_{22} & \lambda_{23} + \beta_{23} \end{pmatrix}$$

has rank 2.

Sharp Six Exponentials Theorem *(exponential form).* *If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2, y_3 *are three complex numbers which are* $\mathbf{Q}$*-linearly independent and if* β_{ij} *are six algebraic numbers such that*

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \textit{for} \quad i = 1, 2,\ j = 1, 2, 3,$$

then $x_i y_j = \beta_{ij}$ *for* $i = 1, 2$ *and* $j = 1, 2, 3$.

Sharp Six Exponentials Theorem *(exponential form). If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2, y_3 *are three complex numbers which are* $\mathbf{Q}$*-linearly independent and if* β_{ij} *are six algebraic numbers such that*

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \textit{for} \quad i = 1, 2,\ j = 1, 2, 3,$$

then $x_i y_j = \beta_{ij}$ *for* $i = 1, 2$ *and* $j = 1, 2, 3$.

The sharp six exponentials Theorem implies the five exponentials Theorem: set $y_3 = \gamma/x_1$ and use Baker's Theorem for checking that y_1, y_2, y_3 are linearly independent over $\mathbf{Q}$.

A consequence of the sharp six exponentials Theorem: *One at least of the two numbers*

$$e^{\lambda^2} = \alpha^{\log \alpha}, \; e^{\lambda^3} = \alpha^{(\log \alpha)^2}$$

is transcendental.

$$\operatorname{rank} \begin{pmatrix} 1 & \lambda & \lambda^2 \\ \lambda & \lambda^2 & \lambda^3 \end{pmatrix} = 1.$$

First proof in 1970 (also by W.D. Brownawell) as a consequence of a result of algebraic independence.

Conjecture. *The numbers*

$$\Gamma(1/5), \quad \textit{Euler constant}, \quad \zeta(5), \quad e^{\pi^2}, \quad e, \quad \pi$$

are algebraically independent.