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Schanuel's Conjecture:
algebraic independence
of transcendental numbers

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Abstract

Schanuel's conjecture asserts that for linearly independent complex numbers $x_1, ..., x_n$, there are at least n algebraically independent numbers among the $2n$ numbers

$$x_1, \ldots, x_n, \; \exp(x_1), \ldots, \exp(x_n).$$

This simple statement has many remarkable consequences ; we will explain some of them. We will also present the state of the art on this topic.

Note : We write $\exp z$ for e^z.

Numbers : rational, algebraic, transcendental

Goal : given a mathematical constant, decide whether it is a rational, an irrational algebraic number, or else a transcendental number.

Rational integers : $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

Rational numbers :

$$\mathbf{Q} = \{p/q \;\mid\; p \in \mathbf{Z}, q \in \mathbf{Z}, q > 0, \; \gcd(p,q) = 1\}.$$

Algebraic numbers : roots of polynomials with rational coefficients, like p/q (root of $qX - p$), or $\sqrt{2}$ (root of $X^2 - 2$), or i (root of $X^2 + 1$), or $e^{2i\pi/163}$ (root of $X^{163} - 1$).

Transcendental number : a complex number which is not algebraic.

Partition of the set of complex numbers

Rational
Algebraic irrational
Transcendental

The difficulty of the problem lies in the fact that the *constants arising from analysis* are most often given as limits of sequences, sums of series, integrals, infinite products...
We are mainly interested here with numbers related with the exponential function, like

$$e, \quad \pi, \quad \log 2, \quad e^{\sqrt{2}}, \quad e^{\pi}, \quad 2^{\sqrt{2}} \ldots$$

Recall that for complex numbers a and b with $a \neq 0$,

$$a^b = \exp\{b \log a\}$$

when a choice for $\log a$ has been selected.

Known and unknown results

We know the transcendence of numbers like

$$e, \quad \pi, \quad \log 2, \quad e^{\sqrt{2}}, \quad e^{\pi}, \quad 2^{\sqrt{2}} \dots$$

For each of the following numbers

$$e+\pi,\ e\pi,\ \pi^e,\ e^e, e^{e^2}, \dots,\ e^{e^e}, \dots,\ \pi^\pi, \pi^{\pi^2}, \dots\ \pi^{\pi^\pi} \dots$$

$$\log\pi,\ \log(\log 2),\ \pi\log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3} \dots$$

we expect that it is a transcendental number, but we do not know even whether it is an irrational number.

Linear independence over $\mathbf{Q}$

Given complex numbers, we may ask whether they are linearly independent over $\mathbf{Q}$.

For instance given a number x, the linear independence of $1, x$ over $\mathbf{Q}$ is equivalent to the irrationality of x.

As an example, the numbers

$$\log 2, \quad \log 3, \quad \log 5, \dots \quad \log p, \dots$$

are linearly independent over $\mathbf{Q}$: for $b_i \in \mathbf{Z}$,

$$b_1 \log p_1 + \cdots + b_n \log p_n = 0 \quad \Longrightarrow b_1 = \cdots = b_n = 0.$$

$$p_1^{b_1} \cdots p_n^{b_n} = 1 \quad \Longrightarrow b_1 = \cdots = b_n = 0.$$

Linear independence over $\overline{\mathbf{Q}}$

The set of algebraic numbers is a subfield of $\mathbf{C}$ (sums and products of algebraic numbers are algebraic).

Given complex numbers, we may ask whether they are linearly independent over the field $\overline{\mathbf{Q}}$ of algebraic numbers.

For instance, given a number x, the linear independence of $1, x$ over $\overline{\mathbf{Q}}$ is equivalent to the transcendence of x.

It has been proved by A. Baker in 1968 that the numbers

$$1, \quad \log 2, \quad \log 3, \quad \log 5, \dots \quad \log p, \dots$$

are linearly independent over $\overline{\mathbf{Q}}$: for $\beta_i \in \overline{\mathbf{Q}}$,

$$\beta_0 + \beta_1 \log p_1 + \cdots + \beta_n \log p_n = 0 \quad \Longrightarrow \beta_0 = \cdots = \beta_n = 0.$$

Algebraic independence

Given complex numbers $x_1, \dots, x_n$, we may ask whether they are algebraically independent over $\mathbf{Q}$: this means that there is no nonzero polynomial $P \in \mathbf{Q}[x_1, \dots, x_n]$ such that $P(x_1, \dots, x_n) = 0$.

One can prove that this is equivalent to saying that $x_1, \dots, x_n$ are algebraically independent over $\overline{\mathbf{Q}}$: if a nonzero polynomial $Q \in \overline{\mathbf{Q}}[x_1, \dots, x_n]$ satisfies $Q(x_1, \dots, x_n) = 0$, then by taking for P the product of the "conjugates" of Q over $\mathbf{Q}$ one gets a nonzero polynomial $P \in \mathbf{Q}[x_1, \dots, x_n]$ such that $P(x_1, \dots, x_n) = 0$.

If $x_1, \dots, x_n$ are algebraically independent, each of these numbers is transcendental.

Transcendence degree

The transcendence degree of a subfield K of $\mathbf{C}$ is the maximal number of elements in K which are algebraically independent over $\mathbf{Q}$ (or over $\overline{\mathbf{Q}}$, this is the same).

Given numbers $t_1, \ldots, t_m$, the maximal number of algebraic elements in the set $\{t_1, \ldots, t_m\}$ is the same as the transcendence degree of the field $\mathbf{Q}(t_1, \ldots, t_m)$.

The transcendence degree of the field $\mathbf{Q}(t_1, \ldots, t_m)$ is m if and only if $t_1, \ldots, t_m$ are algebraically independent.

For $m = 1$, the transcendence degree of the field $\mathbf{Q}(x)$ is 0 if x is algebraic, 1 if x is transcendental.

Charles Hermite and Ferdinand Lindemann

Charles Hermite
(1822 – 1901)

1873 :
Transcendence of e
$e = 2.718\,281\ldots$

Ferdinand von Lindemann
(1852 – 1939)

1882 :
Transcendence of π
$\pi = 3.141\,592\ldots$

Hermite–Lindemann Theorem

Charles Hermite
(1822 – 1901)

Ferdinand von Lindemann
(1852 – 1939)

For any non-zero complex number z, one at least of the two numbers z and e^z is transcendental.

Corollaries : Transcendence of $\log \alpha$ and of e^β for α and β non-zero algebraic complex numbers, provided $\log \alpha \neq 0$.

Lindemann–Weierstraß Theorem (1885)

Let $\beta_1, \ldots, \beta_n$ be algebraic numbers which are linearly independent over $\mathbf{Q}$. Then the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\mathbf{Q}$.

Ferdinand von Lindemann
(1852 – 1939)

Karl Weierstrass
(1815 - 1897)

Corollary of the Lindemann–Weierstraß Theorem

If $\beta_1, \ldots, \beta_n$ are algebraic, then the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent if and only if $\beta_1, \ldots, \beta_n$ are linearly independent over $\mathbf{Q}$.

Indeed, if $\beta_1, \ldots, \beta_n$ are linearly dependent over $\mathbf{Q}$, say $a_1\beta_1 + \cdots + a_n\beta_n = 0$ with $a_i \in \mathbf{Z}$, then the polynomial

$$\prod_{a_i>0} X_i^{a_i} - \prod_{a_i<0} X_i^{|a_i|}$$

vanishes at the point $(e^{\beta_1}, \ldots, e^{\beta_n})$.
The deep part of the Lindemann–Weierstraß Theorem is the converse.

Hilbert's problems

August 8, 1900

David Hilbert
(1862 - 1943)

Second International Congress of Mathematicians, Paris 1900

Twin primes

Goldbach's Conjecture

Riemann Hypothesis

Transcendence of e^π and $2^{\sqrt{2}}$

A.O. Gel'fond and Th. Schneider

Solution of *Hilbert's seventh Problem* (1934) : *Transcendence of α^β and $(\log\alpha_1)/(\log\alpha_2)$ for α, β, α_1 and α_2 algebraic.*

A.O. Gel'fond
(1906 - 1968)

Th. Schneider
(1911 - 1988)

Transcendence of α^β and $\log\alpha_1/\log\alpha_2$: examples

The following numbers are transcendental :

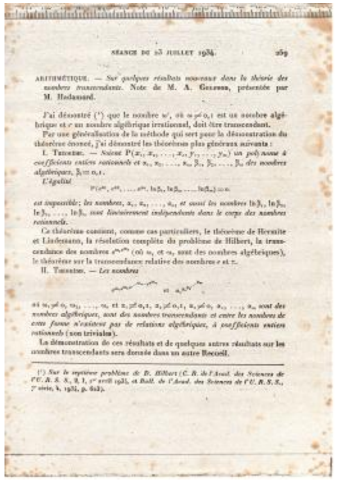
$$2^{\sqrt{2}} = 2.665\,144\,1\ldots$$

$$\frac{\log 2}{\log 3} = 0.630\,929\,\ldots$$

$$e^\pi = 23.140\,692\,\ldots \qquad (e^\pi = (-1)^{-i})$$

$$e^{\pi\sqrt{163}} = 262\;537\;412\;640\;768\;743.\;999\;999\;999\;999\;25\ldots$$

A.O. Gel'fond CRAS 1934



SÉANCE DU 23 JUILLET 1934. 269

ARITHMÉTIQUE. — *Sur quelques résultats nouveaux dans la théorie des nombres transcendants.* Note de M. A. Gelfond, présentée par M. Hadamard.

J'ai démontré (¹) que le nombre ω^r, où $\omega \neq 0, 1$ est un nombre algébrique et r un nombre algébrique irrationnel, doit être transcendant.

Par une généralisation de la méthode qui sert pour la démonstration du théorème énoncé, j'ai démontré les théorèmes plus généraux suivants :

I. Théorème. — *Soient* $P(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$ *un polynome à coefficients entiers rationnels et* $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ *des nombres algébriques,* $\beta_i \neq 0, 1$.

L'égalité

est impossible; les nombres, $\alpha_1, \alpha_2, \ldots, \alpha_n$, *et aussi les nombres* $\ln\beta_1, \ln\beta_2, \ldots, \ln\beta_m$ *sont linéairement indépendants dans le corps des nombres rationnels.*

Ce théorème contient, comme cas particuliers, le théorème de Hermite et Lindemann, la résolution complète du problème de Hilbert, la transcendance des nombres $e^{\omega_1 e^{\omega_2}}$ (où ω_1 et ω_2 sont des nombres algébriques), le théorème sur la transcendance relative des nombres e et π.

II. Théorème. — *Les nombres*

où $\omega_1 \neq 0, \omega_2, \ldots, \omega_n$ et $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_3 \neq 0$, $\ldots, \alpha_m$ *sont des nombres algébriques, sont des nombres transcendants et entre les nombres de cette forme n'existent pas de relations algébriques, à coefficients entiers rationnels (non triviales).*

La démonstration de ces résultats et de quelques autres résultats sur les nombres transcendants sera donnée dans un autre Recueil.

(¹) *Sur le septième problème de D. Hilbert* (*C. R. de l'Acad. des Sciences de l'U. R. S. S.*, 2, 1, 1er avril 1934, et *Bull. de l'Acad. des Sciences de l'U. R. S. S.*, 7e série, 4, 1934, p. 623).

Statement by Gel'fond (1934)

Let $P(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$ *be a polynomial with rational integer coefficients and* $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ *algebraic numbers,* $\beta_i \neq 0, 1$.
The equality

$$P(e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}, \ln\beta_1, \ln\beta_2, \ldots, \ln\beta_m) = 0$$

is impossible ; the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, *as well as the numbers* $\ln\beta_1, \ln\beta_2, \ldots, \ln\beta_m$ *are linearly independent in the rational numbers field.*

Statement by Gel'fond (1934)

This theorem includes as special cases, the theorems of Hermite and Lindemann, the complete solution of Hilbert's problem, the transcendence of numbers $e^{\omega_1 e^{\omega_2}}$ (where ω_1 and ω_2 are algebraic numbers), the theorem on the relative transcendence of the numbers e and π.

Second statement by A.O. Gel'fond

The numbers

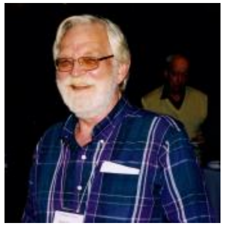
$$e^{\omega_1 e^{\omega_2 e^{\cdot^{\cdot^{\cdot^{\omega_{n-1} e^{\omega_n}}}}}}} \quad \textit{and} \quad \alpha_1^{\alpha_2^{\alpha_3^{\cdot^{\cdot^{\cdot^{\alpha_m}}}}}},$$

where $\omega_1 \neq 0, \omega_2, \ldots, \omega_n$ *and* $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_3 \neq 0$, $\ldots, \alpha_m$ *are algebraic numbers, are transcendental numbers, and among numbers of this form there is no nontrivial algebraic relations with rational integer coefficients.*

The proof of this result and a few other results on transcendental numbers will be given in another journal.

Remark by Mathilde Herblot : the condition on α_2 should be that it is irrational.

Schanuel's Conjecture



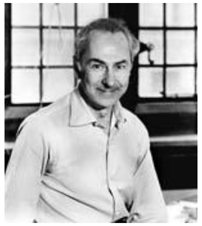
If $x_1, \ldots, x_n$ are $\mathbf{Q}$–linearly independent complex numbers, then n at least of the $2n$ numbers $x_1, \ldots, x_n$, $e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.

Equivalently :
If $x_1, \ldots, x_n$ are $\mathbf{Q}$-linearly independent complex numbers, then

$$\operatorname{tr}\deg_{\mathbf{Q}} \mathbf{Q}(x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n}) \geq n.$$

Origin of Schanuel's Conjecture

Course given by Serge Lang (1927–2005) at Columbia in the 60's



S. Lang – *Introduction to transcendental numbers*, Addison-Wesley 1966.

Formal analogs

W.D. Brownawell
(was a student of Schanuel)

J. Ax's Theorem (1968) :
Version of Schanuel's Conjecture for power series over $\mathbf{C}$
(and R. Coleman for power series over $\overline{\mathbf{Q}}$)
Work by W.D. Brownawell and K. Kubota on the elliptic analog of Ax's Theorem.

Dale Brownawell and Stephen Schanuel

Methods from logic

Ehud Hrushovski   Boris Zilber   Jonathan Kirby



"predimension" function (E. Hrushovski)

B. Zilber : "pseudoexponentiation"

Aussi : A. Macintyre, D.E. Marker, G. Terzo, A.J. Wilkie, D. Bertrand...

Daniel Bertrand

Daniel Bertrand,

Schanuel's conjecture for non-isoconstant elliptic curves over function fields.

Model theory with applications to algebra and analysis. Vol. 1, 41–62, London Math. Soc. Lecture Note Ser., **349**, Cambridge Univ. Press, Cambridge, 2008.

Lindemann–Weierstraß Theorem (1885)

According to the Lindemann–Weierstraß Theorem, Schanuel's Conjecture is true for algebraic $x_1, \dots, x_n$: in this case the transcendence degree of the field $\mathbf{Q}(x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n})$ is n.

Ferdinand von Lindemann (1852 – 1939)

Karl Weierstrass (1815 - 1897)

Transcendence degree $\leq n$

If we select $e^{x_1}, \dots, e^{x_s}$ to be algebraic (this means that the x_i's are logarithms of algebraic numbers) $x_{s+1}, \dots, x_n$ also to be algebraic, then the transcendence degree of the field

$$\mathbf{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$$

is the same as the transcendence degree of the field

$$\mathbf{Q}(x_1, \dots, x_s, e^{x_{s+1}}, \dots, e^{x_n})$$

hence is $\leq n$.

Baire and Lebesgue

René Baire
1874 – 1932

Henri Léon Lebesgue
1875 – 1941

The set of tuples $(x_1, \dots, x_n)$ in $\mathbf{C}^n$ such that the $2n$ numbers $x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}$ are algebraically independent
• is a G_δ set (countable intersection of dense open sets) in Baire's classification (a *generic set* for dynamical systems)
• and has full Lebesgue measure.

True for any transcendental function in place of the exponential function.

Mathematical genealogy

René Baire (1899)
|
Arnaud Denjoy (1909)
|
Charles Pisot (1938)
|
Yvette Amice (1965)
|
Jean Fresnel (1967)
|
Michel Waldschmidt (1972)

`http://genealogy.math.ndsu.nodak.edu`

Joint work with Senthil Kumar and Thangadurai

Given two integers m and n with $1 \le m \le n$, there exist uncountably many tuples $(x_1, \dots, x_n)$ in $\mathbf{R}^n$ such that $x_1, \dots, x_n$ and $e^{x_1}, \dots, e^{x_n}$ are all Liouville numbers and the transcendence degree of the field

$$\mathbf{Q}(x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n})$$

is $n + m$.

$m = 0$?

$1 \le m \le n$:

$$\operatorname{tr}\deg_{\mathbf{Q}} \mathbf{Q}(x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}) = n + m.$$

We do not know whether there are Liouville numbers x such that e^x is also a Liouville number and the two numbers x and e^x are algebraically dependent.

Schanuel's Conjecture for $n = 1$

For $n = 1$, Schanuel's Conjecture is the Hermite–Lindemann Theorem :

If x is a non–zero complex numbers, then one at least of the two numbers x, e^x is transcendental.

Equivalently, *if x is a non–zero algebraic number, then e^x is a transcendental number.*
Another equivalent statement is that *if α is a non–zero algebraic number and $\log \alpha$ any non–zero logarithm of α, then $\log \alpha$ is a transcendental number.*
Consequence : transcendence of numbers like

$$e, \quad \pi, \quad \log 2, \quad e^{\sqrt{2}}.$$

`Proof:` take

$$x = 1, \quad i\pi, \quad \log 2, \quad \sqrt{2}.$$

Schanuel's Conjecture for $n = 2$

For $n = 2$, Schanuel's Conjecture is not yet known :

? If x_1, x_2 are $\mathbf{Q}$–linearly independent complex numbers, then among the 4 numbers x_1, x_2, e^{x_1}, e^{x_2}, at least two are algebraically independent.

A few consequences (open problems) :
With $x_1 = 1$, $x_2 = i\pi$: algebraic independence of e and π.
With $x_1 = 1$, $x_2 = e$: algebraic independence of e and e^e.
With $x_1 = \log 2$, $x_2 = (\log 2)^2$: algebraic independence of $\log 2$ and $2^{\log 2}$.
With $x_1 = \log 2$, $x_2 = \log 3$: algebraic independence of $\log 2$ and $\log 3$.

Alan Baker 1968

Transcendence of numbers like :

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

or

$$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_1^{\beta_1}$$

for algebraic α_i and β_j.

Example (Siegel) :

$$\int_0^1 \frac{dx}{1 + x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right) = 0.835\,648\ldots$$

is transcendental.

Baker's linear independence Theorem

Let $\lambda_1, \ldots, \lambda_n$ be $\mathbf{Q}$–linearly independent logarithms of algebraic numbers. Then the numbers $1, \lambda_1, \ldots, \lambda_n$ are linearly independent over the field $\overline{\mathbf{Q}}$ of algebraic numbers.

Schanuel's Conjecture deals with algebraic independence (over $\mathbf{Q}$ or $\overline{\mathbf{Q}}$), Baker's Theorem deals with linear independence.
Baker's Theorem is a special case of Schanuel's Conjecture.

Serre's reformulation of Baker's Theorem

Denote by $\mathcal{L}$ the set of complex numbers λ for which e^{λ} is algebraic (set of logarithms of algebraic numbers). Hence $\mathcal{L}$ is a $\mathbf{Q}$-vector subspace of $\mathbf{C}$.

J-P. Serre
(Bourbaki seminar) :
the injection of $\mathcal{L}$ into $\mathbf{C}$ extends to a $\overline{\mathbf{Q}}$–linear map $\iota : \overline{\mathbf{Q}} + \mathcal{L} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} \to \mathbf{C}$, and Baker's Theorem means that ι is an injective map.

Algebraic independence

Towards Schanuel's Conjecture :

Ch. Hermite, F. Lindemann, C.L. Siegel, A.O. Gel'fond, Th. Schneider, A. Baker, S. Lang, W.D. Brownawell, D.W. Masser, D. Bertrand, G.V. Chudnovsky, P. Philippon, G. Wüstholz, Yu.V. Nesterenko, D. Roy…

Algebraic independence : A.O. Gel'fond 1948

The two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent.

More generally, if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Algebraic independence

G.V. Chudnovsky (1978)

The numbers π and $\Gamma(1/4) = 3.625\,609\,908\,2\ldots$ are algebraically independent.

Also π and $\Gamma(1/3) = 2.678\,938\,534\,7\ldots$ are algebraically independent.

On the number e^{π}

Yu.V.Nesterenko (1996)
Algebraic independence of $\Gamma(1/4)$, π and e^{π}.
Also : Algebraic independence of $\Gamma(1/3)$, π and $e^{\pi\sqrt{3}}$.

Corollary : *The numbers* $\pi = 3.141\,592\,653\,5\ldots$ *and* $e^{\pi} = 23.140\,692\,632\,7\ldots$ *are algebraically independent.*

The proof uses modular functions.

On the number e^{π}

Open problem : e^{π} is not a Liouville number :

$$\left| e^{\pi} - \frac{p}{q} \right| > \frac{1}{q^{\kappa}}.$$

Algebraic independence of π and e^{π} : Nesterenko

Chudnosvki : algebraic independence of π and $\Gamma(1/4)$

Nesterenko : Algebraic independence of π, $\Gamma(1/4)$ and e^{π}

Open problem : algebraic independence of π and e.

Expected : e, π and e^{π} are algebraic independent.

Easy consequence of Schanuel's Conjecture

According to Schanuel's Conjecture, the following numbers are algebraically independent :

$$e+\pi,\ e\pi,\ \pi^e,\ e^e, e^{e^2}, \ldots,\ e^{e^e}, \ldots,\ \pi^{\pi}, \pi^{\pi^2}, \ldots\ \pi^{\pi^{\pi}}\ldots$$

$$\log\pi,\ \log(\log 2),\ \pi\log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3}\ldots$$

Proof : Use Schanuel's Conjecture several times.

Conjecture of algebraic independence of logarithms of algebraic numbers

The most important special case of Schanuel's Conjecture is :

Conjecture. *Let* $\lambda_1, \ldots, \lambda_n$ *be* $\mathbf{Q}$*-linearly independent complex numbers. Assume that the numbers* $e^{\lambda_1}, \ldots, e^{\lambda_n}$ *are algebraic. Then the numbers* $\lambda_1, \ldots, \lambda_n$ *are algebraically independent over* $\mathbf{Q}$.

Not yet known that the transcendence degree is ≥ 2.

Reformulation by D. Roy

Instead of taking logarithms of algebraic numbers and looking for the algebraic independence relations, D. Roy fixes a polynomial and looks at the points, with coordinates logarithms of algebraic numbers, on the corresponding hypersurface.

Recall that $\mathcal{L}$ is the set of complex numbers λ for which e^λ is algebraic (logarithms of algebraic numbers).

The Conjecture on (homogeneous) algebraic independence of logarithms of algebraic numbers is equivalent to :

Conjecture (Roy). *For any algebraic subvariety V of $\mathbf{C}^n$ defined over the field $\overline{\mathbf{Q}}$ of algebraic numbers, the set $V \cap \mathcal{L}^n$ is the union of the sets $E \cap \mathcal{L}^n$, where E ranges over the set of vector subspaces of $\mathbf{C}^n$ which are contained in V.*

Points with coordinates logarithms of algebraic numbers

Damien Roy : Grassmanian varieties.

Stéphane Fischler : orbit of an affine algebraic group G over $\overline{\mathbf{Q}}$ related to a linear representation of G on a vector space with a $\overline{\mathbf{Q}}$–structure.

Quadratic relations among logarithms of algebraic numbers

One does not know yet how to prove that there is no nontrivial quadratic relations among logarithms of algebraic numbers, like

$$(\log \alpha_1)(\log \alpha_2) = \log \beta.$$

`Example`: Assume $e^{\pi^2} = \beta$ is algebraic. Then

$$(-i\pi)(i\pi) = \log \beta.$$

• **Open problem :** *is the number e^{π^2} transcendental ?*

e^{π^2}, e and π (1972)

W.D. Brownawell
(was a student of Schanuel)

One at least of the two following statements is true :
• the number e^{π^2} is transcendental
• the two numbers e and π are algebraically independent.

Schanuel's Conjecture implies that both statements are true !

Homogeneous quadratic relations among logarithms of algebraic numbers

Any homogeneous quadratic relation among logarithms of algebraic numbers

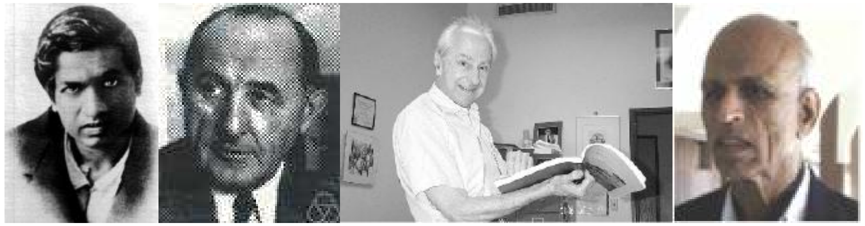
$$(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$$

should be trivial.

Example of a trivial relation : $(\log 2)(\log 9) = (\log 4)(\log 3)$.

The *Four Exponentials Conjecture* can be stated as : *any quadratic relation* $(\log \alpha_1)(\log \alpha_4) = (\log \alpha_2)(\log \alpha_3)$ *among logarithms of algebraic numbers is trivial : either* $\log \alpha_1 / \log \alpha_2$*is rational, or* $\log \alpha_1 / \log \alpha_3$ *is rational.*

S. Ramanujan, C.L. Siegel, S. Lang, K. Ramachandra



Ramanujan : Highly composite numbers.

Alaoglu and Erdős (1944), Siegel.

Four exponentials conjecture (special case)

Let t *be a positive real number. Assume* 2^t *and* 3^t *are both integers. Prove that* t *is an integer.*

Set $n = 2^t$. Then $t = (\log n)/(\log 2)$ and

$$3^t = e^{t \log 3} = e^{(\log n)(\log 3)/(\log 2)} = n^{(\log 3)/(\log 2)}.$$

Equivalently :
If n *is a positive integer such that*

$$n^{(\log 3)/(\log 2)}$$

is an integer, then n *is a power of* 2 *:*

$$2^{k(\log 3)/(\log 2)} = 3^k.$$

Further consequences of Schanuel's Conjecture

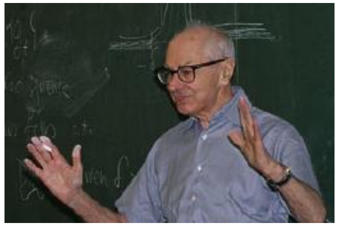
Ram Murty

Kumar Murty

N. Saradha

Purusottam Rath, Ram Murty, Sanoli Gun

The Rohrlich–Lang Conjecture



The Rohrlich–Lang Conjecture implies that for any $q > 1$, the transcendence degree of the field generated by numbers

$$\pi, \quad \Gamma(a/q) \quad 1 \leq a \leq q, \ (a, q) = 1$$

is $1 + \varphi(q)/2$.

Variant of the Rohrlich–Lang Conjecture

Conjecture of S. Gun, R. Murty, P. Rath (2009) : for any $q > 1$, the numbers

$$\log \Gamma(a/q) \quad 1 \leq a \leq q, \ (a, q) = 1$$

are linearly independent over the field $\overline{\mathbf{Q}}$ of algebraic numbers.

A consequence is that for any $q > 1$, there is at most one primitive odd character χ modulo q for which

$$L'(1, \chi) = 0.$$

Peter Bundschuh (1979)

.

For $p/q \in \mathbf{Q}$ with $0 < |p/q| < 1$, the sum of the series

$$\sum_{n=2}^{\infty} \zeta(n)(p/q)^n$$

is a transcendental number.

For $p/q \in \mathbf{Q} \setminus \mathbf{Z}$,

$$\frac{\Gamma'}{\Gamma}\left(\frac{p}{q}\right) + \gamma$$

is transcendental

Peter Bundschuh (1979)

(P. Bundschuh) : *As a consequence of Nesterenko's Theorem, the number*

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} = 2.076\,674\,047\,4\ldots$$

is transcendental, while

$$\sum_{n=0}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$$

(telescoping series).
Hence the number

$$\sum_{n=2}^{\infty} \frac{1}{n^s - 1}$$

is transcendental over $\mathbf{Q}$ for $s = 4$. The transcendence of this number for even integers $s \geq 4$ would follow as a consequence of Schanuel's Conjecture.

$\sum_{n\geq 1} A(n)/B(n)$

Arithmetic nature of
$$\sum_{n\geq 1} \frac{A(n)}{B(n)}$$
where
$$A/B \in \mathbf{Q}(X).$$
In case B has distinct zeroes, by decomposing A/B in simple fractions one gets linear combinations of logarithms of algebraic numbers (Baker's method).
The example $A(X)/B(X) = 1/X^3$ shows that the general case is hard :
$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3}.$$

S.D. Adhikari, N. Saradha, T.N. Shorey and R. Tijdeman (2001),




Catalan's Constant

Catalan's constant is
$$\sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2} = 0.915\,965\ \ldots$$
Is it an irrational number ?

Eugène Catalan
(1814 - 1894)

S. Gun, R. Murty, P. Rath

Assuming Schanuel's Conjecture, one at least of the two next statements is true :
(i) *The two numbers π and G are algebraically independent.*
(ii) *The number $\Gamma_2(1/4)/\Gamma_2(3/4)$ is transcendental.*
The multiple Gamma function of Barnes is defined by $\Gamma_0(z) = 1/z$, $\Gamma_1(z) = \Gamma(z)$,
$$\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)},$$
with $\Gamma_n(1) = 1$.

Damien Roy

Strategy suggested by D. Roy in 1999, Journées Arithmétiques, Roma : Conjecture equivalent to Schanuel's Conjecture.

Roy's approach to Schanuel's Conjecture (1999)

Let $\mathcal{D}$ denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

over the ring $\mathbf{C}[X_0, X_1]$. The *height* of a polynomial $P \in \mathbf{C}[X_0, X_1]$ is defined as the maximum of the absolute values of its coefficients.
Let k be a positive integer, $y_1, \dots, y_k$ complex numbers which are linearly independent over $\mathbf{Q}$, $\alpha_1, \dots, \alpha_k$ non-zero complex numbers and s_0, s_1, t_0, t_1, u positive real numbers satisfying

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$$

and

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Roy's Conjecture

Assume that, for any sufficiently large positive integer N, there exists a non-zero polynomial $P_N \in \mathbf{Z}[X_0, X_1]$ with partial degree $\le N^{t_0}$ in X_0, partial degree $\le N^{t_1}$ in X_1 and height $\le e^N$ which satisfies

$$\left| \left(\mathcal{D}^k P_N\right)\left(\sum_{j=1}^{k} m_j y_j, \prod_{j=1}^{k} \alpha_j^{m_j} \right) \right| \le \exp(-N^u)$$

for any non-negative integers $k, m_1, \dots, m_k$ with $k \le N^{s_0}$ and $\max\{m_1, \dots, m_k\} \le N^{s_1}$. Then

$$\operatorname{tr\,deg}\mathbf{Q}(y_1, \dots, y_k, \alpha_1, \dots, \alpha_k) \ge k.$$

Equivalence between Schanuel and Roy

Let $(y, \alpha) \in \mathbf{C} \times \mathbf{C}^\times$, and let s_0, s_1, t_0, t_1, u be positive real numbers satisfying the inequalities of Roy's Conjecture. *Then the following conditions are equivalent :*
(a) The number αe^{-y} is a root of unity.
(b) For any sufficiently large positive integer N, there exists a nonzero polynomial $Q_N \in \mathbf{Z}[X_0, X_1]$ with partial degree $\le N^{t_0}$ in X_0, partial degree $\le N^{t_1}$ in X_1 and height $\mathrm{H}(Q_N) \le e^N$ such that

$$\left|(\partial^k Q_N)(my, \alpha^m)\right| \le \exp(-N^u)$$

for any $k, m \in \mathbf{N}$ with $k \le N^{s_0}$ and $m \le N^{s_1}$.

Recent progress by D. Roy

$\mathbf{G}_a$, $\mathbf{G}_m$, $\mathbf{G}_a \times \mathbf{G}_m$.

Small value estimates for the additive group. Int. J. Number Theory **6** (2010), 919–956.

Small value estimates for the multiplicative group. Acta Arith. **135** (2008), 357–393.

A small value estimate for $\mathbf{G}_a \times \mathbf{G}_m$. Mathematika **59** (2013), 333–363

Recent developments

Roy's Conjecture deals with polynomials vanishing on some subsets of $\mathbf{C} \times \mathbf{C}^\times$ with multiplicity along the space associated with the derivation $\partial/\partial X + Y\partial/\partial Y$.
D. Roy conjecture depends on parameters s_0, s_1, t_0, t_1, u in a certain range. D. Roy proved that if his conjecture is true for one choice of values of these parameters in the given range, then Schanuel's Conjecture is true, and that conversely, if Schanuel's Conjecture is true, then his conjecture is true for all choices of parameters in the same range.

Nguyen Ngoc Ai Van
extended the range of these parameters.

Ubiquity of Schanuel's Conjecture

Other contexts : p–adic numbers, Leopoldt's Conjecture on the p–adic rank of the units of an algebraic number field
Non-vanishing of Regulators
Non–degenerescence of heights
Conjecture of B. Mazur on rational points
Diophantine approximation on tori

Dipendra Prasad

Gopal Prasad

Centro di Ricerca Matematica Ennio De Giorgi — Scuola Normale Superiore

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Schanuel's Conjecture:
algebraic independence
of transcendental numbers

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http://www.math.jussieu.fr/~miw/