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A course on linear recurrent sequences
African Institute for Mathematical Sciences (AIMS)

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Quizz (10')

A word on the alphabet with two letters $\{a, b\}$ is a finite sequence of letters, like $aaba$, $abab$.

The weight of the letter a is 1, the weight of the letter b is 2. The weight of a word is the sum of the weights of its letters.

For instance the word $aaba$ has 3 letters a and 1 letter b, hence its weight is $3+2=5$. The word $abab$ has 2 letters a and 2 letters b, its weight is $2+2\times 2=6$.

Given a positive integer n, how many words of weight n are there ?

`Solution.`

Denote by S_n be the set of words of weight n, by s_n the number of elements of S_n.

Next, denote by S'_n the subset the words in S_n which end with a, by s'_n the number of elements of S'_n.

Finally, denote by S''_n the subset of the words in S_n which end with b, by s''_n the number of elements of S''_n.

It follows that S_n is the disjoint union of S'_n and S''_n and therefore

$$s_n = s'_n + s''_n.$$

$S_1=\{a\}$, $S_2=\{aa, b\}$, $S_3=\{aaa, ab, ba\}$, $S_4=\{aaaa, aab, aba, baa, bb\}$,
$S_5=\{aaaaa, aaab, aaba, abaa, baaa, bba, abb, bab\}$, ...
$s_1=1$, $s_2=2$, $s_3=3$, $s_4=5$, $s_5=8$,...
$S'_1=\{a\}$, $S'_2=\{aa\}$, $S'_3=\{aaa, ba\}$, $S'_4=\{aaaa, aba, baa\}$,
$S'_5=\{aaaaa, aaba, abaa, baaa, bba\}$, ...
$s'_1=1$, $s'_2=1$, $s'_3=2$, $s'_4=3$, $s'_5=5$,...
$S''_1=\emptyset$, $S''_2=\{b\}$, $S''_3=\{ab\}$, $S''_4=\{aab, bb\}$, $S''_5=\{aaab, abb, bab\}$, ...
$s''_1=0$, $s''_2=1$, $s''_3=1$, $s''_4=2$, $s''_5=3$,...

For $n\ge 2$, the map $S_{n-1}\to S'_n$ which maps a word w to wa is bijective, so that $s'_n=s_{n-1}$.

For $n\ge 3$, the map $S_{n-2}\to S''_n$ which maps a word w to wb is also bijective, so that $s''_n=s_{n-2}$

Hence $s_n=s_{n-1}+s_{n-2}$ for $n\ge 3$. Since $s_1=1$ and $s_2=2$, we deduce that $s_n=F_{n+1}$ for $n\ge 1$, where $(F_n)_{n\ge 0}$ is the Fibonacci sequence.

We also have $s'_n=F_n$ and $s''_n=F_{n-1}$ for $n\ge 1$.

`Comment.` Compare with the slides

`Reflections of a ray of light`
`Levels of energy of an electron of an atom of hydrogen`
`Rhythmic patterns`

of the course `Linear recurrence sequences: part I`.