Transcendental Numbers and Hopf Algebras

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Algebraic groups (commutative, linear, over $\overline{\mathbf{Q}}$)

Exponential polynomials

Transcendence of values of exponential polynomials

Hopf algebras (commutative, cocommutative, of finite type)

Algebra of multizeta values

Commutative linear algebraic groups over $\overline{\mathbf{Q}}$

$$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \qquad d = d_0 + d_1$$

$$G(\overline{\mathbf{Q}}) = \overline{\mathbf{Q}}^{d_0} \times (\overline{\mathbf{Q}}^{\times})^{d_1}$$

$$(\beta_1, \ldots, \beta_{d_0}, \alpha_1, \ldots, \alpha_{d_1})$$

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$$\exp_G : T_e(G) = \mathbf{C}^d \longrightarrow G(\mathbf{C}) = \mathbf{C}^{d_0} \times (\mathbf{C}^{\times})^{d_1}$$

$$(z_1, \ldots, z_d) \longmapsto (z_1, \ldots, z_{d_0}, e^{z_{d_0+1}}, \ldots, e^{z_d})$$

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For α_j and β_i in $\overline{\mathbf{Q}}$,

$$\exp_G(\beta_1, \ldots, \beta_{d_0}, \log \alpha_1, \ldots, \log \alpha_{d_1}) \in G(\overline{\mathbf{Q}})$$

Baker's Theorem. *If*

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n = 0$$

with algebraic β_i *and* α_j*, then*

1. $\beta_0 = 0$

2. If $(\beta_1, \ldots, \beta_n) \neq (0, \ldots, 0)$*, then* $\log \alpha_1, \ldots, \log \alpha_n$ *are* $\mathbf{Q}$-*linearly dependent.*

3.If $(\log \alpha_1, \ldots, \log \alpha_n) \neq (0, \ldots, 0)$*, then* $\beta_1, \ldots, \beta_n$ *are* $\mathbf{Q}$-*linearly dependent.*

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Corollaries.

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3. *Example with* $n=2,\ \beta_0\neq 0$: transcendence of

$$\int_0^1 \frac{dx}{1+x^3}=\frac{1}{3}\log 2+\frac{\pi}{3\sqrt{3}}\cdot$$

Strong Six Exponentials Theorem. *If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2, y_3 *are three complex numbers which are* $\mathbf{Q}$*-linearly independent and if* β_{ij} *are six algebraic numbers such that*

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \textit{for} \quad i = 1, 2,\ j = 1, 2, 3,$$

then $x_i y_j = \beta_{ij}$ *for* $i = 1, 2,\ j = 1, 2, 3.$

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Corollary 1 (Six Exponentials Theorem). *One at least of the six numbers*

$$e^{x_i y_j} \quad (i = 1, 2,\ j = 1, 2, 3)$$

is transcendental.

Strong Four Exponentials Conjecture. *If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2, *are two complex numbers which are* $\mathbf{Q}$*-linearly independent and if* $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$, *are four algebraic numbers such that the four numbers*

$$e^{x_1y_1-\beta_{11}},\ e^{x_1y_2-\beta_{12}},\ e^{x_2y_1-\beta_{21}},\ e^{x_2y_2-\beta_{22}}$$

are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2,\ j = 1, 2.$

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Special case (Four Exponentials Conjecture). *One at least of the four numbers*

$$e^{x_1y_1},\; e^{x_1y_2},\; e^{x_2y_1},\; e^{x_2y_2}$$

is transcendental.

Corollary 2 (Strong Five Exponentials Theorem). *If* x_1, x_2 *are* $\mathbf{Q}$*-linearly independent, if* y_1, y_2 *are* $\mathbf{Q}$*-linearly independent and if* $\alpha, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \gamma$ *are six algebraic numbers such that*

$$e^{x_1y_1-\beta_{11}},\; e^{x_1y_2-\beta_{12}},\; e^{x_2y_1-\beta_{21}},\; e^{x_2y_2-\beta_{22}},\; e^{(\gamma x_2/x_1)-\alpha}$$

are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2,\; j = 1, 2$ *and also* $\gamma x_2 = \alpha x_1$.

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are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2,\ j = 1, 2$ *and also* $\gamma x_2 = \alpha x_1$.

Proof. Set $y_3 = \gamma/x_1$, $\beta_{13} = \gamma$, $\beta_{23} = \alpha$, so that

$$x_1y_3 - \beta_{13} = 0 \quad \text{and} \quad x_2y_3 - \beta_{23} = (\gamma x_2/x_1) - \alpha.$$

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are algebraic, then $x_iy_j = \beta_{ij}$ *for* $i = 1, 2,\; j = 1, 2.$

Five Exponentials Theorem. *If* x_1, x_2 *are* $\mathbf{Q}$*-linearly independent,* y_1, y_2 *are* $\mathbf{Q}$*-linearly independent and* γ *is a non zero algebraic number, then one at least of the five numbers*

$$e^{x_1y_1},\; e^{x_1y_2},\; e^{x_2y_1},\; e^{x_2y_2},\; e^{\gamma x_2/x_1}$$

is transcendental.

More general result (D. Roy). *If* x_1, x_2 *are* $\overline{\mathbf{Q}}$*-linearly independent and if* y_1, y_2, y_3 *are* $\overline{\mathbf{Q}}$*-linearly independent, then one at least of the six numbers*

$$x_i y_j \quad (i = 1, 2,\ j = 1, 2, 3)$$

is not of the form

$$\beta_{ij} + \sum_{h=1}^{\ell} \beta_{ijh} \log \alpha_h.$$

Very Strong Four Exponentials Conjecture. *If* x_1, x_2 *are* $\overline{\mathbf{Q}}$-*linearly independent and if* y_1, y_2, *are* $\overline{\mathbf{Q}}$-*linearly independent, then one at least of the four numbers*

$$x_1y_1,\ x_1y_2,\ x_2y_1,\ x_2y_2$$

does not belong to the $\overline{\mathbf{Q}}$-*vector space*

$$\left\{\beta_0 + \sum_{h=1}^{\ell} \beta_h \log \alpha_h\ ;\ \alpha_i \text{ and } \beta_j \text{ algebraic}\right\}$$

spanned by 1 *and* $\exp^{-1}(\overline{\mathbf{Q}}^\times)$.

Values of exponential polynomials

Proof of Baker's Theorem. Assume

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n$$

(B_1) (Gel'fond–Baker's Method)

Functions: $z_0,\ e^{z_1},\ \ldots,\ e^{z_{n-1}},\ e^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$

Points: $\mathbf{Z}(1, \log \alpha_1, \ldots, \log \alpha_{n-1}) \in \mathbf{C}^n$

Derivatives: $\partial/\partial z_i$, $(0 \le i \le n-1)$.

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Derivatives: $\partial/\partial z_i$, $(0 \le i \le n-1)$.

$n+1$ functions, n variables, 1 point, n derivatives

Another proof of Baker's Theorem. Assume again

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n$$

(B_2) (Generalization of Schneider's method)

Functions: $z_0,\ z_1,\ \ldots, z_{n-1},$

$$e^{z_0}\alpha_1^{z_1} \cdots \alpha_{n-1}^{z_{n-1}} = \exp\{z_0 + z_1 \log \alpha_1 + \cdots + z_{n-1} \log \alpha_{n-1}\}$$

Points: $\{0\} \times \mathbf{Z}^{n-1} + \mathbf{Z}(\beta_0, , \ldots, \beta_{n-1}) \in \mathbf{C}^n$

Derivative: $\partial / \partial z_0$.

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Derivative: $\partial/\partial z_0$.

$n+1$ functions, n variables, n points, 1 derivative

Proof of the strong six exponentials Theorem

Assume $x_1, \ldots, x_a$ are $\mathbf{Q}$-linearly independent, $y_1, \ldots, y_b$ are $\mathbf{Q}$-linearly independent and β_{ij} are algebraic numbers such that

$$e^{x_i y_j - \beta_{ij}} \in \overline{\mathbf{Q}} \quad \text{for} \quad i = 1, \ldots, a, , \; j = 1, \ldots, b$$

with $ab > a + b$.

Functions: $z_i, \; e^{x_i(z_{a+1}+z_1)-z_i} \quad (1 \le i \le a)$

Points: $(\beta_{1j}, \ldots, \beta_{aj}, y_j - \beta_{1j}) \in \mathbf{C}^{a+1} \quad (1 \le j \le b)$

Derivatives: $\partial/\partial z_i \; (2 \le i \le a)$ et $\partial/\partial z_{a+1} - \partial/\partial z_1$.

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$2a$ functions, $a+1$ variables, b points, a derivatives

Linear Subgroup Theorem

$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}, \qquad d = d_0 + d_1.$

$W \subset T_e(G)$ a $\mathbf{C}$-subspace which is rational over $\overline{\mathbf{Q}}$. Let ℓ_0 be its dimension.

$Y \subset T_e(G)$ a finitely generated subgroup with $\Gamma = \exp(Y)$ contained in $G(\overline{\mathbf{Q}}) = \overline{\mathbf{Q}}^{d_0} \times (\overline{\mathbf{Q}}^{\times})^{d_1}$. Let ℓ_1 be the $\mathbf{Z}$-rank of Γ.

$V \subset T_e(G)$ a $\mathbf{C}$-subspace containing both W and Y. Let n be the dimension of V.

Hypothesis:

$$\boxed{n(\ell_1 + d_1) < \ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0}$$

$$n(\ell_1 + d_1) < \ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0$$

$d_0 + d_1$ is the number of functions

d_0 are linear

d_1 are exponential

n is the number of variables

ℓ_0 is the number of derivatives

ℓ_1 is the number of points

	d_0	d_1	ℓ_0	ℓ_1	n
Baker B_1	1	n	n	1	n
Baker B_2	n	1	1	n	n
Six exponentials	a	a	a	b	$a+1$

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Six exponentials	a	a	a	b	$a+1$

Baker:

$$n(\ell_1 + d_1) = n^2 + n$$

$$\ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0 = n^2 + n + 1$$

	d_0	d_1	ℓ_0	ℓ_1	n
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Baker:

$$n(\ell_1 + d_1) = n^2 + n$$

$$\ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0 = n^2 + n + 1$$

Six exponentials: $a + b < ab$

$$n(\ell_1 + d_1) = a^2 + ab + a + b$$

$$\ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0 = a^2 + 2ab$$

duality:

$$(d_0, d_1, \ell_0, \ell_1) \longleftrightarrow (\ell_0, \ell_1, d_0, d_1)$$

$$\left(\frac{d}{dz}\right)^s \left(z^t e^{xz}\right)_{z=y} = \left(\frac{d}{dz}\right)^t \left(z^s e^{yz}\right)_{z=x}.$$

Fourier-Borel duality:

$$(d_0, d_1, \ell_0, \ell_1) \longleftrightarrow (\ell_0, \ell_1, d_0, d_1)$$

$$\left(\frac{d}{dz}\right)^s \left(z^t e^{xz}\right)_{z=y} = \left(\frac{d}{dz}\right)^t \left(z^s e^{yz}\right)_{z=x}.$$

$$\mathsf{L}_{sy} : f \longmapsto \left(\frac{d}{dz}\right)^s f(y).$$

$$f_\zeta(z) = e^{z\zeta}, \qquad \mathsf{L}_{sy}(f_\zeta) = \zeta^s e^{y\zeta}.$$

$$\mathsf{L}_{sy}(z^t f_\zeta) = \left(\frac{d}{d\zeta}\right)^t \mathsf{L}_{sy}(f_\zeta).$$

For $\underline{v} = (v_1, \ldots, v_n) \in \mathbf{C}^n$, set

$$D_{\underline{v}} = v_1 \frac{\partial}{\partial z_1} + \cdots + v_n \frac{\partial}{\partial z_n}.$$

Let $\underline{w}_1, \ldots, \underline{w}_{\ell_0}$, $\underline{u}_1, \ldots, \underline{u}_{d_0}$, $\underline{x}$ and $\underline{y}$ in $\mathbf{C}^n$, $\underline{t} \in \mathbf{N}^{d_0}$ and $\underline{s} \in \mathbf{N}^{\ell_0}$. For $\underline{z} \in \mathbf{C}^n$, write

$$(\mathbf{u}\underline{z})^{\underline{t}} = (\underline{u}_1\underline{z})^{t_1} \cdots (\underline{u}_{d_0}\underline{z})^{t_{d_0}} \quad \text{and} \quad D_{\mathbf{w}}^{\underline{s}} = D_{\underline{w}_1}^{s_1} \cdots D_{\underline{w}_{\ell_0}}^{s_{\ell_0}}.$$

Then

$$\boxed{D_{\mathbf{w}}^{\underline{s}}\Big((\mathbf{u}\underline{z})^{\underline{t}} e^{\underline{x}\underline{z}}\Big)\Big|_{\underline{z}=\underline{y}} = D_{\mathbf{u}}^{\underline{t}}\Big((\mathbf{w}\underline{z})^{\underline{s}} e^{\underline{y}\underline{z}}\Big)\Big|_{\underline{z}=\underline{x}}}$$

Hopf Algebras (over $k = \mathbf{C}$ or $k = \overline{\mathbf{Q}}$)

Algebras

A k-**algebra** (A, m, η) is a k-vector space A with a **product** $m : A \otimes A \to A$ and a **unit** $\eta : k \longrightarrow A$ which are k-linear maps such that the following diagrams commute:

(Associativity)

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{Id}} & A \otimes A \\ {\scriptstyle \mathrm{Id} \otimes m} \downarrow & & \downarrow {\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array}$$

(Unit)

$$\begin{array}{ccccc} k \otimes A & \xrightarrow{\eta \otimes \mathrm{Id}} & A \otimes A & \xleftarrow{\mathrm{Id} \otimes \eta} & A \otimes k \\ \downarrow & & \downarrow {\scriptstyle m} & & \downarrow \\ A & = & A & = & A \end{array}$$

Coalgebras

A k-**coalgebra** (A, Δ, ϵ) is a k-vector space A with a **coproduct** $\Delta : A \to A \otimes A$ and a **counit** $\epsilon : A \longrightarrow k$ which are k-linear maps such that the following diagrams commute:

(Coassociativity)

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Delta \downarrow & & \downarrow \Delta \otimes \mathrm{Id} \\ A \otimes A & \xrightarrow[\mathrm{Id} \otimes \Delta]{} & A \otimes A \otimes A \end{array}$$

(Counit)

$$\begin{array}{ccccc} A & = & A & = & A \\ \downarrow & & \downarrow \Delta & & \downarrow \\ k \otimes A & \xleftarrow[\epsilon \otimes \mathrm{Id}]{} & A \otimes A & \xrightarrow[\mathrm{Id} \otimes \epsilon]{} & A \otimes k \end{array}$$

Bialgebras

A **bialgebra** $(A, m, \eta, \Delta, \epsilon)$ is a k-algebra (A, m, η) together with a coalgebra structure (A, Δ, ϵ) which is *compatible*: Δ and ϵ are algebra morphisms

$$\Delta(xy) = \Delta(x)\Delta(y), \quad \epsilon(xy) = \epsilon(x)\epsilon(y).$$

Hopf Algebras

A **Hopf algebra** $(H, m, \eta, \Delta, \epsilon, S)$ is a bialgebra $(H, m, \eta, \Delta, \epsilon)$ with an *antipode* $S : H \to H$ which is a k-linear map such that the following diagram commutes:

$$\begin{array}{ccccc} H \otimes H & \xleftarrow{\Delta} & H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle \mathrm{Id}\otimes S}\downarrow & & {\scriptstyle \eta\circ\epsilon}\downarrow & & \downarrow{\scriptstyle S\otimes\mathrm{Id}} \\ H \otimes H & \xrightarrow[m]{} & H & \xleftarrow[m]{} & H \otimes H \end{array}$$

In a Hopf Algebra the *primitive* elements

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

satisfy $\epsilon(x) = 0$ and $S(x) = -x$; they form a Lie algebra for the bracket

$$[x, y] = xy - yx.$$

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The *group-like* elements

$$\Delta(x) = x \otimes x, \quad x \neq 0$$

are invertible, they satisfy $\epsilon(x) = 1$, $S(x) = x^{-1}$ and form a multiplicative group.

Example 1.

Let G be a finite multiplicative group, kG the algebra of G over k which is a k vector-space with basis G. The mapping

$$m : kG \otimes kG \rightarrow kG$$

extends the product

$$(x, y) \mapsto xy$$

of G by linearity. The unit

$$\eta : k \rightarrow kG$$

maps 1 to 1_G.

Define a coproduct and a counit

$$\Delta : kG \to kG \otimes kG \text{ and } \epsilon : kG \to k$$

by extending

$$\Delta(x) = x \otimes x \text{ and } \epsilon(x) = 1 \text{ for } x \in G$$

by linearity. The antipode

$$S : kG \to kG$$

is defined by

$$S(x) = x^{-1} \text{ for } x \in G.$$

Since $\Delta(x) = x \otimes x$ for $x \in G$ this Hopf algebra kG is cocommutative.

It is a commutative algebra if and only if G is commutative.

The set of group like elements is G: one recovers G from kG.

Example 2.

Again let G be a finite multiplicative group. Consider the k-algebra k^G of mappings $G \to k$, with basis $\delta_g \quad (g \in G)$, where

$$\delta_g(g') = \begin{cases} 1 & \text{for } g' = g, \\ 0 & \text{for } g' \neq g. \end{cases}$$

Define m by

$$m(\delta_g \otimes \delta_{g'}) = \delta_g \delta_{g'}.$$

Hence m is commutative and $m(\delta_g \otimes \delta_g) = \delta_g$ for $g \in G$. The unit $\eta : k \to k^G$ maps 1 to $\sum_{g \in G} \delta_g$.

Define a coproduct $\Delta : k^G \to k^G \otimes k^G$ and a counit $\epsilon : k^G \to k$ by

$$\Delta(\delta_g) = \sum_{g'g''=g} \delta_{g'} \otimes \delta_{g''} \quad \text{and} \quad \epsilon(\delta_g) = \delta_g(1_G).$$

The coproduct Δ is cocommutative if and only if the group G is commutative.

Define an antipode S by

$$S(\delta_g) = \delta_{g^{-1}}.$$

Remark. One may identify $k^G \otimes k^G$ and $k^{G\times G}$ with

$$\delta_g \otimes \delta_{g'} = \delta_{g,g'}.$$

Duality of Hopf Algebras

The Hopf algebras kG from example 1 and k^G from example 2 are *dual* from each other:

$$\begin{array}{ccc} kG \times k^G & \longrightarrow & k \\ (g_1, \delta_{g_2}) & \longmapsto & \delta_{g_2}(g_1) \end{array}$$

The basis G of kG is dual to the basis $(\delta_g)_{g \in G}$ of k^G.

Example 3.

Let G be a topological compact group over $\mathbf{C}$. Denote by $\mathfrak{R}(G)$ the set of continuous functions $f : G \to \mathbf{C}$ such that the translates $f_t : x \mapsto f(tx)$, for $t \in G$, span a finite dimensional vector space.

Define a coproduct Δ, a counit ϵ and an antipode S on $\mathfrak{R}(G)$ by

$$\Delta f(x, y) = f(xy), \quad \epsilon(f) = f(1), \quad Sf(x) = f(x^{-1})$$

for x, $y \in G$.

Hence $\mathfrak{R}(G)$ is a commutative Hopf algebra.

Example 4.

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{U}(\mathfrak{g})$ its universal envelopping algebra, namely $\mathfrak{T}(\mathfrak{g})/\mathfrak{I}$ where $\mathfrak{T}(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ and $\mathfrak{I}$ the two sided ideal generated by $XY - YX - [X, Y]$.

Define a coproduct Δ, a counit ϵ and an antipode S on $\mathfrak{U}(\mathfrak{g})$ by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -x$$

for $x \in \mathfrak{g}$.

Hence $\mathfrak{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.

The set of primitive elements is $\mathfrak{g}$: one recovers $\mathfrak{g}$ from $\mathfrak{U}(\mathfrak{g})$.

Duality of Hopf Algebras *(again)*

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then the two Hopf algebras $\mathfrak{R}(G)$ and $\mathfrak{U}(\mathfrak{g})$ are dual from each other.

Abelian Hopf algebras of finite type

1.

$$H = k[X], \;\; \Delta(X) = X \otimes 1 + 1 \otimes X, \;\; \epsilon(X) = 0, \;\; S(X) = -X.$$

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$$k[X] \otimes k[X] \simeq k[T_1, T_2], \quad X \otimes 1 \mapsto T_1, \quad 1 \otimes X \mapsto T_2$$

$$\Delta P(X) = P(T_1 + T_2), \quad \epsilon P(X) = P(0), \quad SP(X) = P(-X).$$

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$$\mathbf{G}_a(K) = \mathrm{Hom}_k(k[X], K), \quad k[\mathbf{G}_a] = k[X]$$

$k[\mathbf{G}_a]$ is an abelian Hopf algebra of finite type.

Abelian Hopf algebras of finite type

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$$H = k[Y, Y^{-1}], \quad \Delta(Y) = Y \otimes Y, \quad \epsilon(Y) = 1, \quad S(Y) = Y^{-1}.$$

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$$\mathbf{G}_m(K) = \mathrm{Hom}_k(k[Y, Y^{-1}], K), \quad k[\mathbf{G}_m] = k[Y, Y^{-1}],$$

$k[\mathbf{G}_m]$ is an abelian Hopf algebra of finite type.

Abelian Hopf algebras of finite type

3.

$$H = k[X_1, \ldots, X_{d_0}, Y_1, Y_1^{-1}, \ldots, Y_{d_1}, Y_{d_1}^{-1}]$$

$$\simeq k[X]^{\otimes d_0} \otimes k[Y, Y^{-1}]^{\otimes d_1}$$

Primitive elements: k-space $kX_1 + \cdots + kX_{d_0}$,
dimension d_0.

Group-like elements: multiplicative group $\langle Y_1, \ldots, Y_{d_1} \rangle$,
rank d_1.

$$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_a^{d_1}$$

$$k[G] = H, \quad G(K) = \mathrm{Hom}_k(H, K).$$

Abelian Hopf algebras of finite type

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$$\simeq k[X]^{\otimes d_0} \otimes k[Y, Y^{-1}]^{\otimes d_1}$$

The category of commutative linear algebraic groups over k $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}$ *is anti-equivalent to the category of Hopf algebras of finite type which are abelian (commutative and cocomutative)*

$$H = k[G].$$

The vector space of primitive elements has dimension d_0 while the rank of the group-like elements is d_1.

Other examples

If W is a k-vector space of dimension ℓ_0, $\mathrm{Sym}(W)$ is an abelian Hopf algebra of finite type, anti-isomorphic to $k[\mathbf{G}_a^{\ell_0}]$:

For a basis $\partial_1, \ldots, \partial_{\ell_0}$ of W, $\mathrm{Sym}(W) \simeq k[\partial_1, \ldots, \partial_{\ell_0}]$.

Other examples

If W is a k-vector space of dimension ℓ_0, $\mathrm{Sym}(W)$ is an abelian Hopf algebra of finite type, anti-isomorphic to $k[\mathbf{G}_a^{\ell_0}]$.

If Γ is a torsion free finitely generated $\mathbf{Z}$-module of rank ℓ_1, then the group algebra $k\Gamma$ is again an abelian Hopf algebra of finite type, anti-isomorphic to $k[\mathbf{G}_m^{\ell_1}]$:

For a basis $\gamma_1, \ldots, \gamma_{\ell_1}$ of Γ, $k\Gamma \simeq k[\gamma_1, \gamma_1^{-1}, \ldots, \gamma_{\ell_1}, \gamma_{\ell_1}^{-1}]$.

Other examples

If W is a k-vector space of dimension ℓ_0, $\mathrm{Sym}(W)$ is an abelian Hopf algebra of finite type, anti-isomorphic to $k[\mathbf{G}_a^{\ell_0}]$.

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The category of abelian Hopf algebras of finite type is equivalent to the category of pairs (W, Γ) *where* W *is a* k*-vector space and* Γ *is a finitely generated* $\mathbf{Z}$*-module:*

$$H = \mathrm{Sym}(W) \otimes k\Gamma.$$

Interpretation of the duality in terms of Hopf algebras

following Stéphane Fischler

Let $\mathfrak{C}_1$ be the category with

objects: (G, W, Γ) where $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}$, $W \subset T_e(G)$ is rational over $\overline{\mathbf{Q}}$ and $\Gamma \in G(\overline{\mathbf{Q}})$ is finitely generated

morphisms: $f : (G_1, W_1, \Gamma_1) \rightarrow (G_2, W_2, \Gamma_2)$ where $f : G_1 \rightarrow G_2$ is a morphism of algebraic groups such that $f(\Gamma_1) \subset \Gamma_2$ and f induces a morphism

$$df : T_e(G_1) \longrightarrow T_e(G_2)$$

such that $df(W_1) \subset W_2$.

Let H be an abelian Hopf algebra over $\overline{\mathbf{Q}}$ of finite type. Denote by d_0 the dimension of the $\overline{\mathbf{Q}}$-vector space of primitive elements and by d_1 the rank of the group of group-like elements.

Let H' be another abelian Hopf algebra over $\overline{\mathbf{Q}}$ of finite type, ℓ_0 the dimension of the space of primitive elements and ℓ_1 the rank of the group-like elements.

Let $\langle\cdot\rangle : H \times H' \longrightarrow \overline{\mathbf{Q}}$ be a bilinear product such that

$$\langle x, yy'\rangle = \langle \Delta x, y \otimes y'\rangle \quad \text{and} \quad \langle xx', y\rangle = \langle x \otimes x', \Delta y\rangle.$$

Let $\mathfrak{C}_2$ be the category with

objects: $(H, H', \langle\cdot\rangle)$ pair of Hopf algebras with a bilinear product as above.

morphisms: $(f, g) : (H_1, H_1', \langle\cdot\rangle_1) \longrightarrow (H_2, H_2', \langle\cdot\rangle_2)$ where $f : H_1 \longrightarrow H_2$ and $g : H_2' \longrightarrow H_1'$ are Hopf algebras morphisms such that

$$\langle x_1, g(x_2')\rangle_1 = \langle f(x_1), x_2'\rangle_2.$$

Stéphane Fischler: *The categories* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *are equivalent. Further, Fourier-Borel duality amounts to permute* H *and* H'*.*

Consequence: interpolation lemmas are equivalent to zero estimates.

Stéphane Fischler: *The categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are equivalent.*

For $R \in \mathbf{C}[G]$, $\partial_1, \ldots, \partial_k \in W$ and $\gamma \in \Gamma$, set

$$\langle R, \gamma \otimes \partial_1 \cdot \ldots \cdot \partial_k \rangle = \partial_1 \cdot \ldots \cdot \partial_k R(\gamma).$$

Conversely, for $H_1 = \mathbf{C}[G]$ and $H_2 = \mathrm{Sym}(W) \otimes k\Gamma$, consider

$$\begin{array}{ccc} \Gamma & \longrightarrow & G(\mathbf{C}) \\ \gamma & \longmapsto & \big(R \mapsto \langle R, \gamma \rangle\big) \end{array}$$

and

$$\begin{array}{ccc} W & \longrightarrow & T_e(G) \\ \partial & \longmapsto & \big(R \mapsto \langle R, \partial \rangle\big) \end{array}$$

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Open Problems:

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- Extend to non abelian Hopf algebras (of finite type to start with)
- (?) Transcendence results on non commutative algebraic groups

Algebra of multizeta values

Denote by $\mathfrak{S}$ the set of sequences $\underline{s} = (s_1, \ldots, s_k) \in \mathbf{N}^k$ with $k \geq 1$, $s_1 \geq 2$, $s_i \geq 1$ $(2 \leq i \leq k)$.

The *weight* $|\underline{s}|$ of $\underline{s}$ is $s_1 + \cdots + s_k$, while k is the *depth.*

For $\underline{s} \in \mathfrak{S}$ set

$$\zeta(\underline{s}) = \sum_{n_1 > \cdots > n_k \geq 1} n_1^{-s_1} \cdots n_k^{-s_k}.$$

Depth 1: values of Riemann zeta function at positive integers (Euler).

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Let $\mathfrak{Z}$ denote the $\mathbf{Q}$-vector space spanned in $\mathbf{C}$ by the numbers

$$(2i\pi)^{-|s|}\zeta(\underline{s}) \quad (\underline{s} \in \mathfrak{S}).$$

For $\underline{s}$ and $\underline{s}'$ in $\mathfrak{S}$, the product $\zeta(\underline{s})\zeta(\underline{s}')$ is *in two ways* a linear combination of numbers $\zeta(\underline{s}'')$.

The product of series is one way (*quadratic relations arising from the series expansions*) - for instance:

$$\zeta(s)\zeta(s') = \zeta(s, s') + \zeta(s', s) + \zeta(s + s').$$

$$\sum_{n}\sum_{m} = \sum_{n>m} + \sum_{m>n} + \sum_{n=m}$$

Hence $\mathfrak{Z}$ is a $\mathbf{Q}$-subalgebra of $\mathbf{C}$ bifiltered by the weight and by the depth.

For a graded Lie algebra $C_\bullet$ denote by $\mathfrak{U}C_\bullet$ its universal envelopping algebra and by

$$\mathfrak{U}C_\bullet^\vee = \bigoplus_{n\geq 0}(\mathfrak{U}C)_n^\vee$$

its graded dual, which is a commutative Hopf algebra.

Conjecture (Goncharov). *There exists a free graded Lie algebra $C_\bullet$ and an isomorphism of algebras*

$$\mathfrak{Z} \simeq \mathfrak{U}C_\bullet^\vee$$

filtered by the weight on the left and by the degree on the right.

Reference:

Goncharov A.B. – Multiple polylogarithms, cyclotomy and modular complexes. *Math. Research Letter* **5** (1998), 497–516.

Let $X = \{x_0, x_1\}$ be an alphabet with two letters. Consider the free monoid (concatenation product)

$$X^* = \{x_{\epsilon_1} \cdots x_{\epsilon_k} \,;\, \epsilon_i \in \{0,1\},\ (1 \leq i \leq k),\ \ k \geq 0\}$$

on X (non commutative monomials $=$ words on the alphabet X) including the empty word e.

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For $\underline{s} \in \mathfrak{S}$ set

$$y_{\underline{s}} = x_0^{s_1-1} x_1 \cdots x_0^{s_k-1} x_1.$$

Hence

$$y_s = x_0^{s-1} x_1, \quad y_{\underline{s}} = y_{s_1} \cdots y_{s_k}.$$

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For $\underline{s} \in \mathfrak{S}$ set

$$y_{\underline{s}} = x_0^{s_1 - 1} x_1 \cdots x_0^{s_k - 1} x_1.$$

This is a one-to-one correspondance between $\mathfrak{S}$ and the set $x_0 X^* x_1$ of words which start with x_0 and end with x_1.

The depth k is the number of x_1, the weight $|\underline{s}|$ is the number of letters.

For $\underline{s} = (s_1, \ldots, s_k) \in \mathfrak{S}$ set $p = |\underline{s}|$ and define $(\epsilon_1, \ldots, \epsilon_p)$ in $\{0,1\}^p$ by

$$y_{\underline{s}} = x_{\epsilon_1} \cdots x_{\epsilon_p}.$$

Hence $\epsilon_1 = 0$ and $\epsilon_p = 1$.

For $\underline{s} = (s_1, \dots, s_k) \in \mathfrak{S}$ set $p = |\underline{s}|$ and define $(\epsilon_1, \dots, \epsilon_p)$ in $\{0,1\}^p$ by

$$y_{\underline{s}} = x_{\epsilon_1} \cdots x_{\epsilon_p}.$$

Let

$$\omega_0(t) = \frac{dt}{t} \quad \text{and} \quad \omega_1(t) = \frac{dt}{1-t}.$$

Integral representation of multizeta values:

$$\zeta(\underline{s}) = \int_{1>t_1>\cdots>t_p>0} \omega_{\epsilon_1}(t_1) \cdots \omega_{\epsilon_p}(t_p).$$

Examples:

$y_3 = x_0^2 x_1$:

$$\zeta(3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_3}{1 - t_3}.$$

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$y_{21} = y_2 y_1 = x_0 x_1^2$:

$$\zeta(2,1) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1 - t_2} \int_0^{t_2} \frac{dt_3}{1 - t_3}.$$

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Remark. From $(t_1, t_2, t_3) \mapsto (1-t_3, 1-t_2, 1-t_1)$ one deduces

$$\zeta(2,1) = \zeta(3).$$

Quadratic relations arising from the integral representation

The product of two such integrals is a linear combination of similar integrals.

Indeed the integral

$$\int_{\substack{1>t_1>\cdots>t_p>0\\ 1>t'_1>\cdots>t'_{p'}>0}} \omega_{\epsilon_1}(t_1)\cdots\omega_{\epsilon_p}(t_p)\omega_{\epsilon'_1}(t'_1)\cdots\omega_{\epsilon_{p'}}(t'_{p'})$$

is the integral over

$$1 > u_1 > \cdots > u_{p+p'} > 0$$

of the *shuffle* of $\omega_{\epsilon_1}\cdots\omega_{\epsilon_p}$ with $\omega_{\epsilon'_1}\cdots\omega_{\epsilon'_{p'}}$.

Example: From

$$(x_0x_1) ш (x_0x_1) = 2x_0x_1x_0x_1 + 4x_0^2x_1^2$$

one deduces

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1).$$

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one deduces

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1).$$

For $y_{\underline{s}} \in x_0X^*x_1$, set $\hat{\zeta}(y_{\underline{s}}) = \zeta(\underline{s})$.

This defines $\hat{\zeta} : x_0X^*x_1 \to \mathbf{R}$.

Then for w and w' in $x_0X^*x_1$ we have

$$\boxed{\hat{\zeta}(w)\hat{\zeta}(w') = \hat{\zeta}(w ш w')}$$

Let $\mathfrak{H}$ be the free algebra $\overline{\mathbf{Q}}\langle X\rangle$ over X (non commutative polynomials in x_0 and x_1, $\overline{\mathbf{Q}}$-vector space with basis X^*, concatenation product).

The subalgebra $\mathfrak{H}^0 = \overline{\mathbf{Q}}e + x_0\mathfrak{H}x_1$ of $\mathfrak{H}$ is also the free algebra $\overline{\mathbf{Q}}\langle Y\rangle$ over $Y = \{y_2, y_3, \ldots\}$.

The shuffle ш endows both $\mathfrak{H}$ and $\mathfrak{H}^0$ with commutative algebra structures $\mathfrak{H}_ш$ and $\mathfrak{H}^0_ш$: for x and y in X, u and v in X^*,

$$xu ш yv = x(u ш yv) + v(xu ш v).$$

Extend $\hat{\zeta}$ as a $\overline{\mathbf{Q}}$-linear map $\mathfrak{H}^0 \to \mathbf{R}$.

Then $\hat{\zeta} : \mathfrak{H}^0_ш \to \mathbf{R}$ is a morphism of commutative algebras.

A non commutative but cocommutative Hopf algebra structure on $\mathfrak{H}$ is given by the coproduct

$$\Delta P = P(x_0 \otimes 1 + 1 \otimes x_0, x_1 \otimes 1 + 1 \otimes x_1)$$

the counit $\epsilon(P) = \langle P \mid e \rangle$ and the antipode

$$S(x_1 \cdots x_n) = (-1)^n x_n \cdots x_1$$

for $n \geq 1$ and $x_1, \ldots, x_n$ in X.

Concatenation (or Decomposition) Hopf algebra:

$$(\mathfrak{H}, \cdot, e, \Delta, \epsilon, S)$$

Writing

$$P = \sum_{u \in X^*} (P|u)u$$

we have

$$\Delta P = \sum_{u,v \in X^*} (P|u ш v) u \otimes v.$$

Friedrichs' Criterion. *The set of primitive elements in* $\mathfrak{H}$ *is the free Lie algebra* $\mathrm{Lie}(X)$ *on* X*.*

Hence

$$P \in \mathrm{Lie}(X) \Longleftrightarrow (P|u ш v) = 0 \quad \text{for all } u,\ v \text{ in } X^* \setminus \{e\}.$$

Dual of the concaténation product: $\Phi : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ defined by

$$\langle \Phi(w) \mid u \otimes v \rangle = \langle uv \mid w \rangle.$$

Hence

$$\Phi(w) = \sum_{\substack{u,v \in X^* \\ uv = w}} u \otimes v.$$

Shuffle (or factorization) Hopf algebra:

$$(\mathfrak{H}, ш, e, \Phi, \epsilon, S).$$

Commutative, not cocommutative.

Harmonic Algebra *M. Hoffmann*

The *quasi-shuffle* (or *stuffle*) product:

$$\star : \mathfrak{H}^0 \times \mathfrak{H}^0 \longrightarrow \mathfrak{H}^0$$

with

$$y_s u \star y_t v = y_s(u \star y_t v) + y_t(y_s u \star y_t v) + y_{s+t}(u \star v).$$

endows $\mathfrak{H}^0$ with a commutative algebra structure $\mathfrak{H}^0_\star$ and the quadratic relations arising from series expansions show that

$\hat{\zeta} : \mathfrak{H}^0_\star \rightarrow \mathbf{R}$ **is a morphism of commutative algebras.**

Cocommutative *quasi-shuffle Hopf algebra* $\overline{\mathbf{Q}}\langle Y\rangle$*:*

$$\Delta(y_i) = y_i \otimes e + e \otimes y_i,$$

$$\epsilon(P) = \langle P \mid e\rangle,$$
$$S(y_{s_1} \cdots y_{s_k}) = (-1)^k y_{s_k} \cdots y_{s_1}.$$

Remark. Combining both quadratic relations yields

$$\hat{\zeta}(w ш w' - w \star w') = 0 \quad \text{for } w \text{ and } w' \text{ in } x_0 X^* x_1.$$

From

$$y_1 ш y_2 = x_1 ш x_0 x_1 = x_1 x_0 x_1 + 2 x_0 x_1^2 = y_1 y_2 + 2 y_2 y_1$$

and

$$y_1 \star y_2 = y_1 y_2 + y_2 y_1 + y_3$$

one deduces

$$y_1 ш y_2 - y_1 \star y_2 = y_2 y_1 - y_3.$$

As we know $\zeta(2,1) = \zeta(3)$, hence $y_1 ш y_2 - y_1 \star y_2$ lies in the kernel of $\hat{\zeta}$. But $y_1 \notin x_0 X^* x_1$.

Conjecture *(Ihara-Kaneko). The kernel of* $\hat{\zeta}$ *as a* $\mathbf{Q}$*-linear map* $\mathfrak{H}^0 \to \mathbf{R}$ *is spanned by the* regularized double shuffle relations.

Results on the formal algebra by J. Écalle.