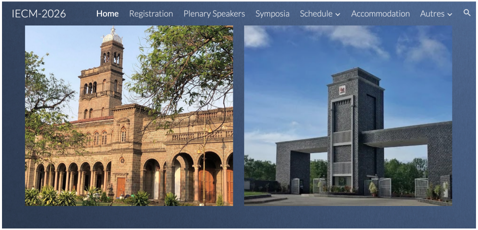

The abc of Number Theory

Michel Waldschmidt

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Institut de Mathématiques de Jussieu, Paris
http://www.imj-prg.fr/~michel.waldschmidt/

Abstract

One of the most fascinating feature of number theory is the production of statements which are easy to formulate, but are either very hard to prove or even remain as conjectures so far. The abc conjecture of Oesterlé and Masser is a good example. The consequences of it cover a surprisingly large range of topics – we will only mention a few of them.

As an introduction we propose a brief overview of the cooperation in mathematics between India and Europe, with an accent on the Indo–French cooperation, especially in number theory.

Indo-European cooperation in mathematics

First example

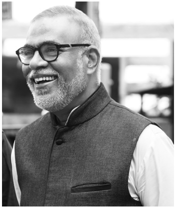
Godfrey Harold Hardy
1877–1947

Srinivasa Ramanujan
1887–1920

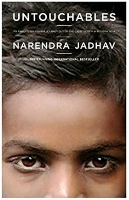
Credit Photo
https://mathshistory.st-andrews.ac.uk/Biographies/

Taxi cab number 1729

French Science Today
November 4-26, 2006 : Chennai, Indore, Bhopal, Chandigarh, Delhi, Ranchi, Kolkata, Belur, Bangalore
October 21 - November 10, 2007 : Mumbai, **Pune**, Kolkata, Guwahati, Trivandrum, Chennai
Bhaskaracharya Pratishthana (Shreeram Abhyankar)



Narendra Jadhav



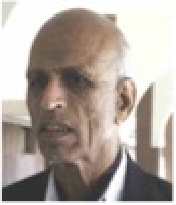
Outcaste – A Memoir : Life and Triumphs of an Untouchable Family in India (Penguin, India, 2003)

2006–2009 : Vice Chancellor of Savitribai Phule Pune University

International Monetary Fund (IMF), headed economic research at the Reserve Bank of India, economist, educationist, public policy expert, professor and writer in English, Marathi and Hindi.

Indo-European cooperation in mathematics

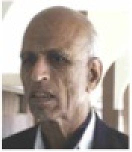
Carl Ludwig Siegel
1896–1981



Kanakanahalli Ramachandra
1933–2011

Credit Photo Siegel : Bhāvanā

K. Ramachandra



Kanakanahalli Ramachandra
(1933 – 2011)

PhD 1965

1965 –1995 Tata Institute of Fundamental Research Bombay

1995 – 2011 National Institute of Advanced Studies, Bangalore

K. Ramachandra, *Contributions to the theory of transcendental numbers*. Acta Arith. **14**, (1968), pp. 65–88.

https://en.wikipedia.org/wiki/Kanakanahalli_Ramachandra

Indo–Italian cooperation in mathematics

Muhammad Abdus Salam

1926–1996

Shreeram Shankar Abhyankar

1930–2012

Indo–French cooperation in mathematics

André Weil

1906–1998

ALIGARH MUSLIM UNIVERSITY

G20 2023 INDIA

(1930–1931)

In 1929 Syed Ross Masood, Vice-Chancelor of Aligarh Muslim University, proposed a chair of French civilization to André Weil, who was recommended to him by a specialist of Indology, Sylvain Levi. A few months later this offer was converted into a chair of mathematics.

Father Racine

Fr Charles Racine
(1897 – 1976)

Father Racine reached India in 1937 as a Jesuit missionary after having taken his Doctorate in Mathematics in 1934 under Élie Cartan.

He taught mathematics first at St Joseph's College in Tiruchirappally (Trichy, Tamil Nadu) and from 1939 onwards at Loyola College (Madras). He spent forty-two years in India.

He had connections with many important French mathematicians of that time like J. Hadamard, J. Leray, A. Weil, H. Cartan.

Charles Racine's thesis

Élie Cartan

1869–1951

Doctorate in Mathematics in 1934 under Élie Cartan.
Le problème des N corps dans la théorie de la relativité.
Thèse 1934

http://www.numdam.org/item?id=THESE_1934__158__1_0

M. Santiago, *International conference on teaching and research in mathematics*, in Birth Centenary Celebrations of Father Charles Racine, S.J. Loyola College, Racine Research Centre, Chennai, January, 1997.

Madhuri Katti, Two Jesuits who introduced modern education in India : Father Charles Racine and Father Eugène Lafont, Bhāvanā vol. 6 Issue 2 April 2022.

Students of Father Racine

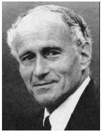
K. S. Chandrasekharan
1920–2017

K.G. Ramanathan
1920–1992

Father Racine encouraged his best students to join the newly founded Tata Institute of Fundamental Research (TIFR) in Bombay with K.S. Chandrasekharan and Kollagunta Gopalaiyer Ramanathan.
C.S. Seshadri, K.S. Chandrasekharan (1920 – 2017), Bhāvanā Vol. 1 Issue 1 July 2017.

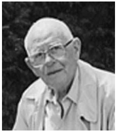
Visited the TATA Institute early

Jean Dieudonné
1906–1992



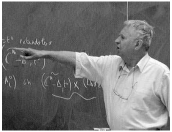
Laurent Schwartz
1915–2002

Jean-Louis Koszul
1921–2018



Pierre Samuel
1921–2009

J-P. Serre



Bernard Malgrange
1928–2024

Alexander Grothendieck
1928–2014

François Bruhat
1929–2007

Adrien Douady
1935–2006

M.S. Narasimhan

M.S. Narasimhan
1932–2021

Recipient of the Padma Bhushan in 1990
Ordre national du Mérite in 1989
King Faisal International Prize for Science in 2006

Head of the research group in Mathematics at the International Centre for Theoretical Physics (ICTP) in Trieste from 1993 to 1999.

R. Narasimhan, *The coming of age of mathematics in India*, in Miscellanea mathematica, Springer, Berlin, 1991, pp. 235–258.

https://www.ictp.it/about-ictp/media-centre/news/2021/5/in-memoriam-narasimhan.aspx

C.S. Seshadri

C.S. Seshadri
1932 – 2020

Paris 1957–1960

Doctorat honoris causa, Université Pierre-et-Marie-Curie (UPMC), Paris, 2013

Recipient of the Padma Bhushan in 2009

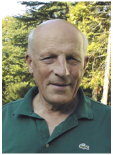
M.S. Narasimhan and C.S. Seshadri were among the first graduate students of the School of Mathematics, headed by K.S. Chandrasekharan.

https://fr.wikipedia.org/wiki/C._S._Seshadri

Agreement CMI – ENS

Etienne Guyon

1935–2023



Pierre Cartier

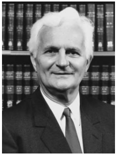
1932–2024

Thanks to an agreement (MoU) between the CMI (*Chennai Mathematical Institute*) and ENS (*École Normale Supérieure*, rue d'Ulm, Paris), every year since 2000, some three young students from ENS visit CMI for two months and deliver courses to the undergraduate students of CMI, and three students from CMI visit ENS for two months. The French students are accommodated in the guest house of IMSc, which participates in this cooperation.

https://fr.wikipedia.org/wiki/Etienne_Guyon

https://mathshistory.st-andrews.ac.uk/Biographies/Cartier/

Indo-French cooperation in mathematics



Jacques-Louis Lions
1928–2001

Jean–Louis Verdier
1935–1989

Gilles Lachaud
1946–2018

French frequent visitors to India also include :
Marc Chardin, Laurent Clozel, Jean-Louis Colliot-Thélène, Sinnou David, Jean-Pierre Demailly (1957–2022), Maria Esteban, Joseph Oesterlé, Patrice Philippon, Olivier Pironneau, Jacques Tilouine, Pascal Weil...

About CEFIPRA

Indo-French Centre for the Promotion of Advanced Research (IFCPAR/CEFIPRA) is a model for international collaborative research in advanced areas of Science & Technology. The Centre was established in 1987 and is being supported by Department of Science & Technology, Government of India and the Ministry for Europe & Foreign Affairs, Government of France .

https://www.cefipra.org/

Two examples of success stories for the Indo-French Cooperation in Number Theory

- Serre's Modularity Conjecture
(Chandrashekhar Khare, Jean-Pierre Wintenberger),

- Waring's Problem
(R. Balasubramanian, Jean-Marc Deshouillers, François Dress).

Serre's Modularity Conjecture

Chandrashekhar Khare

J-P. Wintenberger
1954–2019

J-P. Serre

2006 joint work by Chandrashekhar Khare and Jean-Pierre Wintenberger

Chandrashekhar Khare,
TIFR and a conjecture of Jean-Pierre Serre, Bhāvanā, volume 7 issue 3 July 2023.

Serre's Modularity Conjecture

Let

$$\rho : G_{\mathbb{Q}} \to GL_2(F).$$

be an absolutely irreducible, continuous, and odd two-dimensional representation of $G_{\mathbb{Q}}$ *over a finite field* $F = \mathbb{F}_{\ell^r}$ *of characteristic* ℓ*.*
There exists a normalized modular eigenform

$$f = q + a_2 q^2 + a_3 q^3 + \cdots$$

of level $N = N(\varrho)$, *weight* $k = k(\varrho)$, *and some Nebentype character* $\chi : \mathbb{Z}/N\mathbb{Z} \to F^*$ *such that for all prime numbers* p, *coprime to* $N\ell$, *we have*

$$\mathrm{Trace}(\rho(\mathrm{Frob}_p)) = a_p \quad \textit{and} \quad \det(\rho(\mathrm{Frob}_p)) = p^{k-1}\chi(p).$$

Waring's Problem

Any positive integer is the sum of at most 19 *biquadrates*
R. Balasubramanian, J-M. Deshouillers, F. Dress (1986).

$$n = x_1^4 + \cdots + x_{19}^4$$

François Dress, R. Balasubramanian, Jean-Marc Deshouillers

19 is optimal

If a number n is less than $3^4 = 81$, then any x_i involved in a formula

$$n = x_1^4 + \cdots + x_g^4$$

is either 1 or 2. In a minimal such equation, the number of 1's is at most 15. Since

$$81 = 5 \cdot 2^4 + 1,$$

the integer < 81 which requests the largest number of summands is

$$4 \cdot 2^4 + 15 \times 1^4 = 4 \cdot 16 + 15 = 79$$

with $g = 19$.

Lower bound for $g(k)$

Given k, let $g(k)$ be minimal such that any integer is the sum of at most $g(k)$ terms x^k. For instance $g(4) = 19$.
Divide 3^k by 2^k :

$$3^k = q_k 2^k + r_k \text{ with } 0 < r_k < 2^k, \qquad q_k = \lfloor (3/2)^k \rfloor.$$

Any integer $< 3^k$ written as $x_1^k + \cdots + x_g^k$ requires $x_i \in \{1, 2\}$.
We look for such an integer which has $2^k - 1$ summands 1^k.
The largest such $n < 3^k$ is

$$n_k = (q_k - 1)2^k + (2^k - 1)$$

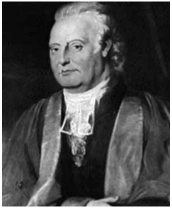
for which the number of summands is

$$\boxed{I_k = 2^k + q_k - 2.}$$

Hence $g(k) \geqslant I_k$ for all $k \geqslant 2$.

Waring's Problem

In 1770, a few months before J.L. Lagrange solved a conjecture of Bachet (1621) and Fermat (1640) by proving that every positive integer is the sum of at most four squares of integers, E. Waring wrote :



Edward Waring

1736–1798

"Omnis integer numerus vel est cubus, vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, &, usque ad novemdecim compositus, & sic deinceps"

"Every integer is a cube or the sum of two, three, ...nine cubes ; every integer is also the square of a square, or the sum of up to nineteen such ; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree."

Waring's functions $g(k)$ and $G(k)$

- Waring's function g is defined as follows : *For any integer $k \geqslant 2$, $g(k)$ is the least positive integer s such that any positive integer N can be written $x_1^k + \cdots + x_s^k$.*
- Waring's function G is defined as follows : *For any integer $k \geqslant 2$, $G(k)$ is the least positive integer s such that any sufficiently large positive integer N can be written $x_1^k + \cdots + x_s^k$.*

Known : $G(1) = 1$, $G(2) = 4$ and $G(4) = 16$.

$$G(k) \leqslant g(k).$$

Conjectured sequence $G(k)$, $k \geqslant 1$: https://oeis.org/A079611

$$1, 4, 4, 16, 6, 9, 8, 32, 13, 12, 12, 16, 14, 15, 16, 64, 18, 27, 20, 25$$

J.L. Lagrange : $g(2) = G(2) = 4$

$g(2) \leqslant 4$: any positive number is a sum of at most 4 squares : $n = x_1^2 + x_2^2 + x_3^2 + x_4^2$.

$g(2) = G(2) \geqslant 4$: there are (infinitely many) positive numbers (7 is the smallest of them) which are not sum of 3 squares.

Joseph-Louis Lagrange
1736 – 1813

Lower bounds are easy, not upper bounds.

Evaluations of $g(k)$ for $k = 2, 3, 4, \ldots$

$g(2) = 4$	Lagrange	1770
$g(3) = 9$	Kempner	1912
$g(4) = 19$	Balusubramanian,Deshouillers, Dress	1986
$g(5) = 37$	Chen Jingrun	1964
$g(6) = 73$	Pillai	1940
$g(7) = 143$	Dickson	1936

Sequence $g(k)$, $k \geqslant 1$: https://oeis.org/A002804

$$1, 4, 9, 19, 37, 73, 143, 279, 548, 1079, 2132, 4223, 8384, \ldots$$

The Ideal Waring Theorem

$g(k) \geqslant I(k)$ for any $k \geqslant 2$.

(J. A. Euler, son of Leonhard Euler).

Johann Albrecht Euler
(1734–1800)

Conjecture. (C.A. Bretschneider, 1853) :
$g(k) = I(k)$ for any $k \geqslant 2$.

True for $4 \leqslant k \leqslant 471\,600\,000$.

Mahler's contribution (1957)

The ideal Waring's Theorem

$$g(k) = 2^k + q_k - 2$$

holds for all sufficiently large k.

Kurt Mahler
1903–1988

However Mahler's proof involves non effective results of Diophantine approximation, we do not know how large k should be for the upper bound for $g(k)$ to be proven.

Waring's Problem and the abc Conjecture

Sinnou David

Shanta Laishram

S. David : The ideal Waring's Theorem $g(k) = 2^k + q_k - 2$ for large k follows from the abc Conjecture.

S. Laishram : The ideal Waring's Theorem for all k follows from the explicit abc Conjecture.

As simple as abc

The ABC's of salvation.
How to go to Heaven is as simple as ABC

American Broadcasting Company

http://fr.wikipedia.org/wiki/American_Broadcasting_Company https://abcathome.com/

The woman/parenting/homeschooling/entrepreneur resource brought to you by a busy, but efficient mother!

Smart Strategies for Parents Wanting to Head Back to School

ABC Stores

https://abcstores.com/
https://sites.google.com/view/emstmc2026/

Annapurna Base Camp, October 22, 2014

Mt. Annapurna (8091m) is the 10th highest mountain in the world and the journey to its base camp is one of the most popular treks on earth.

http://www.himalayanglacier.com/trekking-in-nepal/160/annapurna-base-camp-trek.htm

The radical of a positive integer

According to the fundamental theorem of arithmetic, any integer $n \geqslant 2$ can be written as a product of prime numbers :

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}.$$

The *radical* (also called *kernel*) $\mathrm{Rad}(n)$ of n is the product of the distinct primes dividing n :

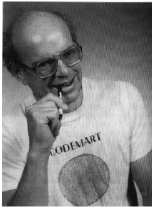
$$\mathrm{Rad}(n) = p_1 p_2 \cdots p_t.$$

$\mathrm{Rad}(n)$ divides n, it is the largest *squarefree* factor of n.
Examples :

$$\mathrm{Rad}(2^a) = 2$$
$$\mathrm{Rad}(60\,500) = \mathrm{Rad}(2^2 \cdot 5^3 \cdot 11^2) = 2 \cdot 5 \cdot 11 = 110,$$
$$\mathrm{Rad}(82\,852\,996\,681\,926) = 2 \cdot 3 \cdot 23 \cdot 109 = 15\,042.$$

The sequence of radicals

$n =$	1	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{Rad}(n)=$	1	2	3	2	5	6	7	2	3	10	11	6



Neil J. A. Sloane

Neil J. A. Sloane's encyclopaedia
`http://oeis.org/A007947`
`Largest squarefree number dividing` n`: the squarefree kernel of` n`,` $\mathrm{rad}(n)$`, radical of` n`.`

$$1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 2, 17, 6, 19, 10, 21, 22, 23, 6, 5, 26, 3, 14, 29, 30, 31, 2, 33, 34, 35, 6, 37, 38, 39, 10, 41, 42, 43, 22, 15, 46, 47, 6, 7, 10, 51, 26, 53, 6, 55, 14, 57, 58, 59, 30, 61, 62, 21, 2, \ldots$$

abc–triples

An abc–triple is a triple of three positive integers a, b, c which are coprime, $a < b$ and that $a + b = c$.

Examples:

$$1 + 2 = 3, \quad 1 + 8 = 9,$$

$$1 + 80 = 81, \quad 4 + 121 = 125,$$

$$2 + 3^{10} \cdot 109 = 23^5, \qquad 11^2 + 3^2 5^6 7^3 = 2^{21} \cdot 23.$$

There are thirteen abc–triples with $c < 10$

a, b, c are coprime, $1 \leqslant a < b$, $a + b = c$ and $c \leqslant 9$.

$1 + 2 = 3$			
$1 + 3 = 4$			
$1 + 4 = 5$	$2 + 3 = 5$		
$1 + 5 = 6$			
$1 + 6 = 7$	$2 + 5 = 7$	$3 + 4 = 7$	
$1 + 7 = 8$		$3 + 5 = 8$	
$1 + 8 = 9$	$2 + 7 = 9$		$4 + 5 = 9$

Radical of the abc–triples with $c < 10$

$\mathrm{Rad}(1\cdot 2\cdot 3) = 6$		
$\mathrm{Rad}(1\cdot 3\cdot 4) = 6$		
$\mathrm{Rad}(1\cdot 4\cdot 5) = 10$	$\mathrm{Rad}(2\cdot 3\cdot 5) = 30$	
$\mathrm{Rad}(1\cdot 5\cdot 6) = 30$		
$\mathrm{Rad}(1\cdot 6\cdot 7) = 42$	$\mathrm{Rad}(2\cdot 5\cdot 7) = 70$	$\mathrm{Rad}(3\cdot 4\cdot 7) = 42$
$\mathrm{Rad}(1\cdot 7\cdot 8) = 14$		$\mathrm{Rad}(3\cdot 5\cdot 8) = 30$
$\boxed{\mathrm{Rad}(1\cdot 8\cdot 9) = 6}$	$\mathrm{Rad}(2\cdot 7\cdot 9) = 54$	$\mathrm{Rad}(4\cdot 5\cdot 9) = 30$

$$a = 1,\ b = 8,\ c = 9,\ a + b = c,\ \mathrm{gcd} = 1,\ \mathrm{Rad}(abc) < c.$$

A single example in this range with $\mathrm{Rad}(abc) < c$.

abc–hits

Following F. Beukers, an abc–hit is an abc–triple such that $\mathrm{Rad}(abc) < c$.

Frits Beukers

http://www.staff.science.uu.nl/~beuke106/ABCpresentation.pdf

Example: $(1, 8, 9)$ is an abc–hit since $1 + 8 = 9$, $\gcd(1, 8, 9) = 1$ and

$$\mathrm{Rad}(1 \cdot 8 \cdot 9) = \mathrm{Rad}(2^3 \cdot 3^2) = 2 \cdot 3 = 6 < 9.$$

On the condition that a, b, c are relatively prime

Starting with $a + b = c$, multiply by a power of a divisor $d > 1$ of abc and get

$$ad^\ell + bd^\ell = cd^\ell.$$

The radical did not increase : the radical of the product of the three numbers ad^ℓ, bd^ℓ and cd^ℓ is nothing else than $\mathrm{Rad}(abc)$; but c is replaced by cd^ℓ.

For ℓ sufficiently large, cd^ℓ is larger than $\mathrm{Rad}(abc)$.

But $(ad^\ell, bd^\ell, cd^\ell)$ is not an abc–hit.

It would be too easy to get examples without the condition that a, b, c are relatively prime.

Some abc–hits

$(1, 80, 81)$ is an abc–hit since $1 + 80 = 81$, $\gcd(1, 80, 81) = 1$ and

$$\mathrm{Rad}(1 \cdot 80 \cdot 81) = \mathrm{Rad}(2^4 \cdot 5 \cdot 3^4) = 2 \cdot 5 \cdot 3 = 30 < 81.$$

$(4, 121, 125)$ is an abc–hit since $4 + 121 = 125$, $\gcd(4, 121, 125) = 1$ and

$$\mathrm{Rad}(4 \cdot 121 \cdot 125) = \mathrm{Rad}(2^2 \cdot 5^3 \cdot 11^2) = 2 \cdot 5 \cdot 11 = 110 < 125.$$

Further abc–hits

- $$(2, 3^{10} \cdot 109, 23^5) = (2, 6\,436\,341, 6\,436\,343)$$
  is an abc–hit since $2 + 3^{10} \cdot 109 = 23^5$ and
  $\mathrm{Rad}(2 \cdot 3^{10} \cdot 109 \cdot 23^5) = 15\,042 < 23^5 = 6\,436\,343.$

- $$(11^2, 3^2 \cdot 5^6 \cdot 7^3, 2^{21} \cdot 23) = (121, 48\,234\,275, 48\,234\,496)$$
  is an abc–hit since $11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 23$ and
  $\mathrm{Rad}(2^{21} \cdot 3^2 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 23) = 53\,130 < 2^{21} \cdot 23 = 48\,234\,496.$

- $$(1, 5 \cdot 127 \cdot (2 \cdot 3 \cdot 7)^3, 19^6) = (1, 47\,045\,880, 47\,045\,881)$$
  is an abc–hit since $1 + 5 \cdot 127 \cdot (2 \cdot 3 \cdot 7)^3 = 19^6$ and
  $\mathrm{Rad}(5 \cdot 127 \cdot (2 \cdot 3 \cdot 7)^3 \cdot 19^6) = 5 \cdot 127 \cdot 2 \cdot 3 \cdot 7 \cdot 19 = 506\,730.$

abc–triples and abc–hits

Among $15 \cdot 10^6$ abc–triples with $c < 10^4$, there are 120 abc–hits.

Among $380 \cdot 10^6$ abc–triples with $c < 5 \cdot 10^4$, there are 276 abc–hits.

More abc–hits

Recall the abc–hit $(1, 80, 81)$, where $81 = 3^4$.

$$(1, 3^{16} - 1, 3^{16}) = (1, 43\,046\,720, 43\,046\,721)$$

is an abc–hit.

Proof.

$$\begin{aligned}
3^{16} - 1 &= (3^8 - 1)(3^8 + 1) \\
&= (3^4 - 1)(3^4 + 1)(3^8 + 1) \\
&= (3^2 - 1)(3^2 + 1)(3^4 + 1)(3^8 + 1) \\
&= (3 - 1)(3 + 1)(3^2 + 1)(3^4 + 1)(3^8 + 1)
\end{aligned}$$

is divisible by 2^6. (Quotient : $672\,605$).

Hence

$$\mathrm{Rad}((3^{16} - 1) \cdot 3^{16}) \leqslant \frac{3^{16} - 1}{2^6} \cdot 2 \cdot 3 < 3^{16}.$$

Infinitely many abc–hits

Proposition. *There are infinitely many abc–hits.*

Take $k \geqslant 1$, $a = 1$, $c = 3^{2^k}$, $b = c - 1$.

Lemma. 2^{k+2} *divides* $3^{2^k} - 1$.

`Proof` : Induction on k using

$$3^{2^k} - 1 = (3^{2^{k-1}} - 1)(3^{2^{k-1}} + 1).$$

`Consequence` :

$$\mathrm{Rad}((3^{2^k} - 1) \cdot 3^{2^k}) \leqslant \frac{3^{2^k} - 1}{2^{k+1}} \cdot 3 < 3^{2^k}.$$

Hence

$$(1, 3^{2^k} - 1, 3^{2^k})$$

is an abc–hit.

Infinitely many abc–hits

This argument shows that there exist infinitely many abc–triples such that

$$c > \frac{1}{6\log 3} R \log R$$

with $R = \mathrm{Rad}(abc)$.

Question : *Does there exist an abc–triples for which $c \geqslant \mathrm{Rad}(abc)^2$?*

`We do not know the answer.`

It is expected that the answer is no : always for an abc–triple one should have

$$\mathrm{Rad}(abc) > c^{1/2}.$$

Two best known examples

When a, b and c are three positive relatively prime integers satisfying $a+b=c$, define

$$\lambda(a,b,c)=\frac{\log c}{\log \mathrm{Rad}(abc)} \text{ and } \varrho(a,b,c)=\frac{\log abc}{\log \mathrm{Rad}(abc)}.$$

The largest known values are (É. Reyssat and A. Nitaj)

$\lambda(a,b,c)=1.629912\ldots$ with $2+3^{10}\cdot 109=23^5$

$\varrho(a,b,c)=4.41901\ldots$ with $13\cdot 19^6+2^{30}\cdot 5=3^{13}\cdot 11^2\cdot 31.$

Éric Reyssat

Abderrahmane Nitaj

Resource on abc : two references

Abderrahmane Nitaj
https://nitaj.users.lmno.cnrs.fr/abc.html

Bart de Smit
http://www.math.leidenuniv.nl/~desmit/abc/

Explicit abc Conjecture

Shanta Laishram

Tarlok Shorey

Alan Baker
1939–2018

According to S. Laishram and T. N. Shorey, an explicit version, due to A. Baker, of the abc Conjecture, yields

$$c < \mathrm{Rad}(abc)^{7/4}$$

for any abc–triple (a, b, c).
In other terms for an abc triple one should have

$$\lambda(a, b, c) < 1.75.$$

The abc Conjecture

Recall that for a positive integer n, the *radical* of n is

$$\mathrm{Rad}(n) = \prod_{p|n} p.$$

abc **Conjecture**. *Let $\varepsilon > 0$. Then the set of abc triples for which*

$$c > \mathrm{Rad}(abc)^{1+\varepsilon}$$

is finite.

Equivalent statement : *For each $\varepsilon > 0$ there exists $\kappa(\varepsilon)$ such that, if a, b and c in $\mathbb{Z}_{>0}$ are relatively prime and satisfy $a + b = c$, then*

$$c < \kappa(\varepsilon)\mathrm{Rad}(abc)^{1+\varepsilon}.$$

Lower bound for the radical of abc

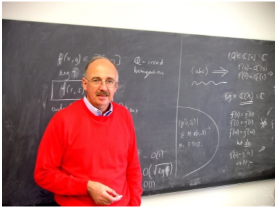
The abc Conjecture is a **lower bound** for the radical of the product abc :

abc **Conjecture**. *For any* $\varepsilon > 0$, *there exist* $\kappa'(\varepsilon)$ *such that, if* a, b *and* c *are relatively prime positive integers which satisfy* $a + b = c$, *then*

$$\mathrm{Rad}(abc) > \kappa'(\varepsilon) c^{1-\varepsilon}.$$

The abc Conjecture of Oesterlé and Masser

Joseph Oesterlé



David Masser

The abc Conjecture resulted from a discussion between J. Oesterlé and D. W. Masser in the mid 1980's.

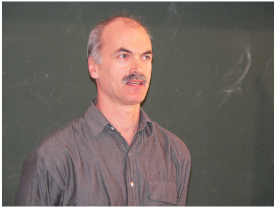
C.L. Stewart and Yu Kunrui

Best known non conditional result : C.L. Stewart and Yu Kunrui (1991, 2001) :

$$\log c \leqslant \kappa R^{1/3} (\log R)^3$$

with $R = \mathrm{Rad}(abc)$:

$$c \leqslant e^{\kappa R^{1/3} (\log R)^3}.$$



Cam. L. Stewart

Yu Kunrui

Szpiro's Conjecture

J. Oesterlé and A. Nitaj proved that the abc Conjecture implies a previous conjecture by L. Szpiro on the conductor of elliptic curves.

Lucien Szpiro
(1941–2020)

Given any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that, for every elliptic curve with minimal discriminant Δ and conductor N,

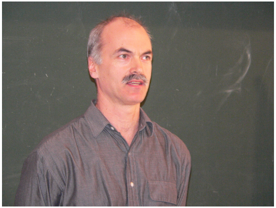
$$|\Delta| < C(\varepsilon) N^{6+\varepsilon}.$$

Szpiro's Conjecture

Conversely, J. Oesterlé proved in 1988 that the conjecture of L. Szpiro implies a weak form of the abc conjecture with $1 - \epsilon$ replaced by $(5/6) - \epsilon$.

Joseph Oesterlé

Is abc Conjecture optimal ?



Cam. L. Stewart

Rob Tijdeman

Let $\delta > 0$. In 1986, C.L. Stewart and R. Tijdeman proved that there are infinitely many abc–triples for which

$$c > R \exp\left((4-\delta) \frac{(\log R)^{1/2}}{\log \log R} \right).$$

Better than $c > R \log R$.

Why should the abc Conjecture be true ?

Heuristic assumption : *Any knowledge of* $\mathrm{Rad}(a)$ *and* $\mathrm{Rad}(b)$ *for* a *and* b *coprime positive integers should not give any non trivial information on* $\mathrm{Rad}(a+b)$.

For instance if a and b are (odd) primes, $a+b$ may be any (even) integer (Goldbach's Conjecture).

Christian Goldbach
1690–1764

Heuristic assumption

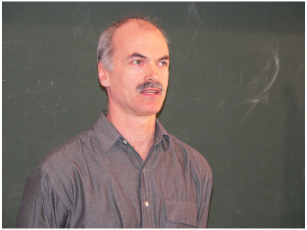
Whenever a and b are coprime positive integers, $\mathrm{Rad}(a+b)$ is independent of $\mathrm{Rad}(a)$ and $\mathrm{Rad}(b)$.

O. Robert, C.L. Stewart and G. Tenenbaum, *A refinement of the abc conjecture*, Bull. London Math. Soc., Bull. London Math. Soc. (2014) **46** (6) : 1156-1166.
http://blms.oxfordjournals.org/content/46/6/1156.full.pdf

http://iecl.univ-lorraine.fr/~Gerald.Tenenbaum/PUBLIC/Prepublications_et_publications/abc.pdf

Machiel van Frankenhuijsen, Olivier Robert, Cam Stewart and Gérald Tenenbaum






Conjectures by Machiel van Frankenhuijsen, Olivier Robert, Cam Stewart and Gérald Tenenbaum

Let $\varepsilon > 0$.

? There exists $\kappa(\varepsilon) > 0$ such that for any abc triple with $R = \mathrm{Rad}(abc) > 8$,

$$c < \kappa(\varepsilon) R \exp\left((4\sqrt{3} + \varepsilon) \left(\frac{\log R}{\log\log R} \right)^{1/2} \right).$$

? Further, there exist infinitely many abc–triples for which

$$c > R \exp\left((4\sqrt{3} - \varepsilon) \left(\frac{\log R}{\log\log R} \right)^{1/2} \right).$$

Fermat's Last Theorem $x^n + y^n = z^n$ for $n \geqslant 6$

Pierre de Fermat

(1601 – 1665)

Andrew Wiles

Solution in 1993–1994 published in 1995

Fermat's last Theorem as a consequence of the explicit abc Conjecture

Assume $x^n + y^n = z^n$ with $\gcd(x,y,z) = 1$ and $x < y$. Then (x^n, y^n, z^n) is an abc–triple with

$$\mathrm{Rad}(x^n y^n z^n) \leqslant xyz < z^3.$$

If the explicit abc Conjecture $c < \mathrm{Rad}(abc)^2$ is true, then one deduces

$$z^n < z^6,$$

hence $n \leqslant 5$ (and therefore $n \leqslant 2$).

Square, cubes. . .

- A perfect power is an integer of the form a^b where $a \geqslant 1$ and $b > 1$ are positive integers.
- Squares :

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100,\ 121,\ 144,\ 169,\ 196, \dots$$

- Cubes :

$$1,\ 8,\ 27,\ 64,\ 125,\ 216,\ 343,\ 512,\ 729,\ 1\,000,\ 1\,331, \dots$$

- Fifth powers :

$$1,\ 32,\ 243,\ 1\,024,\ 3\,125,\ 7\,776,\ 16\,807,\ 32\,768, \dots$$

Perfect powers

$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, 196, 216, 225, 243, 256, 289, 324, 343, 361, 400, 441, 484, 512, 529, 576, 625, 676, 729, 784, \ldots$

http://oeis.org/A001597

Nearly equal perfect powers

- Difference 1 : $(8, 9)$
- Difference 2 : $(25, 27), \ldots$
- Difference 3 : $(1, 4)$, $(125, 128), \ldots$
- Difference 4 : $(4, 8)$, $(32, 36)$, $(121, 125), \ldots$
- Difference 5 : $(4, 9)$, $(27, 32)$,...

Two conjectures

Subbayya Sivasankaranarayana Pillai (1901-1950)

Eugène Charles Catalan (1814 – 1894)

• Catalan's Conjecture : In the sequence of perfect powers, $8, 9$ is the only example of consecutive integers.

• Pillai's Conjecture : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

Pillai's Conjecture :

- Pillai's Conjecture : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

- Alternatively : Let k be a positive integer. The equation

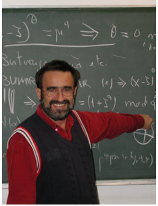
$$x^p - y^q = k,$$

where the unknowns x, y, p and q take integer values, all $\geqslant 2$, has only finitely many solutions (x, y, p, q).

$3^2 - 2^3 = 1$

P. Mihăilescu, 2002.

Catalan was right : *the equation* $x^p - y^q = 1$ *where the unknowns* x, y, p *and* q *take integer values, all* $\geqslant 2$, *has only one solution* $(x, y, p, q) = (3, 2, 2, 3)$.



Preda Mihailescu

Previous work on Catalan's Conjecture

J.W.S. Cassels
(1922–2015)

Rob Tijdeman

Michel Langevin

$$y^q < x^p < \exp\exp\exp\exp(730)$$

Previous work on Catalan's Conjecture

Maurice Mignotte

Yuri Bilu

Pillai's conjecture and the abc Conjecture

There is no value of $k \geqslant 2$ for which one knows that Pillai's equation $x^p - y^q = k$ has only finitely many solutions.

Pillai's conjecture as a consequence of the abc Conjecture : *if* $x^p \neq y^q$, *then*

$$|x^p - y^q| \geqslant c(\epsilon) \max\{x^p, y^q\}^{\kappa - \epsilon}$$

with

$$\kappa = 1 - \frac{1}{p} - \frac{1}{q} \cdot$$

Lower bounds for linear forms in logarithms

- A special case of my conjectures with S. Lang for

$$|q \log y - p \log x|$$

yields

$$|x^p - y^q| \geqslant c(\epsilon) \max\{x^p, y^q\}^{\kappa-\epsilon}$$

with

$$\kappa = 1 - \frac{1}{p} - \frac{1}{q} \cdot$$

Serge Lang
1927–2005

S. Lang, Elliptic curves Diophantine Analysis, Grundlehren der mathematischen Wissenschaften (GL, volume 231) Springer 1978.

Not a consequence of the abc Conjecture

$p = 3$, $q = 2$

Hall's Conjecture (1971) :

if $x^3 \neq y^2$, *then*

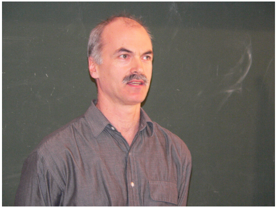
$$|x^3 - y^2| \geqslant c \max\{x^3, y^2\}^{1/6}.$$

Marshall Hall
(1910–1990)

https://en.wikipedia.org/wiki/Marshall_Hall_(mathematician)

Conjecture of F. Beukers and C.L. Stewart (2010)

Frits Beukers



Cam L. Stewart

? *Let p, q be coprime integers with $p > q \geqslant 2$. Then, for any $c > 0$, there exist infinitely many positive integers x, y such that*

$$0 < |x^p - y^q| < c \max\{x^p, y^q\}^\kappa$$

with $\kappa = 1 - \dfrac{1}{p} - \dfrac{1}{q}$.

Generalized Fermat's equation $x^p + y^q = z^r$

Consider the equation $x^p + y^q = z^r$ in positive integers (x, y, z, p, q, r) such that x, y, z relatively prime and p, q, r are $\geqslant 2$.

If

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geqslant 1,$$

then (p, q, r) is a permutation of one of

$$(2,2,k), \quad (2,3,3), \quad (2,3,4), \quad (2,3,5),$$
$$(2,4,4), \quad (2,3,6), \quad (3,3,3)$$

and in each case the set of solutions (x, y, z) is known (for some of these values there are infinitely many solutions).

Frits Beukers and Don Zagier

For

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

10 primitive solutions (x, y, z, p, q, r) (up to obvious symmetries) to the equation

$$x^p + y^q = z^r$$

are known.

Frits Beukers

Don Zagier

Primitive solutions to $x^p + y^q = z^r$

Condition : x, y, z are relatively prime

Trivial example of a non primitive solution : $2^p + 2^p = 2^{p+1}$.

`Exercise` (Henri Darmon, Claude Levesque) : for any pairwise relatively prime integers (p, q, r), there exist positive integers x, y, z with $x^p + y^q = z^r$.

`Hint` :

$$17^3 + 2^7 = 71^2,$$

$$\left(17 \times 71^{21}\right)^3 + \left(2 \times 71^9\right)^7 = \left(71^{13}\right)^5.$$

Generalized Fermat's equation

For

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

the equation

$$x^p + y^q = z^r$$

has the following 10 solutions with x, y, z relatively prime :

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2,$$
$$3^5 + 11^4 = 122^2, \quad 33^8 + 1\,549\,034^2 = 15\,613^3,$$
$$1\,414^3 + 2\,213\,459^2 = 65^7, \quad 9\,262^3 + 15\,312\,283^2 = 113^7,$$
$$17^7 + 76\,271^3 = 21\,063\,928^2, \quad 43^8 + 96\,222^3 = 30\,042\,907^2.$$

Conjecture of Beal, Granville and Tijdeman–Zagier



The equation $x^p + y^q = z^r$ has no solution in positive integers (x, y, z, p, q, r) with each of p, q and r at least 3 and with x, y, z relatively prime.

http://mathoverflow.net/

Andrew Beal

Find a solution with all exponents at least 3, or prove that there is no such solution.

http://www.forbes.com/2009/04/03/banking-andy-beal-business-wall-street-beal.html

Beal's Prize

Mauldin, R. D. – *A generalization of Fermat's last theorem : the Beal Conjecture and prize problem.* Notices Amer. Math. Soc. **44** N°11 (1997), 1436–1437.

The prize. Andrew Beal is very generously offering a prize of $5,000 for the solution of this problem. The value of the prize will increase by $5,000 per year up to $50,000 until it is solved. The prize committee consists of Charles Fefferman, Ron Graham, and R. Daniel Mauldin, who will act as the chair of the committee. All proposed solutions and inquiries about the prize should be sent to Mauldin.

Beal's Prize : $1,000,000\$$ US

An AMS-appointed committee will award this prize for either a proof of, or a counterexample to, the Beal Conjecture published in a refereed and respected mathematics publication. The prize money – currently US$1,000,000 – is being held in trust by the AMS until it is awarded. Income from the prize fund is used to support the annual Erdős Memorial Lecture and other activities of the Society.

One of Andrew Beal's goals is to inspire young people to think about the equation, think about winning the offered prize, and in the process become more interested in the field of mathematics.

http://www.ams.org/profession/prizes-awards/ams-supported/beal-prize

Henri Darmon, Andrew Granville

"Fermat-Catalan" Conjecture (H. Darmon and A. Granville), consequence of the abc Conjecture : *the set of solutions (x, y, z, p, q, r) to $x^p + y^q = z^r$ with x, y, z relatively prime and $(1/p) + (1/q) + (1/r) < 1$ is finite.*

Henri Darmon

Andrew Granville

Hint: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ implies $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leqslant \frac{41}{42}$.

1995 (H. Darmon and A. Granville) : unconditionally, for fixed (p, q, r), only finitely many (x, y, z).

Fermat's Little Theorem, Wieferich primes

For $a > 1$, any prime p not dividing a divides $a^{p-1} - 1$.

Hence if p is an odd prime, then p divides $2^{p-1} - 1$.

Pierre de Fermat
(1601 – 1665)

Wieferich primes (1909) : p^2 divides $2^{p-1} - 1$

The only known *Wieferich primes* are 1093 and 3511. These are the only ones below $4 \cdot 10^{12}$.

Assuming abc : Infinitely many primes are not Wieferich

Joseph H. Silverman

J.H. Silverman : if the abc Conjecture is true, given a positive integer $a > 1$, there exist infinitely many primes p such that p^2 does not divide $a^{p-1} - 1$.

Nothing is known about the finiteness of the set of Wieferich primes.

Consecutive integers with the same radical

Notice that

$$75 = 3 \cdot 5^2 \quad \text{and} \quad 1215 = 3^5 \cdot 5,$$

hence

$$\mathrm{Rad}(75) = \mathrm{Rad}(1215) = 3 \cdot 5 = 15.$$

But also

$$76 = 2^2 \cdot 19 \quad \text{and} \quad 1216 = 2^6 \cdot 19$$

have the same radical

$$\mathrm{Rad}(76) = \mathrm{Rad}(1216) = 2 \cdot 19 = 38.$$

Consecutive integers with the same radical

For $k \geqslant 1$, the two numbers

$$x = 2^k - 2 = 2(2^{k-1} - 1)$$

and

$$y = (2^k - 1)^2 - 1 = 2^{k+1}(2^{k-1} - 1)$$

have the same radical, and also

$$x + 1 = 2^k - 1 \quad \text{and} \quad y + 1 = (2^k - 1)^2$$

have the same radical.

Consecutive integers with the same radical

Are there further examples of $x \neq y$ with

$$\mathrm{Rad}(x) = \mathrm{Rad}(y) \quad \text{and} \quad \mathrm{Rad}(x+1) = \mathrm{Rad}(y+1)?$$

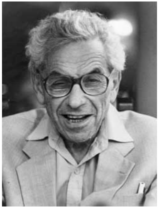
Is–it possible to find two distinct integers x, y such that

$$\mathrm{Rad}(x) = \mathrm{Rad}(y),$$
$$\mathrm{Rad}(x+1) = \mathrm{Rad}(y+1)$$

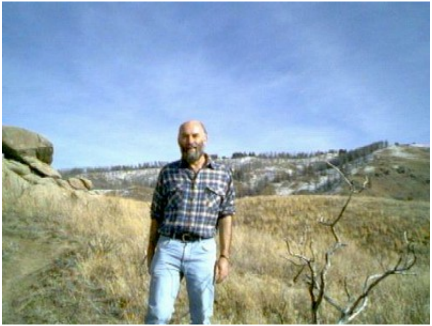
and

$$\mathrm{Rad}(x+2) = \mathrm{Rad}(y+2)?$$

Erdős – Woods Conjecture



Paul Erdős
(1913–1996)



http://school.maths.uwa.edu.au/~woods/

There exists an absolute constant k such that, if x and y are positive integers satisfying

$$\mathrm{Rad}(x+i) = \mathrm{Rad}(y+i)$$

for $i = 0, 1, \ldots, k-1$, then $x = y$.

Erdős – Woods as a consequence of abc

M. Langevin : The abc Conjecture implies that there exists an absolute constant k such that, if x and y are positive integers satisfying

$$\mathrm{Rad}(x+i) = \mathrm{Rad}(y+i)$$

for $i = 0, 1, \ldots, k-1$, then $x = y$.

Already in 1975 M. Langevin studied the radical of $n(n+k)$ with $\gcd(n,k) = 1$ using lower bounds for linear forms in logarithms (Baker's method).

A factorial as a product of factorials

For $n > a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t > 1$, $t > 1$, consider

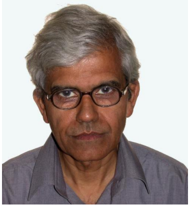
$$a_1!a_2!\cdots a_t! = n!$$

Trivial solutions : $2^r! = (2^r - 1)!2!^r$ with $r \geqslant 2$.
Non trivial solutions :

$7!3!^2 2! = 9!$, $7!6! = 10!$, $7!5!3! = 10!$, $14!5!2! = 16!$.

Saranya Nair and Tarlok Shorey : The effective abc conjecture implies Hickerson's conjecture that the largest non-trivial solution is given by $n = 16$.

Saranya Nair



Tarlok Shorey

Erdős Conjecture on $2^n - 1$

In 1965, P. Erdős conjectured that the greatest prime factor $P(2^n - 1)$ satisfies

$$\frac{P(2^n-1)}{n} \to \infty \quad \text{when} \quad n \to \infty.$$

In 2002, R. Murty and S. Wong proved that this is a consequence of the abc Conjecture.
In 2012, C.L. Stewart proved Erdős Conjecture (in a wider context of Lucas and Lehmer sequences) :

$$P(2^n-1) > n \exp\bigl(\log n / 104 \log\log n\bigr).$$

Conjecture of Alan Baker (1996)

Let (a, b, c) be an abc–triple and let $\epsilon > 0$. Then

$$c \leqslant \kappa \left(\epsilon^{-\omega} R\right)^{1+\epsilon}$$

where κ is an absolute constant, $R = \mathrm{Rad}(abc)$ and $\omega = \omega(abc)$ is the number of distinct prime factors of abc.

Remark of Andrew Granville : the minimum of the function on the right hand side over $\epsilon > 0$ occurs essentially with $\epsilon = \omega / \log R$. This yields a slightly sharper form of the conjecture :

$$c \leqslant \kappa R \frac{(\log R)^\omega}{\omega!}.$$

Alan Baker : explicit abc Conjecture (2004)

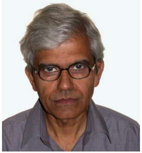
Let (a, b, c) be an abc–triple. Then

$$c \leqslant \frac{6}{5} R \frac{(\log R)^\omega}{\omega!}$$

with $R = \mathrm{Rad}(abc)$ the radical of abc and $\omega = \omega(abc)$ the number of distinct prime factors of abc.

Alan Baker
(1939–2018)

Shanta Laishram and Tarlok Shorey



The Nagell–Ljunggren equation is the equation

$$y^q = \frac{x^n - 1}{x - 1}$$

in integers $x > 1$, $y > 1$, $n > 2$, $q > 1$.

This means that in basis x, all the digits of the perfect power y^q are 1.
If the explicit abc–conjecture of Baker is true, then the only solutions are

$$11^2 = \frac{3^5 - 1}{3 - 1}, \quad 20^2 = \frac{7^4 - 1}{7 - 1}, \quad 7^3 = \frac{18^3 - 1}{18 - 1}.$$

The abc conjecture for number fields

P. Vojta (1987) – variants due to D.W. Masser and K. Győry

The abc conjecture for number fields (continued)

Survey by J. Browkin.

Jerzy Browkin
(1934 – 2015)

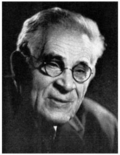
The abc– conjecture for
Algebraic Numbers
Acta Mathematica Sinica,
Jan., 2006, Vol. 22, No. 1,
pp. 211–222

http://dx.doi.org/10.1007/s10114-005-0624-3

Mordell's Conjecture (Faltings's Theorem)

Using an effective extension of the abc Conjecture for a number field, N. Elkies deduces an effective version of Faltings's Theorem on the finiteness of the set of rational points on an algebraic curve of genus $\geqslant 2$ over the same number field.

L.J. Mordell (1922) G. Faltings (1984) N. Elkies (1991)



http://www.math.harvard.edu/~elkies/

Mordell (1888–1972)

The abc conjecture for number fields

Andrea Surroca
(1973–2022)

The effective abc Conjecture implies an effective version of Siegel's Theorem on the finiteness of the set of integer points on a curve.

A. Surroca, *Méthodes de transcendance et géométrie diophantienne*, Thèse, Université de Paris 6, 2003.

Thue–Siegel–Roth Theorem (Bombieri)

Using the abc Conjecture for number fields, E. Bombieri (1994) deduces a refinement of the Thue–Siegel–Roth Theorem on the rational approximation of algebraic numbers

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{q^{2+\varepsilon}}$$

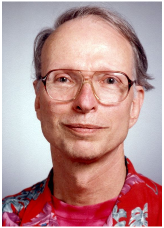
where he replaces ε by

$$\kappa(\log q)^{-1/2}(\log\log q)^{-1}$$

where κ depends only on the algebraic number α.

Siegel's zeroes (A. Granville and H.M. Stark)

The uniform abc Conjecture for number fields implies a lower bound for the class number of an imaginary quadratic number field, and K. Mahler has shown that this implies that the associated L–function has no Siegel zero.



abc and Vojta's height Conjecture

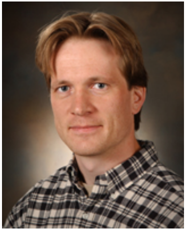
Paul Vojta

Vojta stated a conjectural inequality on the height of algebraic points of bounded degree on a smooth complete variety over a global field of characteristic zero which implies the *abc* Conjecture.

Further consequences of the abc Conjecture

- Erdős's Conjecture on consecutive powerful numbers.
- Dressler's Conjecture : between two positive integers having the same prime factors, there is always a prime (Cochrane and Dressler 1999).
- Squarefree and powerfree values of polynomials (Browkin, Filaseta, Greaves and Schinzel, 1995).
- Lang's conjectures : lower bounds for heights, number of integral points on elliptic curves (Frey 1987, Hindry Silverman 1988).
- Bounds for the order of the Tate–Shafarevich group (Goldfeld and Szpiro 1995).
- Greenberg's Conjecture on Iwasawa invariants λ and μ in cyclotomic extensions (Ichimura 1998).
- Lower bound for the class number of imaginary quadratic fields (Granville and Stark 2000), hence no Siegel zero for the associated L–function (Mahler).
- Fundamental units of certain quadratic and biquadratic fields (Katayama 1999).
- The height conjecture and the degree conjecture (Frey 1987, Mai and Murty 1996)

The n–Conjecture



Nils Bruin, Generalization of the ABC-conjecture, Master Thesis, Leiden University, 1995.

http://www.cecm.sfu.ca/~nbruin/scriptie.pdf

Let $n \geqslant 3$. There exists a positive constant κ_n such that, if $x_1, \dots, x_n$ are relatively prime rational integers satisfying $x_1 + \cdots + x_n = 0$ and if no proper subsum vanishes, then

$$\max\{|x_1|, \dots, |x_n|\} \leqslant \mathrm{Rad}(x_1 \cdots x_n)^{\kappa_n}.$$

? Should hold for all but finitely many $(x_1, \dots, x_n)$ with $\kappa_n = 2n - 5 + \epsilon$.

A consequence of the n–Conjecture

Open problem : for $k \geqslant 5$, no positive integer can be written in two essentially different ways as sum of two k–th powers.

It is not even known whether such a k exists.

Reference : Hardy and Wright : §21.11

For $k = 4$ (Euler) :

$$59^4 + 158^4 = 133^4 + 134^4 = 635\,318\,657$$

A parametric family of solutions of $x_1^4 + x_2^4 = x_3^4 + x_4^4$ is known

Reference : http://mathworld.wolfram.com/DiophantineEquation4thPowers.html

abc and meromorphic function fields

Rolf Nevanlinna
(1895–1980)
Nevanlinna value distribution theory.

Recent work of Hu, Pei–Chu, Yang, Chung-Chun and P. Vojta.

ABC Theorem for polynomials

Let K be an algebraically closed field. The *radical* of a monic polynomial

$$P(X) = \prod_{i=1}^{n} (X - \alpha_i)^{a_i} \in K[X]$$

with α_i pairwise distinct is defined as

$$\mathrm{Rad}(P)(X) = \prod_{i=1}^{n} (X - \alpha_i) \in K[X].$$

ABC Theorem for polynomials

ABC **Theorem** (A. Hurwitz, W.W. Stothers, R. Mason). Let A, B, C be three relatively prime polynomials in $K[X]$ with $A + B = C$ and let $R = \mathrm{Rad}(ABC)$. Then

$$\max\{\deg(A), \deg(B), \deg(C)\} < \deg(R).$$

Adolf Hurwitz (1859–1919)

This result can be compared with the abc Conjecture, where the degree replaces the logarithm.

The radical of a polynomial as a gcd

The common zeroes of

$$P(X) = \prod_{i=1}^{n} (X - \alpha_i)^{a_i} \in K[X]$$

and P' are the α_i with $a_i \geqslant 2$. They are zeroes of P' with multiplicity $a_i - 1$. Hence

$$\mathrm{Rad}(P) = \frac{P}{\gcd(P, P')}.$$

Proof of the ABC Theorem for polynomials

Now suppose $A + B = C$ with A, B, C relatively prime.

Notice that

$$\mathrm{Rad}(ABC) = \mathrm{Rad}(A)\mathrm{Rad}(B)\mathrm{Rad}(C).$$

We may suppose A, B, C to be monic and, say, $\deg(A) \leqslant \deg(B) \leqslant \deg(C)$.

Write

$$A + B = C, \qquad A' + B' = C',$$

and

$$AB' - A'B = AC' - A'C.$$

Proof of the ABC Theorem for polynomials

Recall $\gcd(A, B, C) = 1$. Since $\gcd(C, C')$ divides $AC' - A'C = AB' - A'B$, it divides also

$$\frac{AB' - A'B}{\gcd(A, A')\gcd(B, B')}$$

which is a polynomial of degree

$$< \deg\bigl(\mathrm{Rad}(A)\bigr) + \deg\bigl(\mathrm{Rad}(B)\bigr) = \deg\bigl(\mathrm{Rad}(AB)\bigr).$$

Hence

$$\deg\bigl(\gcd(C, C')\bigr) < \deg\bigl(\mathrm{Rad}(AB)\bigr)$$

and

$$\deg(C) < \deg\bigl(\mathrm{Rad}(C)\bigr) + \deg\bigl(\mathrm{Rad}(AB)\bigr) = \deg\bigl(\mathrm{Rad}(ABC)\bigr).$$

https://en.wikipedia.org/wiki/Abc_conjecture 2012

In August 2012, Shinichi Mochizuki released a series of four preprints announcing a proof of the abc Conjecture.

When an error in one of the articles was pointed out by Vesselin Dimitrov and Akshay Venkatesh in October 2012, Mochizuki posted a comment on his website acknowledging the mistake, stating that it would not affect the result, and promising a corrected version in the near future.

2017

Not Even Wrong
Latest on abc
Posted on December 16, 2017 by Peter Woit
http://www.math.columbia.edu/~woit/wordpress/?p=9871

The ABC conjecture has (still) not been proved
Posted on December 17, 2017 by Frank Calegari
https://galoisrepresentations.wordpress.com/2017/12/17/the-abc-conjecture-has-still-not-been-proved/

Hector Pasten
Shimura curves and the abc conjecture
https://arxiv.org/abs/1705.09251

2022 : Explicit estimates

June 2022

Explicit estimates in inter-universal Teichmüller theory

Shinichi Mochizuki, Ivan Fesenko, Yuichiro Hoshi, Arata Minamide, Wojciech Porowski

Author Affiliations +

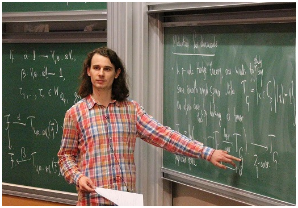
Kodai Math. J. 45(2): 175-236 (June 2022). DOI: 10.2996/kmj45201

https://doi.org/10.2996/kmj45201

Mochizuki – Fesenko vs Scholze – Stix

Shinichi Mochizuki

Ivan Fesenko



Peter Scholze

Jakob Stix

Review by Peter Scholze

Together with J. Stix, the reviewer has spent a week in Kyoto to discuss these issues with the author, and has detailed the findings in a manuscript entitled
"Why ABC is still a conjecture"
https://www.math.uni-bonn.de/people/scholze/WhyABCisStillaConjecture.pdf
that discusses the issues in slightly more detail.

The concerns expressed in this manuscript have not been addressed in the published version.

https://zbmath.org/?q=an:1465.14002

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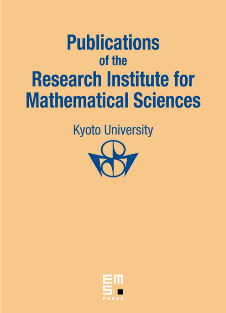


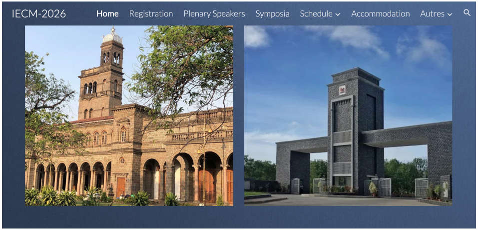
Mochizuki - zbMath

☐ Mochizuki, Shinichi
Inter-universal Teichmüller theory. IV: Log-volume computations and set-theoretic foundations. (English) Zbl 1465.14005
Publ. Res. Inst. Math. Sci. 57, No. 1-2, 627-723 (2021).
MSC: 14-02 14H25 14H30 14G40 11Gxx Reviewer: Peter Scholze (Bonn)
Cite Review PDF Full Text: DOI Backlinks: MO

☐ Mochizuki, Shinichi
Inter-universal Teichmüller theory. III: Canonical splittings of the log-theta-lattice. (English)
Zbl 1465.14004
Publ. Res. Inst. Math. Sci. 57, No. 1-2, 403-626 (2021).
MSC: 14-02 14H25 14H30 14A21 Reviewer: Peter Scholze (Bonn)
Cite Review PDF Full Text: DOI Backlinks: MO

☐ Mochizuki, Shinichi
Inter-universal Teichmüller theory. II: Hodge-Arakelov-theoretic evaluation. (English)
Zbl 1465.14003
Publ. Res. Inst. Math. Sci. 57, No. 1-2, 209-401 (2021).
MSC: 14-02 14H25 14H30 14G32 14G40 Reviewer: Peter Scholze (Bonn)
Cite Review PDF Full Text: DOI Backlinks: MO

☐ Mochizuki, Shinichi
Inter-universal Teichmüller theory. I: Construction of Hodge theaters. (English) Zbl 1465.14002
Publ. Res. Inst. Math. Sci. 57, No. 1-2, 3-207 (2021).
MSC: 14-02 14H25 14H30 14G32 Reviewer: Peter Scholze (Bonn)
Cite Review PDF Full Text: DOI Backlinks: MO MO



The abc of Number Theory

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