November 09 – 19, 2021 Limbe (Cameroon)

Name: ..........................................................................................................................................

**Number Theory**
**II. Prime Numbers**
**African Institute for Mathematical Sciences (AIMS)**

*Michel Waldschmidt, Sorbonne Université*

**Quizz 2** Date: *18/11/2021*

(a) Prove that any positive integer $> 1$ of the form $3n - 1$ has at least one prime divisor congruent to 2 modulo 3.

(b) Deduce that there are infinitely many prime numbers congruent to 2 modulo 3.

**Solution**

(a) The product of prime numbers congruent to 1 modulo 3 is congruent to 1 modulo 3. Hence an integer $> 1$ which is congruent to $-1$ modulo 3 has at least one prime divisor congruent to 2 modulo 3.

(b) We repeat the proof given in the course of the fact that there are infinitely many prime numbers congruent to 3 modulo 4: let $\{p_1, p_2, p_3, \ldots, p_s\}$ be a finite set of prime number congruent to 2 modulo 3. Consider their product $N = p_1 p_2 \cdots p_s$. According to (a), the number $3N - 1$ has a prime divisor congruent to 2 modulo 3, hence this prime divisor is not one of $\{p_1, p_2, p_3, \ldots, p_s\}$ since it does not divide $3N - (3N - 1) = 1$. Hence there is a prime number congruent to 2 modulo 3 which is not in the set $\{p_1, p_2, p_3, \ldots, p_s\}$.