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Strategies of transcendence proofs

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Abstract

The purpose of this course is to explain the ideas behind transcendence proofs, avoiding technical details. We start with the Schneider-Lang Theorem. A reference for this part of the course is

http://www.imj-prg.fr/~michel.waldschmidt/articles/pdf/Strategy.pdf

The two solutions (in 1934) by Gel'fond and Schneider to Hilbert's seventh problem are linked by a duality related with the Fourier-Borel transform. This leads to the replacement of auxiliary functions with auxiliary functionals. The next step is to introduce the interpolation determinants of Michel Laurent. Finally we will introduce a conjecture by D. Roy which he proved to be equivalent to Schanuel's Conjecture.

1 *The method of Gel'fond –Schneider*

- Hermite, Lindemann
- Gel'fond's method; Sketch of proof
- Schneider – Lang Theorem; Sketch of proof
- Meromorphic functions
- Schneider's Theorem on $j(\tau)$ (Gel'fond's method)
- Schneider's method
- Gel'fond vs Schneider. Duality
- Six Exponentials and Four Exponentials
- Baker's method

Hermite– Lindemann

Hermite 1873 transcendence of e

Lindemann 1882 transcendence of π

Theorem (Hermite– Lindemann).

Let α be a non–zero complex number. Then one at least of the two numbers α, e^{α} is transcendental.

For $\alpha = 1$, this is Hermite's result. For $\alpha = \mathrm{i}\pi$, this is Lindemann's result.

Also: the numbers $\log 2$, $\mathrm{e}^{\sqrt{2}}$ are transcendental.

Hermite– Lindemann: ideas of the proof

The function $f(z) = \mathrm{e}^z$ is entire, it satisfies the differential equation $f' = f$ and the functional equation $f(z_1 + z_2) = f(z_1)f(z_2)$;
in particular $f(mz) = f(z)^m$ for $m \in \mathbb{Z}$.

The two functions $f(z)$ and $g(z) = z$ are algebraically independent: the function f is transcendental over the field $\mathbb{C}(z)$ of rational functions.

If $\alpha \in \mathbb{C}$ is such that α and e^α are algebraic, then the two functions f and g take their values in the number field $K := \mathbb{Q}(\alpha, \mathrm{e}^\alpha)$ when $z = m\alpha$, with $m \in \mathbb{Z}$.

If $\alpha \neq 0$, then the set of $m\alpha$, with $m \in \mathbb{Z}$ is infinite.

Gel'fond's method 1

Assume $\alpha \neq 0$ and e^{α} are algebraic. We wish to get a contradiction with the algebraic independence of the two functions z, e^{z}, hence to prove the existence of a non zero polynomial $P \in \mathbb{Z}[X, Y]$ such that the function $F(z) := P(z, \mathrm{e}^{z})$ is the zero function.

If

$$P(X, Y) = \sum_{\tau=0}^{T_0-1} \sum_{t=0}^{T_1-1} a_{\tau,t} X^{\tau} Y^{t}$$

with $a_{\tau,t} \in \mathbb{Z}$, then

$$F(z) = P(z, \mathrm{e}^{z}) = \sum_{\tau=0}^{T_0-1} \sum_{t=0}^{T_1-1} a_{\tau,t} f_{\tau,t}(z) \text{ where } f_{\tau,t}(z) := z^{\tau} \mathrm{e}^{tz}.$$

Gel'fond's method 2

Notice that the numbers

$$f_{\tau,t}^{(\sigma)}(s\alpha) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\sigma} (z^{\tau}\mathrm{e}^{tz})(s\alpha)$$

($\tau \geqslant 0$, $t \geqslant 0$, $\sigma \geqslant 0$, $s \geqslant 0$) are all in the number field K.

We first request $P(X, Y)$ to satisfy finitely many conditions

$$F^{(\sigma)}(s\alpha) = 0 \quad \text{for} \quad 0 \leqslant \sigma < S_0,\ 0 \leqslant s < S_1.$$

The existence of such a $P \neq 0$ follows from linear algebra as soon as $T_0T_1 > \delta S_0S_1$ with $\delta = [K : \mathbb{Q}]$.

We will need an upper bound for the coefficients $a_{\tau,t}$ - such an estimate follows from Dirichlet's box principle and this estimate is sharp as soon as $T_0T_1 \geqslant 2\delta S_0S_1$. We refer to this argument as the Thue–Siegel Lemma.

Gel'fond's method 3

Recall that the goal is to prove $F = 0$. It suffices to prove

$$F^{(\sigma)}(s\alpha) = 0 \quad \text{for all} \quad \sigma \geqslant 0 \quad \text{and} \quad 0 \leqslant s < S_1.$$

Assume that this is true for $0 \leqslant \sigma < U$ with $U \geqslant S_0$. Let j satisfy $0 \leqslant j < S_1$. Consider the number $\gamma := F^{(U)}(j\alpha)$. From Cauchy's inequality we deduce

$$|\gamma| \leqslant U!|F|_r$$

for any $r \geqslant 1 + S_1|\alpha|$, where

$$|F|_r := \sup_{|z|=r} |F(z)|.$$

The idea is that, since F has many zeroes in the disc $|z| = r$, the number $|F|_r$ is small. More precisely we will use Schwarz's Lemma below.

Gel'fond's method 4

The last tool is due to Liouville, it is a lower bound for a non zero algebraic number.

The main such instance is the lower bound $|n| \geqslant 1$ for a nonzero rational number n.

In general, if γ is a nonzero algebraic number, multiplying by a denominator and taking the norm over $\mathbb{Q}$ yields a nonzero rational integer, hence of absolute value at least 1. This gives a lower bound for $|\gamma|$, which we call Liouville's estimate.

The initial example is the lower bound

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{c(\alpha)}{q^d}$$

when α is an algebraic number of degree d and p/q a rational number $\neq \alpha$, which enabled Liouville to give the first examples of transcendental numbers in 1844.

Schwarz's Lemma

Let f be an analytic function in a disc $|z| \leqslant R$ of $\mathbb{C}$, with at least N zeroes in a disc $|z| \leqslant r$ with $r < R$, counting multiplicities. Then

$$|f|_r \leqslant \left(\frac{3r}{R}\right)^N |f|_R.$$

When $R > 3r$, this improves the maximum modulus bound $|f|_r \leqslant |f|_R$.

Proof.

Let $\zeta_1, \ldots, \zeta_N$ be zeroes of f in the disc $|z| \leqslant r$. Then the function

$$g(z) := f(z) \prod_{j=1}^{N} (X - \zeta_j)^{-1}$$

is analytic in the disc $|z| \leqslant R$, hence satisfies $|g|_r \leqslant |g|_R$. $\square$

Schneider–Lang (simplified)

An entire function f is said to be of finite order if there exists $\varrho > 0$ and $R_0 > 0$ such that $|f|_R \leqslant \mathrm{e}^{R^\varrho}$ for all $R > R_0$. In other terms

$$\limsup_{R\to\infty} \frac{\log\log|F|_R}{\log R} < \infty.$$

Theorem (Schneider–Lang).

Let f_1, f_2 be two entire functions of finite order which are algebraically independent over $\mathbb{Q}$. Let K be a number field. Assume that the derivatives f_1' and f_2' belong to $K[f_1, f_2]$. Then the set

$$\{w \in \mathbb{C} \mid f_1(w) \text{ and } f_2(w) \text{ belong to } K\}$$

is finite.

Hilbert's seventh problem

Theorem (Gel'fond –Schneider 1934).

If β is an irrational algebraic number, then for any $\ell \in \mathbb{C} \setminus \{0\}$ one at least of the two numbers e^{ℓ}, $\mathrm{e}^{\beta\ell}$ is transcendental.

Proof.

Assume $\alpha := \mathrm{e}^{\ell}$. Write α^{β} for $\mathrm{e}^{\beta\ell}$. Assume α, β and α^{β} are algebraic. Let K be the number field $\mathbb{Q}(\alpha, \beta, \alpha^{\beta})$

The two functions $f_1(z) = \mathrm{e}^{z}$, $f_2(z) = \mathrm{e}^{\beta z}$ have order 1, are algebraically independent ($\beta \notin \mathbb{Q}$) and satisfy the differential equations $f_1' = f_1$, $f_2' = \beta f_2$ with coefficients in K. These two functions take their values in K for $z = m\ell$, which is an infinite set since $\ell \neq 0$. □

Hilbert's Seventh Problem

Theorem (Gel'fond –Schneider).

Let α be a nonzero algebraic number, $\log \alpha$ a nonzero logarithm of α and β an irrational algebraic number. Then $\alpha^\beta := \mathrm{e}^{\beta \log \alpha}$ is transcendental.

With $\alpha = -1$, $\log \alpha = \mathrm{i}\pi$ and $\beta = \mathrm{i}$ this gives the transcendence of e^π. Proved by A.O. Gel'fond in 1929 using an interpolation method introduced by Pólya to prove that 2^z is the entire function of least growth taking values in $\mathbb{Z}$ for $z \in \mathbb{N}$.

Also if α_1 and α_2 are nonzero algebraic numbers and $\log \alpha_1$, $\log \alpha_2$ nonzero logarithms of α_1 and α_2 respectively, then $\log \alpha_1 / \log \alpha_2$ is either rational or transcendental.

If two logarithms of algebraic numbers are linearly independent over $\mathbb{Q}$, they are linearly independent over the field of algebraic numbers.

Proof of Schneider–Lang Theorem - 1

Let K be a number field, $\delta := [K : \mathbb{Q}]$, $w_1, \ldots, w_m$ satisfy

$$f_i(w_j) \in K \text{ for } i = 1, 2 \text{ and } j = 1, \ldots, m,$$

ϱ_1, ϱ_2 and C satisfy

$$\log |f_i|_R \leqslant C R^{\varrho_i}$$

for $i = 1, 2$ and for all sufficiently large R.
We sketch the proof of the upper bound:

$$m \leqslant (\varrho_1 + \varrho_2)\delta.$$

Let S be a sufficiently large integer. We introduce the following parameters T_1 and T_2:

$$T_1 = S^{\frac{\varrho_2}{\varrho_1+\varrho_2}} \sqrt{\log S}, \quad T_2 = S^{\frac{\varrho_1}{\varrho_1+\varrho_2}} \sqrt{\log S}.$$

Proof of Schneider–Lang Theorem - 2

The first step is to prove the existence of a nonzero polynomial in $\mathbb{Z}[X_1, X_2]$, of degree $< T_1$ in X_1 and $< T_2$ in X_2, such that the function $F := P(f_1, f_2)$ satisfies the mS equations

$$F^{(\sigma)}(w_j) = 0$$

for $0 \leqslant \sigma < S$ and $1 \leqslant j \leqslant m$. Since

$$T_1 T_2 = S \log S,$$

the existence of P is a consequence of linear algebra as soon as $\log S > \delta m$. For sufficiently large S, the Thue–Siegel Lemma implies the existence of P with an upper bound for its height:

$$\log \mathrm{H}(P) = O(S).$$

Proof of Schneider–Lang Theorem - 3

Since the functions f_1 and f_2 are algebraically independent, the numbers $F^{(\sigma)}(w_j)$ for $1 \leqslant j \leqslant m$ and $\sigma \geqslant 0$ are not all zero. Let U be the least integer such that the function F satisfies $F^{(\sigma)}(w_j) = 0$ for $0 \leqslant \sigma < U$ and $1 \leqslant j \leqslant m$. From the first step we infer $U \geqslant S$. Since U is minimal, there exists j_0 which satisfies $1 \leqslant j_0 \leqslant m$ such that the number

$$\gamma := F^{(U)}(w_{j_0})$$

is not 0. Using Schwarz's Lemma with $R = U^{\frac{1}{\varrho_1+\varrho_2}}$, we obtain

$$\log|\gamma| \leqslant \left(1 - \frac{m}{\varrho_1+\varrho_2} + o(1)\right) U \log U.$$

Liouville's estimate for the nonzero element γ of K yields

$$\log|\gamma| \geqslant -(\delta - 1 + o(1))U \log U.$$

The upper bound for m follows.

Full version of Schneider–Lang Theorem

Theorem (Schneider–Lang).

Let $f_1, \ldots, f_m$ be meromorphic functions with $m \geqslant 2$. Assume f_1 and f_2 have finite order $\leqslant \varrho_1$ and ϱ_2 respectively, and are algebraically independent over $\mathbb{Q}$. Let K be a number field. Assume that for $1 \leqslant j \leqslant m$ the derivative f_j' of f_j belongs to $K[f_1, \ldots, f_m]$. Then the set

$$\{w \in \mathbb{C} \mid f_j(w) \in K \text{ for } j = 1, \ldots, m\}$$

has at most $(\varrho_1 + \varrho_2)[K : \mathbb{Q}]$ elements.

The main point is that this set is finite.

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Appendix 1: The transcendence of e and π.

Schneider 1949 Satz 1

Satz I. Es seien $f(z)$ bzw. $g(z)$ ganze Funktionen der Wachstumsordnungen μ_1 bzw. μ_2, d. h. es seien

$$\lim_{R\to\infty} \frac{\log\log \operatorname{Max}_{|z|\leqq R} |f(z)|}{\log R} = \mu_1, \quad \lim_{R\to\infty} \frac{\log\log \operatorname{Max}_{|z|\leqq R} |g(z)|}{\log R} = \mu_2.$$

$\zeta_1, \zeta_2, \ldots, \zeta_m, \ldots$ bezeichne eine unendliche Zahlenfolge. Mit $z_0^{\{m\}} = z_0$, $z_1^{\{m\}} = z_1, \ldots, z_k^{\{m\}} = z_k$ mögen die voneinander Verschiedenen unter den Zahlen $\zeta_1, \zeta_2, \ldots, \zeta_m$ benannt werden, und die Vielfachheit von $z_\varkappa$ in $\zeta_1, \zeta_2, \ldots, \zeta_m$ heiße $l_\varkappa^{\{m\}} + 1 = l_\varkappa + 1$. Dann ist $\sum_{\varkappa=0}^{k} (l_\varkappa + 1) = m$. Nun seien für jedes m die Funktionswerte $f(z_\varkappa)$, $g(z_\varkappa)$ $(\varkappa = 0, \ldots, k)$ und die Werte der Ableitungen von $f(z)$ und $g(z)$

$$\frac{d^\lambda f(z)}{dz^\lambda}\bigg|_{z=z_\varkappa} = f^{(\lambda)}(z_\varkappa), \quad \frac{d^\lambda g(z)}{dz^\lambda}\bigg|_{z=z_\varkappa} = g^{(\lambda)}(z_\varkappa) \qquad \begin{pmatrix}\lambda = 1, \ldots, l_\varkappa;\\ \varkappa = 0, \ldots, k\end{pmatrix}$$

ganzrationale Zahlen. Es werde $\operatorname{Max}_{(\varkappa=0,\ldots,k)} l_\varkappa$ mit l bezeichnet und $l \leqq \frac{m}{\log m}$ vorausgesetzt. r sei das Minimum der Längen der Radien aller Kreise um den Koordinatenursprung, die die Stellen $z_0, \ldots, z_k$ der komplexen Ebene enthalten. Mit der Bezeichnung

$$\alpha = \varliminf_{m\to\infty} \frac{\log m}{\log r}$$

lautet dann die Behauptung:

Ist $\mu_1 + \mu_2 < \alpha$, so sind $f(z)$ und $g(z)$ voneinander algebraisch abhängig.

Schneider 1949 Satz III

Satz III. Gegeben seien n ganze oder meromorphe Funktionen $f_1(z), \ldots, f_n(z)$. Die sämtlichen Ausdrücke

$$f_\nu^{(\lambda)}(z_\varkappa) \qquad (\lambda = 0, \ldots, l_\varkappa;\ \varkappa = 0, \ldots, k;\ \nu = 1, \ldots, n)$$

seien algebraische Zahlen aus K. $H_\nu(z_\varkappa)$ seien solche ganzrationale positive Zahlen, daß für alle $\lambda = 0, \ldots, l_\varkappa$ die Produkte $H_\nu(z_\varkappa) \cdot f_\nu^{(\lambda)}(z_\varkappa)$ $\binom{\varkappa = 0, \ldots, k}{\nu = 1, \ldots, n}$ ganze algebraische Zahlen in K sind. Es sei $l \leqq \frac{m}{\log m}$ und

$$\lim_{m \to \infty} \frac{\log m}{\log r} = \alpha. \tag{1}$$

Die Funktionen $f_\nu(z)$ $(\nu = 1, \ldots, n)$ sollen, sofern sie ganz sind, die Wachstumsordnungen μ_ν besitzen, und sofern sie meromorph sind, gebe es für jedes ν eine ganze Funktion $G_\nu(z)$ der Wachstumsordnung μ_ν derart, daß $G_\nu(z) f(z)$ ganz ist und ebenfalls die Wachstumsordnung μ_ν hat. Es werde gefordert

$$\frac{\mu_1 + \cdots + \mu_n}{n - 1} < \alpha. \tag{2}$$

Daraus folgt $\alpha > 0$, und nun nenne $\frac{\mu_\nu}{\alpha} = \eta_\nu$ $(\nu = 1, \ldots, n)$. Mit diesen η_ν sollen die folgenden Ungleichungen gelten:

$$\overline{\lim} \frac{\log\log \operatorname{Max}_{\varkappa = 0, \ldots, k} (|G_\nu(z_\varkappa)|^{-1})}{\log m} \leqq \eta_\nu \qquad (\nu = 1, \ldots, n) \tag{3}$$

und

$$\overline{\lim_{m \to \infty}} \frac{\log\log \operatorname{Max}_{\substack{\varkappa = 0, \ldots, k \\ \lambda = 0, \ldots, l_\varkappa}} \left(\left|f_\nu^{(\lambda)}(z_\varkappa)\right|, H_\nu(z_\varkappa)\right)}{\log m} \leqq \eta_\nu \qquad (\nu = 1, \ldots, n). \tag{4}$$

Dann, so wird behauptet, sind $f_1(z), \ldots, f_n(z)$ voneinander algebraisch abhängig.

Meromorphic functions

A meromorphic function in a domain D is a function which is analytic outside a discrete subset of D, the points of which are poles (no essential singularity).

It is a fact that a meromorphic function in $\mathbb{C}$ is a quotient of two entire function.

Definition. *A meromorphic function f has order $\leqslant \varrho$ if there exist two entire functions f_1 and f_2 of order $\leqslant \varrho$ such that $f = f_1/f_2$.*

Order of a meromorphic function

Definition. *A meromorphic function f has order $\leqslant \varrho$ if there exist two entire functions f_1 and f_2 of order $\leqslant \varrho$ such that $f = f_1/f_2$.*

Height of a rational fraction.

The height $H(P)$ of a polynomial $P \in \mathbb{C}[X]$ is the maximum of modulus of its coefficients.

"Definition". *A rational fraction $R \in \mathbb{C}(X)$ has height $\leqslant H$ if there exist two polynomials P_1 and P_2 of height $\leqslant H$ such that $R = P_1/P_2$.*

$$X^2 + 2X + 1 = \frac{(X-1)(X+1)^2}{X-1} = \frac{X^3 + X^2 - X - 1}{X-1}.$$

Lemma.

If an entire function f is a quotient of two entire functions of order $\leqslant \varrho$, then f has order $\leqslant \varrho$.

Weierstrass elliptic functions

A Weierstrass elliptic function $\wp$, solution of the differential equation $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, has order $\leqslant 2$ (quotient of two sigma functions).

Elliptic analog of Hermite–Lindemann Theorem:

Corollary (Schneider).

Assume g_2 and g_3 are algebraic. Let α be a nonzero algebraic number. Then α is not a pole of $\wp$ and $\wp(\alpha)$ is transcendental.

Proof.

Let $f_1(z) = z$, $f_2(z) = \wp(z)$, $f_3(z) = \wp'(z)$. Then $f_3'(z) = \wp''(z) = 6\wp^2 - g_2/2$:

$$f_1' = 1, \quad f_2' = f_3, \quad f_3' = 6f_2^2 - g_2/2.$$

Corollaries (Schneider)

Assume g_2 and g_3 are algebraic.

1. *(Elliptic analog of Hermite–Lindemann)*
   Let α be a nonzero algebraic number. Then α is not a pole of $\wp$ and $\wp(\alpha)$ is transcendental.
2. *Elliptic analog of Gel'fond–Schneider*
   Let α be a nonzero algebraic number and $\log \alpha$ a nonzero logarithm of α. Then $\log \alpha$ is not a pole of $\wp$ and $\wp(\log \alpha)$ is transcendental.
   $f_1(z) = \mathrm{e}^z$, $f_2(z) = \wp(z)$, $f_3(z) = \wp'(z)$.
3. Let $\wp^*$ be another Weierstrass elliptic function with g_2^* and g_3^* algebraic. Assume $\wp$ and $\wp^*$ are algebraically independent. Then for any $z \in \mathbb{C}$ not pole of $\wp$ nor of $\wp^*$ one at least of the two numbers $\wp(z)$, $\wp^*(z)$ is transcendental.

Consequences of Schneider's result on $\wp$ and $\wp^*$

Let $\wp$ be a Weierstras elliptic function with algebraic invariants g_2, g_3. Let ω_1, ω_2 be a pair of fundamental periods of $\wp$.

Assume that the quotient $\tau = \omega_2/\omega_1$ is algebraic.

Then the Weierstrass elliptic function $\wp^*(z) := \tau^2 \wp(\tau z)$ has algebraic invariants g_2^*, g_3^* and the two functions $\wp(z)$, $\wp^*(z)$ take values in a number field for $z \in (\omega_1/2) + \mathbb{Z}\omega_1$.

Hence, by Schneider's result, these two functions are algebraically dependent. It follows that $\wp$ has complex multiplication and that τ is quadratic.

The modular function $j(\tau)$

Let j be the modular function, analytic in the upper half plane $\mathfrak{H} := \{\tau \mid \Im\mathrm{m}(\tau) > 0\}$, satisfying

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

having a simple pole at infinity with residue 1.
For $\tau \in \mathfrak{H}$, τ and $j(\tau)$ are both algebraic if and only if τ is quadratic (τ is a special point*).*

Proof.

According to the theory of complex multiplication, if τ is quadratic then $j(\tau)$ is algebraic (Hilbert's class field).
Conversely, let $\Delta(\tau)$ be the discriminant of the elliptic curve of periods $\mathbb{Z} + \mathbb{Z}\tau$, let ω_1 be a 12–th root of $\Delta(\tau)$ and let $\omega_2 = \tau\omega_1$. If $j(\tau)$ is algebraic then the invariants g_2, g_3 of $\wp$ are algebraic. If τ is also algebraic, then Schneider's theorem implies that τ is imaginary quadratic. □

Schneider's method 1

Recall:

Theorem (Gel'fond –Schneider).

If β is an irrational algebraic number, then for any $\ell \in \mathbb{C} \setminus \{0\}$ one at least of the two numbers e^{ℓ}, $\mathrm{e}^{\beta\ell}$ is transcendental.

Let β be an irrational number and let $\ell \in \mathbb{C} \setminus \{0\}$. Assume β, e^{ℓ} and $\mathrm{e}^{\beta\ell}$ are algebraic. Let $\alpha := \mathrm{e}^{\ell}$, $\mathrm{e}^{\beta\ell} = \alpha^{\beta}$, $K = \mathbb{Q}(\alpha, \beta, \alpha^{\beta})$.

Gel'fond :

Uses the two functions $f_1(z) = \mathrm{e}^{z}$, $f_2(z) = \mathrm{e}^{\beta z}$ with derivatives evaluated at points in $\mathbb{Z}\ell$.

Schneider:

Uses the two functions z and $\mathrm{e}^{\ell z}$ which are algebraically independent and take algebraic values at points in $\mathbb{Z} + \mathbb{Z}\beta$.
No differential equation with algebraic coefficients.

Schneider's method 2

We argue by contradiction: let again K be the number field $\mathbb{Q}(\alpha, \beta, \alpha^\beta)$. Notice that the two functions z and $\alpha^z = \mathrm{e}^{z \log \alpha}$ are algebraically independent since $\log \alpha \neq 0$. The auxiliary function G is now a linear combination of the functions

$$g_{\sigma,s}(z) = z^\sigma \alpha^{sz},$$

say

$$G(z) = \sum_{\sigma=0}^{S_0-1} \sum_{s=0}^{S_1-1} b_{\sigma,s} g_{\sigma,s}(z)$$

with $b_{\sigma,s} \in \mathbb{Z}$.
For $\sigma \geqslant 0$, $s \geqslant 0$, $\tau \geqslant 0$, $t \geqslant 0$, we have

$$g_{\sigma,s}(\tau + t\beta) \in K.$$

Schneider's method 3

The first step is the existence of a nonzero polynomial

$$P(X,Y) = \sum_{\sigma=0}^{S_0-1} \sum_{s=0}^{S_1-1} b_{\sigma,s} X^\sigma Y^s \in \mathbb{Z}[X,Y]$$

such that the auxiliary function $G(z) = P(z, \alpha^z)$ satisfies

$$G(\tau + t\beta) = 0$$

for $0 \leqslant \tau < T_0$, $0 \leqslant t < T_1$. The numbers $\tau + t\beta$ are pairwise distinct since β is irrational. The Thue–Siegel Lemma shows the existence of such a polynomial with an upper bound for the height as soon as we request S_0, S_1, T_0, T_1 to satisfy

$$S_0 S_1 \geqslant 2[K : \mathbb{Q}] T_0 T_1.$$

Schneider's method 4

By induction on $T'_0 + T'_1$, one proves that the equations $G(\tau + t\beta) = 0$ are satisfied for $0 \leqslant \tau < T'_0$, $0 \leqslant t < T'_1$.

For proving that a number of the form $\gamma = G(\tau + t\beta)$ is zero, one uses the analytic estimate given by Schwarz's Lemma to show that $|\gamma|$ is small, thanks to the inductive assumptions. One concludes the inductive process by means of Liouville's inequality.

Therefore the equations $G(\tau + t\beta) = 0$ are satisfied for all $\tau \geqslant 0$, and $t \geqslant 0$.

The last step is to deduce that $G = 0$. There are several ways of achieving this goal, the easiest is to remark that in the inductive argument in the second step, we show that $|G|_r$ is small for arbitrarily large values of r, hence $G = 0$ by the maximum modulus principle.

Duality Gel'fond – Schneider

The methods of Gel'fond and Schneider follow the same sketch of proof. There is something more that they share in common: the numbers which occur in Gel'fond 's proof and in Schneider's proof are the same.

Gel'fond 's method:

$$f_{\tau,t}(z) = (\mathrm{e}^z)^\tau (\mathrm{e}^{\beta z})^t = \mathrm{e}^{(\tau+t\beta)z}, \quad z = s\log\alpha.$$

Schneider's method:

$$g_{\sigma,s}(z) = z^\sigma \alpha^{sz}, \quad z = \tau + t\beta.$$

We have

$$f^{(\sigma)}_{\tau,t}(s\log\alpha) = (\tau+t\beta)^\sigma \alpha^{\tau s}(\alpha^\beta)^{st} = g_{\sigma,s}(\tau+t\beta).$$

Connection with Fourier – Borel duality.

Schneider's problems

Wir nennen die folgenden Probleme:

1. Es seien α, β, γ algebraische Zahlen und $\alpha \neq 0, 1$ sowie $\frac{\log \beta}{\log \alpha}$ und $\frac{\log \gamma}{\log \alpha}$ irrational. Es ist $\delta = \exp\left(\frac{\log \beta \log \gamma}{\log \alpha}\right)$ auf Transzendenz zu untersuchen.

2. Der Satz über die Transzendenz der Werte der Modulfunktion $J(\tau)$ soll durch direkte Untersuchung der transzendenten Modulfunktion und nicht durch Untersuchung von $\wp$-Funktionen gezeigt werden.

3. Es ist zu versuchen, Transzendenzresultate über elliptische Integrale dritter Gattung zu beweisen.

4. Die Transzendenzsätze über elliptische Integrale erster und zweiter Gattung sind in weitestmöglichem Umfang auf analoge Sätze über ABELsche Integrale zu verallgemeinern.

5. Es ist die Approximation bekannter transzendenter Zahlen durch solche rationale oder auch algebraische Zahlen zu untersuchen, die arithmetischen Bedingungen genügen.

6. Die Aussagen über das Maß der S-Zahlen (Satz 24) sind in Richtung auf die MAHLERsche Vermutung, wonach die Menge der S-Zahlen vom Typus $\vartheta > 1$ ein LEBESGUEsches Maß Null hat, zu verbessern.

7. Es seien α und $\beta_1, \ldots, \beta_n$ algebraische Zahlen sowie $\alpha \neq 0, 1$ und $\beta_1, \ldots, \beta_n$ irrational und mit rationalen Koeffizienten linear unabhängig voneinander. Dann untersuche die transzendenten Zahlen $\alpha^{\beta_1}, \ldots, \alpha^{\beta_n}$ auf algebraische Unabhängigkeit bei algebraischen Koeffizienten.

8. Zeige, daß mindestens eine der Zahlen e^e und e^{e^2} transzendent sein muß.

Mit dieser kleinen Auswahl wollen wir uns bescheiden. Wir haben nicht gefragt nach der Irrationalität der EULERschen Konstanten,

nach der Transzendenz von Werten der RIEMANNschen ζ-Funktion, von weiteren Werten der Γ-Funktion, nach der algebraischen Unabhängigkeit von e und π, um nur einige von solchen häufig aufgeworfenen Problemen zu nennen, deren Lösung beim derzeitigen Stande der Theorie völlig aussichtslos erscheint, aber natürlich kann der nächste methodische Fortschritt auch hier ein völlig neues Bild erwirken.

Schneider's problems 1 and 2

1. Es seien α, β, γ algebraische Zahlen und $\alpha \neq 0, 1$ sowie $\frac{\log \beta}{\log \alpha}$ und $\frac{\log \gamma}{\log \alpha}$ irrational. Es ist $\delta = \exp\left(\frac{\log \beta \log \gamma}{\log \alpha}\right)$ auf Transzendenz zu untersuchen.

2. Der Satz über die Transzendenz der Werte der Modulfunktion $J(\tau)$ soll durch direkte Untersuchung der transzendenten Modulfunktion und nicht durch Untersuchung von $\wp$-Funktionen gezeigt werden.

Theodor Schneider.
Einführung in die transzendenten Zahlen.
Die Grundlehren der Mathematischen Wissenschaften. Band **81**.
Berlin-Göttingen-Heidelberg: Springer-Verlag (1957).
Zbl 0077.04703
Introduction aux nombres transcendants.
Traduit par P. Eymard. Paris: Gauthier-Villars. (1959).
Zbl 0098.26304

Four Exponentials Conjecture

Conjecture.

Let x_1, x_2 be two $\mathbb{Q}$–linearly independent complex numbers and y_1, y_2 be two $\mathbb{Q}$–linearly independent complex numbers. Then one at least of the four numbers

$$\mathrm{e}^{x_1y_1},\ \mathrm{e}^{x_1y_2},\ \mathrm{e}^{x_2y_1},\ \mathrm{e}^{x_2y_2}$$

is transcendental.

Same as Schneider's first problem:

$$x_1 = \log\alpha,\ x_2 = \log\beta,\ y_1 = 1,\ y_2 = \frac{\log\gamma}{\log\alpha} = \frac{\log\delta}{\log\beta}$$

Open problem:
Assume 2^t and 3^t are integers. Prove that t is an integer.

Six Exponentials and Four Exponentials

Let $x_1, \ldots, x_d$ be $\mathbb{Q}$–linearly independent complex numbers and $y_1, \ldots, y_\ell$ be $\mathbb{Q}$–linearly independent complex numbers.

Theorem (Six Exponentials Theorem: Siegel – Schneider – Lang – Ramachandra).

If $\ell d > \ell + d$ then one at least of the ℓd numbers

$$\mathrm{e}^{x_i y_j} \quad (1 \leqslant i \leqslant d,\ 1 \leqslant j \leq \ell)$$

is transcendental. The relevant case is $\{\ell, d\} = \{2, 3\}$.

Compare with the Four Exponentials Conjecture:
If $\ell d \geqslant \ell + d$ then one at least of the ℓd numbers

$$\mathrm{e}^{x_i y_j} \quad (1 \leqslant i \leqslant d,\ 1 \leqslant j \leq \ell)$$

is transcendental. The relevant case is $\ell = d = 2$.

References


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Variations on the six exponentials theorem.
Tandon, Rajat (ed.), Algebra and number theory. Proceedings of the silver jubilee conference, Hyderabad, India, December 11–16, 2003. New Delhi: Hindustan Book Agency 338-355 (2005).
Zbl 1176.11033


Waldschmidt, Michel
Further variations on the six exponentials theorem.
Hardy-Ramanujan J. 28, 1-9 (2005).
Zbl 1116.11054


Waldschmidt, Michel
The four exponentials problem and Schanuel's conjecture.
Morel, Jean-Michel (ed.) et al., Mathematics going forward. Collected mathematical brushstrokes. Springer. Lect. Notes Math. 2313, 579-592 (2023).
Zbl 1543.11060

Resonance

K. Senthil Kumar, R. Thangadurai & Veekesh Kumar
On a Problem of Alaoglu and Erdős
Resonance, 23, 749–758 (2018),
doi 10.1007/s12045-018-0676-1

Waldschmidt, Michel
Six Exponentials Theorem — Irrationality
Resonance, 27, 599–607 (2022)
doi 10.1007/s12045-022-1351-0

Proof of the Six Exponentials Theorem 1

Since $x_1, \ldots, x_d$ are $\mathbb{Q}$–linearly independent, the functions $\mathrm{e}^{x_1 z}, \ldots, \mathrm{e}^{x_d z}$ are algebraically independent.
Here there is no differential equation with algebraic coefficients.
Let K be the number field $\mathbb{Q}(\mathrm{e}^{x_i y_j} \ ; \ 1 \leqslant i \leqslant d, \ 1 \leqslant j \leq \ell)$.
The auxiliary function is an exponential polynomial with integer coefficients

$$F(z) = P(\mathrm{e}^{x_1 z}, \ldots, \mathrm{e}^{x_d z}).$$

The first step is to show the existence of a nonzero polynomial P of partial degrees $< T$ such that

$$F(s_1 y_1 + \cdots + s_\ell y_\ell) = 0$$

for $0 \leqslant s_j < S$, $(1 \leqslant j \leqslant \ell)$. Linear algebra show that such a polynomial exists as long as $T^d > [K : \mathbb{Q}] S^\ell$.

Proof of the Six Exponentials Theorem 2

Assume $T^d \geqslant 2[K : \mathbb{Q}]S^\ell$. A Lemma due to Thue and Siegel provides a solution with an upper bound for the height of P:

$$\mathrm{H}(P) \leqslant \mathrm{e}^{c_1 ST}.$$

The constant c_1 depends on the algebraic numbers $\mathrm{e}^{x_i y_j}$ but not on S nor T.
By induction on $\max\{s_j\}$ one proves that

$$F(s_1 y_1 + \cdots + s_\ell y_\ell) = 0$$

for all $(s_1, \ldots, s_\ell)$ with $s_j \geqslant 0$.
We explain the first step of the induction: assume $\max\{s_j\} = S$. From Schwarz's Lemma, using the fact that we already know S^ℓ zeroes of F, we deduce

$$|F|_{c_2 S} \leqslant \mathrm{e}^{-c_3 S^\ell}$$

Proof of the Six Exponentials Theorem 3

In particular

$$|F(s_1y_1 + \cdots + s_\ell y_\ell)| \leqslant \mathrm{e}^{-c_3 S^\ell}$$

for $0 \leqslant s_j \leqslant S$, $(1 \leqslant j \leq \ell)$. The left hand side is an algebraic number, Liouville's inequality yields that it is zero.

This is the first step of the inductive argument. Once we complete the induction, one deduces $F = 0$, hence the contradiction.

Assumption $\ell d > \ell + d$:

$$T^d \simeq S^\ell, \quad TS \simeq S^{(\ell+d)/d}, \quad TS < S^\ell.$$

Baker's Theorem

Let $\ell_1, \ldots, \ell_n$ be $\mathbb{Q}$–linearly independent complex numbers such that $\alpha_1 = \mathrm{e}^{\ell_1}, \ldots, \alpha_n = \mathrm{e}^{\ell_n}$ are algebraic. Then the numbers $1, \ell_1, \ldots, \ell_n$ are-linearly independent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

Corollaries:

- Transcendence of numbers like $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$, $e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$.
- Siegel integral

$$\int_0^1 \frac{\mathrm{d}x}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right)$$

Siegel, Carl Ludwig
Transcendental numbers. Annals of Mathematics Studies. 16. Princeton, N. J.: Princeton University Press. 102 p. (1949). Reprint of the 1949 edition. Texts and Readings in Mathematics 60. New Delhi: Hindustan Book Agency 102 p. (2012).
Zbl 0039.04402 Zbl 1283.11003

Baker's method 1

Let $\ell_1, \ldots, \ell_n$ be $\mathbb{Q}$–linearly independent complex numbers such that $\alpha_i = \mathrm{e}^{\ell_i}$, $(i = 1, \ldots, n)$ are algebraic. Write $\ell_i = \log \alpha_i$. Let $\beta_0, \beta_1, \ldots, \beta_{n-1}$ be algebraic numbers such that

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n.$$

Consider the functions

$$z_0, \mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_{n-1}}, \mathrm{e}^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$$

and their derivatives at the points in

$$\mathbb{Z}(1, \log \alpha_1, \ldots, \log \alpha_{n-1}).$$

Baker's method 2

The first step is the existence of a nonzero polynomial $P \in \mathbb{Z}[X_0, \ldots, X_n]$ such that the auxiliary function of n variables

$$F(z_0, \ldots, z_{n-1}) = P(z_0, \mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_{n-1}}, \mathrm{e}^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}})$$

has a zero of high multiplicity, say T, at several points in

$$\mathbb{Z}\underline{u}, \text{ where } \underline{u} = (1, \log \alpha_1, \ldots, \log \alpha_{n-1}).$$

If F has a zero of multiplicity $\geqslant T$ at a point $s\underline{u}$, then for a derivative $D^{\underline{t}_1}$ of multiplicity $\leqslant T/2$ the function of a single variable

$$G(z) = D^{\underline{t}_1} F(z\underline{u})$$

has a zero of multiplicity $\geqslant T/2$ at s. Schwarz's Lemma in a single variable produces an upper bound for $|G|_r$. In the inductive process one decreases the order of derivation, and increases the number of points. To complete the proof (and reach a contradiction) one needs to use a zero estimate.

p–adic transcendence

The series

$$\sum_{n\geqslant 0} \frac{z^n}{n!}$$

which defines the exponential function has a finite radius of convergence in the p–adic case.

The transcendence proofs which require to take a large radius when applying the analytic estimate do not work in the p–adic case.

Two open problems:

- p–adic Lindemann–Weierstrass
- Algebraic independence of $2^{\sqrt[3]{2}}$, $2^{\sqrt[3]{4}}$


Adams, W. W.
Transcendental numbers in the p–adic domain. Am. J. Math. 88, 279-308 (1966). Zbl 0144.29301

Charles Hermite

1822 – 1901

Ferdinand von Lindemann

1852 – 1939

Joseph Liouville

1809 – 1882

David Hilbert

1862 – 1943

Axel Thue

1863 - 1922

Carl Ludwig Siegel

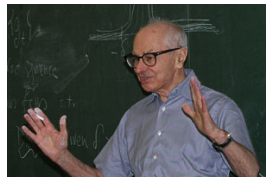
1896 - 1981

Alexander Ossipovitch Gelfond

1906 – 1968

Theodor Schneider

1911 – 1988



Serge Lang

1927 – 2005

Kanakanahalli Ramachandra

1933 – 2011

Alan Baker

1939 – 2018

Indian Institute of Science (IISc), Bengaluru

Strategies of transcendence proofs

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http://www.imj-prg.fr/~michel.waldschmidt/

- Rank of matrices.
- Schwarz's Lemma: one variable;
  Several variables; Cartesian products;
  Bombieri; Hypersurfaces; Commutative algebra;
  Seshadri's constant. Nagata's conjectures.
- Construction of auxiliary functions.
  Masser's zero estimate. Exponentials in several variables.
  The matrix coefficient conjecture.
- Duality. Fourier–Borel transform
- Dual of Baker's proof.
  Matrices with entries linear combinations of logarithms.
- Interpolation determinants.
  Slope inequalities - Bost, Chambert-Loir

Regulators

Let ℓ_{ij} $(1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant \ell)$ be complex numbers such that $\alpha_{ij} := \mathrm{e}^{\ell_{ij}}$ are algebraic numbers. Write

$$\ell_{ij} := \log \alpha_{ij}.$$

Consider the $d \times \ell$ matrix

$$M := \big(\ell_{ij}\big)_{\substack{1 \leqslant i \leqslant d \\ 1 \leqslant j \leq \ell}}.$$

Assume that the rows of M are $\mathbb{Q}$–linearly independent, and that the columns of M are also $\mathbb{Q}$–linearly independent.

The conclusion of the Six Exponentials Theorem is that the rank of the matrix M is at least 2 if $\ell d > \ell + d$, the conclusion of the Four Exponentials Conjecture is the same under the assumption $\ell d \geqslant \ell + d$.

What should one expect?

A consequence of Schanuel's Conjecture is that $\mathbb{Q}$–*linearly independent logarithms of algebraic numbers are algebraically independent.*

Let $\lambda_1, \ldots, \lambda_k$ be a basis of the $\mathbb{Q}$–vector space spanned by the entries ℓ_{ij} of the matrix and let b_{ijh} be the rational numbers such that

$$\ell_{ij} = \sum_{h=1}^{k} b_{ijh}\lambda_h, \quad 1 \leqslant i \leqslant d,\ 1 \leqslant j \leq \ell.$$

Damien Roy defines the *structural rank* $\mathrm{rstr}(M)$ of the matrix M as the rank of the matrix

$$\left(\sum_{h=1}^{k} b_{ijh}X_h\right)_{\substack{1\leqslant i\leqslant d\\ 1\leqslant j\leq \ell}} \in \mathrm{Mat}_{d\times\ell}(\mathbb{Q}(X_1, \ldots, X_k)).$$

Hence the conjecture: *the rank of M should be $\mathrm{rstr}(M)$.*

Rank of matrices

A $d \times \ell$ matrix M has rank $\leqslant n$ if and only if there exists $x_1, \ldots, x_d$ and $y_1, \ldots, y_\ell$ in $\mathbb{C}^n$ such that

$$M = \big(x_i y_j\big)_{\substack{1 \leqslant i \leqslant d \\ 1 \leqslant j \leqslant \ell}}$$

where for u and v in $\mathbb{C}^n$, uv denotes the usual scalar product

$$(u_1, \ldots, u_n)(v_1, \ldots, v_n) = u_1 v_1 + \cdots + u_n v_n.$$

Assume that the numbers $\mathrm{e}^{x_i y_j}$ are algebraic. We wish to prove a lower bound for n (under suitable assumptions on linear independence on $x_1, \ldots, x_d$ and $y_1, \ldots, y_\ell$). We will consider a non–zero polynomial $P \in \mathbb{Z}[X_1, \ldots, X_d]$ and introduce the entire function of n variables

$$F(z) = P(\mathrm{e}^{x_1 z}, \ldots, \mathrm{e}^{x_d z})$$

at the points $z \in \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_\ell$.

Rank of matrices

The first assumptions are that $x_1, \ldots, x_d$ are $\mathbb{Q}$–linearly independent (so that F is not the zero function) and that $y_1, \ldots, y_\ell$ are $\mathbb{Q}$–linearly independent (so that the points in $\mathbb{Z}y_1 + \cdots + \mathbb{Z}y_\ell$ are pairwise distinct).
Comparing with the proof of the six exponentials Theorem (the case $n = 1$), a natural approach would be to start with the existence of a polynomial P such that many numbers $F(s_1y_1 + \cdots + s_\ell y_\ell)$ vanish.
Then, by induction, we would wish to prove that all these numbers, with $(s_1, \ldots, s_\ell) \in \mathbb{Z}^\ell$, vanish.
A lower bound for n would easily follow.
So far we are not able to complete the proof following these lines, because of lack of a suitable Schwarz Lemma in several variables.

Schwarz's Lemma in one variable

Let f be an analytic function in a disc $|z| \leqslant R$ of $\mathbb{C}$, with at least N zeroes in a disc $|z| \leqslant r$ with $r < R$. Then

$$|f|_r \leqslant \left(\frac{3r}{R}\right)^N |f|_R$$

where

$$|f|_r = \sup_{|z|=r} |f(z)|.$$

When $R > 3r$, this improves the maximum modulus bound $|f|_r \leqslant |f|_R$.

Schwarz Lemma in several variables

The zeroes of an analytic function in several variables are not isolated. The number of zeroes is not relevant.

Three solutions:

- Cartesian products
- Lelong number
- Hypersufaces.

Cartesian products (grids)

Let f be an analytic function in a ball $|z| \leqslant R$ of $\mathbb{C}^n$. Assume f vanishes with multiplicity at least t on a set $\mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ where each $\mathcal{S}_i$ is contained in a disc $|z| \leqslant r$ with $r < R$ and has at least s elements.
Then

$$|f|_r \leqslant \left(\frac{3r}{R}\right)^{st} |f|_R.$$

Schwarz Lemma for Cartesian products can be proved by induction.

§4.3 of M.W. *Diophantine Approximation on Linear Algebraic Groups.* Grund. Math. Wiss. **326** Springer-Verlag (2000).

Cartesian products

Another proof, based on integral formulae, yields a weaker result : for $R > 3r$,

$$|f|_r \leqslant \left(\frac{R-3r}{2r}\right)^n \left(\frac{3r}{R}\right)^{st} |f|_R.$$

The conclusion follows from a homogeneity argument: replace f by f^N (and t by Nt) and let $N \to \infty$. (Landau's trick — book by Pólya and Szegő).

Chap. 7 of M.W. *Nombres transcendants et groupes algébriques.* Astérisque, **69–70** (1979).

Landau's trick

Pólya and Szegő mention that a rough estimate may sometimes be transformed into a sharper estimate by making appropriate use of the generality for which the original estimate is valid.

Pólya, George; Szegő, Gabor
Problems and theorems in analysis II. Theory of functions, zeros, polynomials, determinants, number theory, geometry. Transl. from the German by C. E. Billigheimer. Reprint of the 1976 English translation. Classics in Mathematics. Berlin: Springer. 392 p. (1998). Zbl 1024.00003

Schneider – Lang Theorem in several variables: Cartesian products (1941, 1966)

Let $f_1, \ldots, f_m$ *be meromorphic functions in* $\mathbb{C}^n$ *with* $m \geqslant n+1$. *Assume* $f_1, \ldots, f_{n+1}$ *are algebraically independent of finite order. Let* $\mathbb{K}$ *be a number field. Assume* $(\partial/\partial z_i) f_j$ *belongs to* $\mathbb{K}[f_1, \ldots, f_m]$ *for* $j = 1, \ldots, m$ *and* $i = 1, \ldots, n$. *Let*

$$\mathcal{S} = \{w \in \mathbb{C}^n \mid w \text{ not pole of } f_j,\ f_j(w) \in \mathbb{K}\ (j = 1, \ldots, m)\}$$

and let $e_1, \ldots, e_n$ *be a basis of* $\mathbb{C}^n$. *Then* $\mathcal{S}$ *does not contain the Cartesian product*

$$\{s_1 e_1 + \cdots + s_n e_n \mid (s_1, \ldots, s_n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n\}$$

where each $\mathcal{S}_i$ *is infinite.*

Schneider–Lang for Cartesian products

Schneider's Theorem on Euler's Beta function *Let* a, b *be rational numbers, not integers. Then the number*

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental.

Th. Schneider and S. Lang: abelian functions and algebraic groups.

Generalizations of Baker's Theorem to algebraic groups.

Transcendence and linear independence of *periods* (as defined by Kontsevich – Zagier).

Baker, Alan; Wüstholz, Gisbert
Logarithmic forms and Diophantine geometry. New Mathematical Monographs 9. Cambridge: Cambridge University Press 198 p. (2007). Zbl 1145.11004

Schneider–Lang for Cartesian products

Bertrand – Masser: The Schneider – Lang Theorem for Cartesian products implies Baker's Theorem.

Assume

$$\ell_1 + \sqrt[3]{2}\ell_2 + \sqrt[3]{4}\ell_3 = 0,$$

where ℓ_1, ℓ_2, ℓ_3 are nonzero logarithms of algebraic numbers

By Gel'fond – Schneider Theorem, the three numbers ℓ_1, ℓ_2, ℓ_3 are linearly independent over $\mathbb{Q}$.

We multiply the given relation by $\sqrt[3]{2}$ and by $\sqrt[3]{4}$:

$$2\ell_3 + \sqrt[3]{2}\ell_1 + \sqrt[3]{4}\ell_2 = 0 \quad \text{and} \quad 2\ell_2 + 2\sqrt[3]{2}\ell_3 + \sqrt[3]{4}\ell_1 = 0.$$

Therefore the three functions e^{z_1}, e^{z_2} and $e^{\sqrt[3]{2}z_1 + \sqrt[3]{4}z_2}$ take algebraic values at the points $\mathbf{y}_1 = (\ell_2, \ell_3)$, $\mathbf{y}_2 = (\ell_1, \ell_2)$ and $\mathbf{y}_3 = (2\ell_3, \ell_1)$. By the Six Exponentials Theorem, $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ contains a basis of $\mathbb{C}^2$ over $\mathbb{C}$.


Bertrand, Daniel; Masser, David
Linear forms in elliptic integrals. Invent. Math. 58, 283-288 (1980).
Zbl 0425.10041

Schwarz Lemma in several variables

In order to deal with sets of zeroes which do not contain a cartesian product, Bombieri and Lang used the Lelong number.

In the p–adic case an analog is due to Serre, in connexion with the characters of the idèles class groups

Bombieri, Enrico; Lang, Serge
Analytic subgroups of group varieties. Invent. Math. 11, 1-14 (1970).
Zbl 0237.14015

Serre, Jean-Pierre
Dépendance d'exponentielles p–adiques. Théorie des Nombres, Sém. Delange-Pisot-Poitou 7 (1965/66), No. 15, 14 pp. (1967).
Zbl 0254.10033

Bombieri – Lang (1970)

Let f be an analytic function in a ball $|z| \leqslant R$ of $\mathbb{C}^n$. Assume f vanishes at N points z_i (counting multiplicities) in a ball $|z| \leqslant r$ with $r < R$. Assume $\min_{z_i \neq z_k} |z_i - z_k| \geqslant \delta$.

Then

$$|f|_r \leqslant \left(\frac{3r}{R}\right)^M |f|_R$$

with

$$M = N\left(\frac{\delta}{6r}\right)^{2n-2}.$$

Nagata's suggestion (1966)

Masayoshi Nagata (1927 – 2008)
In the conclusion of the Schneider – Lang Theorem, replace the fact that $\mathcal{S}$ does not contain a Cartesian product $\mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ where each $\mathcal{S}_i$ is infinite by the fact that $\mathcal{S}$ is contained in an algebraic hypersurface.

Bombieri's Theorem (1970)

Theorem.

Let $f_1, \ldots, f_m$ be meromorphic functions in $\mathbb{C}^n$ with $m \geqslant n+1$. Assume $f_1, \ldots, f_{n+1}$ are algebraically independent and of finite order. Let $\mathbb{K}$ be a number field. Assume $(\partial/\partial z_i) f_j$ belongs to $\mathbb{K}[f_1, \ldots, f_m]$ for $j = 1, \ldots, m$ and $i = 1, \ldots, n$. Then the set

$$\mathcal{S} := \{w \in \mathbb{C}^n \mid f_j(w) \in \mathbb{K}\ (j = 1, \ldots, m)\}$$

is contained in an algebraic hypersurface.

Bombieri, E.
Algebraic values of meromorphic maps.
Invent. Math. 10, 267-287 (1970); addendum ibid. 11, 163-166 (1970).
Zbl 0214.33702

L^2 – estimates of Hörmander

Existence theorem for the $\overline{\partial}$ operator.

Result of E. Bombieri, improved by H. Skoda.
Let φ be a plurisubharmonic function in $\mathbb{C}^n$ not identically $-\infty$. For any $\epsilon > 0$ there exists a nonzero entire function F such that

$$\int_{\mathbb{C}^n} |F(z)|^2 e^{-\varphi(z)} (1 + |z|^2)^{-n-\epsilon} d\lambda(z) < \infty.$$

Example of a plurisubharmonic function: $c \log |f|$ where f is entire $\neq 0$ and $c \in \mathbb{R}$ a constant.

Bombieri's method yields a Schwarz lemma in several variables.

Schwarz lemma in several variables

Let $\mathcal{S}$ be a finite set of $\mathbb{C}^n$ and t a positive integer.

Denote by $\omega_t(\mathcal{S})$ the smallest degree of a polynomial vanishing at each point of $\mathcal{S}$ with multiplicity $\geqslant t$.

Let t be a positive integer. There exists a real number r such that for $R > r$, if f is an analytic function in the ball $|z| \leqslant R$ of $\mathbb{C}^n$ which vanishes with multiplicity at least t at each point of $\mathcal{S}$, then

$$|f|_r \leqslant \left(\frac{e^n r}{R}\right)^{\omega_t(\mathcal{S})} |f|_R.$$

The exponent $\omega_t(\mathcal{S})$ cannot be improved: take for f a non–zero polynomial of degree $\omega_t(\mathcal{S})$.

Upper bound for $\omega_t(\mathcal{S})$: linear algebra

Given a finite subset $\mathcal{S}$ of $\mathbb{C}^n$ and a positive integer t, if D is a positive integer such that

$$|\mathcal{S}|\binom{t+n-1}{n} < \binom{D+n}{n},$$

then

$$\omega_t(\mathcal{S}) \leqslant D.$$

Consequence:

$$\omega_t(\mathcal{S}) \leqslant (t+n-1)|\mathcal{S}|^{1/n}.$$

For instance $\omega_1(\mathcal{S}) \leqslant n|\mathcal{S}|^{1/n}$. Also $\omega_t(\mathcal{S}) \leqslant t\omega_1(\mathcal{S})$.

The invariant $\Omega(\mathcal{S})$

Since

$$\omega_{t_1+t_2}(\mathcal{S}) \leqslant \omega_{t_1}(\mathcal{S}) + \omega_{t_2}(\mathcal{S}),$$

by Fekete's lemma the sequence

$$\left(\frac{1}{t}\omega_t(\mathcal{S})\right)_{t\geqslant 1}$$

has a limit $\Omega(\mathcal{S})$ as $t \to \infty$ and $\Omega(\mathcal{S}) = \inf_{t\geqslant 1} \omega_t(S)/t$.

Consequence of the L^2 – estimate for the $\overline{\partial}$ operator:

$$\frac{1}{n}\omega_1(\mathcal{S}) \leqslant \Omega(\mathcal{S}) \leqslant \omega_1(\mathcal{S}).$$

The invariant $\Omega(\mathcal{S})$

Connections with

- A Conjecture of Nagata related with his solution of Hilbert's Problem 14
- Seshadri's Constant.

Further works by
G.V. Chudnovskiĭ,
H. Skoda, J-P. Demailly, A. Azhari,
H. Esnault and E. Viehweg, A. Hirschowitz,
B. Harbourne, C. Bocci, L. Ein, R. Lazarfeld, K.E. Smith,
M. Hochster, C. Huneke, M. Dumnicki, T. Szemberg,
H. Tutaj-Gasińska, T. Bauer, M. Baczyńska, A. Habura,
G. Malara, J. Szpond, S.M. Cooper, E. Guardo, L. Farnik.

Zero estimate for $Y(S)$

Let $Y = \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_\ell$ be a finitely generated subgroup of $\mathbb{C}^n$. For $S > 0$, let

$$Y(S) = \{s_1 y_1 + \cdots + s_\ell y_\ell \mid 0 \leqslant s_j \leqslant S\}.$$

Easy: $\omega_1(Y(S)) \leqslant n(S+1)^{\ell/n}$.

The *Generalized Dirichlet exponent* of Y is defined as

$$\mu(Y) = \min_{V \subset \mathbb{C}^n} \frac{\mathrm{rang}_{\mathbb{Z}}(s_V(Y))}{\dim(\mathbb{C}^n/V)}$$

where V runs over the vector subspaces of $\mathbb{C}^n$ distinct form $\mathbb{C}^n$ and $s_V : \mathbb{C}^n \to \mathbb{C}^n/V$ is the canonical surjective map.

Hence

$$\omega_1(Y(S)) \leqslant c(Y) S^{\mu(Y)}.$$

Zero estimate for $Y(S)$

Theorem (D.W. Masser).

$$\omega_1(Y(S)) \geqslant (S/n)^{\mu(Y)}.$$

Masser, D. W.
On polynomials and exponential polynomials in several complex variables. Invent. Math. 63, 81-95 (1981).
Zbl 0436.32005

Schwarz's Lemma for $Y(S)$

Corollary.

There exists a constant c real number r **depending on** S *such that for $R > r$, if f is an analytic function in the ball $|z| \leqslant R$ of $\mathbb{C}^n$ which vanishes at each point of $Y(S)$, then*

$$|f|_r \leqslant \left(\frac{cr}{R}\right)^{(S/n)^{\mu(Y)}} |f|_R.$$

Question: does there exists r independent of S?
A consequence would be

$$\mu(Y) \leqslant \frac{d}{d-n},$$

hence, (under suitable assumptions on the x_i and y_j),

$$n \geqslant \frac{\ell d}{\ell + d}.$$

Schwarz's Lemma for $Y(S)$

My Conjecture 7.1.10 in § 7 of the reference below, according to which there exists r independent of S, was disproved by Damien Roy.

Waldschmidt, Michel
Nombres transcendants et groupes algébriques. Complété par deux appendices de Daniel Bertrand et Jean-Pierre Serre. 2e éd.
Astérisque, 69/70 (1987).
Zbl 0621.10022

Roy., Damien
Interpolation formulas and auxiliary functions. J. Number Theory 94, No. 2, 248-285 (2002).
Zbl 1010.11039

Another strategy

Waldschmidt, Michel
Transcendance et exponentielles en plusieurs variables. Invent. Math. 63, 97-127 (1981). Zbl 0454.10020

Michel Waldschmidt
Diophantine approximation on linear algebraic groups. Transcendence properties of the exponential function in several variables, *Grundlehren Math. Wiss.* **326** Berlin: Springer, 2000. Zbl 1467843 MR 1756786 doi 10.1007/978-3-662-11569-5

Auxiliary function

Proposition.

Let L and n be positive integers, N, U, V, R, r positive real numbers and $\varphi_1, \dots, \varphi_L$ entire functions in $\mathbb{C}^n$. Define $W = N + U + V$ and assume

$$W \geqslant 12n^2, \quad e \leqslant \frac{R}{r} \leqslant e^{W/6}, \quad \sum_{\lambda=1}^{L} |\varphi_\lambda|_R \leqslant e^U$$

and

$$(2W)^{n+1} \leqslant LN\big(\log(R/r)\big)^n.$$

Then there exist rational integers $p_1, \dots, p_L$, with

$$0 < \max_{1 \leq \lambda \leqslant L} |p_\lambda| \leqslant e^N,$$

such that the function $F = p_1\varphi_1 + \cdots + p_L\varphi_L$ satisfies

$$|F|_r \leqslant e^{-V}.$$

Thue–Siegel Lemma

Let X be a positive integer, U, V be positive real numbers and u_{ij} $(1 \leqslant i \leqslant \nu,\ 1 \leqslant j \leq \mu)$ be complex numbers. Assume

$$\sum_{i=1}^{\nu} |u_{ij}| \leqslant e^U, \qquad (1 \leqslant j \leq \mu)$$

and

$$\left(\sqrt{2} X e^{U+V} + 1\right)^{2\mu} \leqslant (X+1)^{\nu}.$$

Then there exists $(\xi_1, \ldots, \xi_\nu) \in \mathbb{Z}^\nu$ satisfying

$$0 < \max_{1 \leqslant i \leq \nu} |\xi_i| \leqslant X$$

and

$$\max_{1 \leqslant j \leqslant \mu} \left| \sum_{i=1}^{\nu} u_{ij} \xi_i \right| \leqslant e^{-V}.$$

Interpolation Formula

Let r and R be real numbers satisfying $0 < r < R$, T a positive integer, and F an entire function in $\mathbb{C}^n$. Then

$$|F|_r \leq (1 + \sqrt{T}) \left(\frac{r}{R}\right)^T |F|_R + \sum_{\|\underline{\tau}\|<T} |D^{\underline{\tau}} F(0)| \frac{r^{\|\underline{\tau}\|}}{\underline{\tau}!}.$$

Consequence
If the the numbers $|D^{\underline{\tau}} F(0)|$ for all $\|\underline{\tau}\| < T$ are small, then $|F|_r$ is small.

Auxiliary function: sketch of proof

Let T be sufficiently large. Consider the linear system of $\binom{T+n-1}{n} \leqslant T^n$ inequalities:

$$\left| \sum_{\lambda=1}^{L} p_\lambda D^{\underline{\tau}} \varphi_\lambda(0) \right| \leqslant \mathrm{e}^{-V_1}, \qquad (\underline{\tau} \in \mathbb{N}^n, \ \ \|\underline{\tau}\| < T).$$

From the Thue–Siegel Lemma we deduce that there exists a nontrivial solution $(p_1, \ldots, p_L) \in \mathbb{Z}^L$ with $\max_{1 \leqslant \lambda \leqslant L} |p_\lambda| \leqslant e^N$. The function $F = p_1\varphi_1 + \cdots + p_L\varphi_L$ then satisfies

$$\sum_{\|\underline{\tau}\| < T} |D^{\underline{\tau}} F(0)| \frac{r^{\|\underline{\tau}\|}}{\underline{\tau}!} \leqslant \mathrm{e}^{-V_2}.$$

The Interpolation Formula provides the conclusion

$$|F|_r \leqslant \mathrm{e}^{-V}.$$

Consequence

Given any $x_1, \ldots, x_d$ in $\mathbb{C}^n$, for sufficiently large T, there exists a nonzero polynomial P such that the function

$$F(z) = P(\mathrm{e}^{x_1 z}, \ldots, \mathrm{e}^{x_d z})$$

satisfies (for some r and c_1)

$$|F|_r \leqslant \mathrm{e}^{-c_1 T^{d/n}}.$$

Assume $\mathrm{e}^{x_i y_j}$ are algebraic $(1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant \ell)$. If

$$|F(s_1 y_1 + \cdots + s_\ell y_\ell)| \leqslant \mathrm{e}^{-c_2 T S},$$

then $F(s_1 y_1 + \cdots + s_\ell y_\ell) = 0$.
One needs more equations than unknowns: $S^\ell > c_3 T^d$. This gives the requirement

$$c_3 T^{d/\ell} < S < c_4 T^{(d/n)-1}, \qquad \frac{d}{\ell} < \frac{d}{n} - 1,$$

i.e. $d\ell > n(d+\ell)$.

Masser's Zero Estimate

Course in Paris, spring 1980

The previous construction shows the existence of a nonzero polynomial such that $P(\mathrm{e}^{x_1 z}, \ldots, \mathrm{e}^{x_d z})$ vanishes at $Y(S)$.

The existence of an auxiliary function gives an upper bound for the degree, and Masser's Zero Estimate for exponential polynomials gives a lower bound for this degree.

Masser, D. W.
On polynomials and exponential polynomials in several complex variables. Invent. Math. 63, 81-95 (1981).
Zbl 0436.32005

Consequence

Square case:
A matrix M with entries logarithms of algebraic numbers has rank $\geqslant \mathrm{rstr}(M)/2$.

Non square case:
If a $d \times \ell$ matrix M with entries logarithms of algebraic numbers has rank $< \ell d/(d+\ell)$, then there exist two regular matrices P and Q with rational coefficients such that

$$PMQ = \begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix}$$

where the 0 bloc is a $d' \times \ell'$ matrix with

$$\frac{d'}{d} + \frac{\ell'}{\ell} > 1.$$

The matrix coefficient conjecture

Conjecture.

Let M be a square matrix with entries logarithms of algebraic numbers and zero determinant. Then there exist u and v in $\mathbb{Q}^n \setminus \{0\}$ such that $uMv = 0$.


Samit Dasgupta
Introduction to Transcendence Theory. Course Math 790, March 2021, Duke University,


Dasgupta, Samit
Ranks of matrices of logarithms of algebraic numbers. I: The theorems of Baker and Waldschmidt-Masser. Essent. Number Theory 2, No. 1, 93-138 (2023).
Zbl 1543.11059


Dasgupta, Samit; Kakde, Mahesh
Ranks of matrices of logarithms of algebraic numbers. II: The matrix coefficient conjecture. Adv. Math. 487, Article ID 110753, 19 p. (2026).
Zbl 08155756

The matrix coefficient conjecture

The matrix coefficient conjecture is a consequence of the conjecture that the rank of M is equal to its structural rank, and it implies a number of conjectures on nonvanishing of regulators (Leopoldt, Gross–Kuz'min).

New tool: Instead of the degree, Samit Dasgupta and Mahesh Kakde introduce the number of nonzero coefficients of the polynomial. They replace Masser's Zero estimate with a result involving a theorem of Bernstein–Kushnirenko.

Fourier–Borel transform

Let $H(\mathbb{C})$ denote the space of entire functions in $\mathbb{C}$ and $H'(\mathbb{C})$ the space of *analytic functionals*, namely linear maps $\eta : H(\mathbb{C}) \to \mathbb{C}$ such that there exists a constant $C = C(\eta)$ such that, for any entire function F,

$$\eta(F) \leqslant C|F|_R.$$

For $F(z) = \sum_{n\geqslant 0} b_n z^n$, we have $\eta(F) = \sum_{n\geqslant 0} b_n \eta(z^n)$.

The Fourier – Borel transform of η is the entire function of finite exponential type

$$\mathcal{F}_\eta(u) = \eta(\mathrm{e}^z) = \sum_{n\geqslant 0} \eta(z^n)\frac{u^n}{n!}.$$

Duality: Fourier – Borel

Conversely, given an entire function

$$\Phi(u) = \sum_{n \geqslant 0} a_n \frac{u^n}{n!}$$

of finite exponential type, i.e. such that there exists C and R positive with

$$|\Phi(u)| \leqslant C \mathrm{e}^{Ru}$$

for all $u \in \mathbb{C}$, one defines an analytic functional $\mathcal{H}_\Phi \in H'(\mathbb{C})$ by

$$\mathcal{H}_\Phi(F) = \sum_{n \geqslant 0} a_n b_n \quad \text{for } F(z) = \sum_{n \geqslant 0} b_n z^n.$$

The maps $\eta \mapsto \mathcal{F}_\eta$ and $\Phi \to \mathcal{H}_\Phi$ are inverse bijective maps from $H'(\mathbb{C})$ to the set of entire function of finite exponential type.

Duality: Fourier – Borel

- For $v \in \mathbb{C}$, the Fourier – Borel transform of $F \mapsto F(v)$ is $\Phi(u) = \mathrm{e}^{vu}$.

- For $\eta \in H'(\mathbb{C})$, the Fourier – Borel transform of $\eta \circ (\mathrm{d}/\mathrm{d}z) : F \mapsto \eta(\mathrm{d}F/\mathrm{d}z)$ is $u\mathcal{F}_\eta(u)$.

- For $\eta \in H'(\mathbb{C})$, the Fourier – Borel transform of $\eta \circ z : F \mapsto \eta(zF)$ is $(\mathrm{d}/\mathrm{d}u)\mathcal{F}_\eta(u)$.

Michel Waldschmidt
La transformation de Fourier-Borel: une dualité en transcendance.
Troisième Rencontre Internationale Mathématique de Delphes, 29 septembre 1989.
Groupe d'études sur les problèmes diophantiens 1988-1989,
Publ. Math. Univ. P. et M. Curie, 90 N°8.

https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/DelphesFourierBorel.pdf

Duality: Fourier – Borel — Gel'fond – Schneider

For s and t non negative integers and $v \in \mathbb{C}$, the Fourier – Borel transform of the analytic functional

$$\eta : F \mapsto \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\sigma} (z^{\tau} F)_{z=v}$$

is

$$\mathcal{F}_{\eta}(u) = \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{\tau} (u^{\sigma} \mathrm{e}^{vu}) \,.$$

In other words

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\sigma} (z^{\tau} \mathrm{e}^{tz})(s) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\tau} (z^{\sigma} \mathrm{e}^{sz})(t).$$

This common value is

$$\left(\frac{\partial^{\sigma+\tau}}{(\partial z_1)^{\sigma}(\partial z_2)^{\tau}} \mathrm{e}^{(z_1+s)(z_2+t)}\right)_{(z_1,z_2)=(0,0)}.$$

Baker's Theorem by Schneider's method

Assume

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n.$$

Recall Baker's method: consider the functions

$$z_0, \mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_{n-1}}, \mathrm{e}^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$$

and their derivatives at the points in

$$\mathbb{Z}(1, \log \alpha_1, \ldots, \log \alpha_{n-1}).$$

The *dual* proof, generalizing Schneider's method to several variables, deals with $n+1$ functions of n variables

$$z_0, z_1, \ldots, z_{n-1}, \mathrm{e}^{z_0} \alpha_1^{z_1} \cdots \alpha_{n-1}^{z_{n-1}}.$$

The auxiliary function is evaluated at the points

$$(t\beta_0, \tau_1 + t\beta_1, \ldots, \tau_{n-1} + t\beta_{n-1}),\ (\tau_1, \ldots, \tau_{n-1}, t) \in \mathbb{Z}^n$$

and only the powers of one derivative, $\partial/\partial z_0$, is used.

Example 1: $\beta_0 = 0$, $n = 3$, $\alpha_1^{\beta_1}\alpha_2^{\beta_2} = \alpha_3$ (no z_0)

Baker

$$f_{\tau_1,\tau_2,t}(z_1, z_2) = \mathrm{e}^{(\tau_1+t\beta_1)z_1}\mathrm{e}^{(\tau_2+t\beta_2)z_2}$$

Schneider

$$g_{\sigma_1,\sigma_2,s}(z_1, z_2) = z_1^{\sigma_1} z_2^{\sigma_2}(\alpha_1^{z_1}\alpha_2^{z_2})^s$$

Duality

$$\left(\frac{\partial}{\partial z_1}\right)^{\sigma_1}\left(\frac{\partial}{\partial z_2}\right)^{\sigma_2} f_{\tau_1,\tau_2,t}(s\log\alpha_1, s\log\alpha_2)$$

$$=$$

$$(\tau_1 + t\beta_1)^{\sigma_1}(\tau_2 + t\beta_2)^{\sigma_2}(\alpha_1^{\tau_1}\alpha_2^{\tau_2}\alpha_3^t)^s$$

$$=$$

$$g_{\sigma_1,\sigma_2,s}(\tau_1 + t\beta_1, \tau_2 + t\beta_2).$$

Example 2: $\beta_0 \neq 0$, $n = 2$, $\mathrm{e}^{\beta_0}\alpha_1^{\beta_1} = \alpha_2$

Baker:

$$f_{\tau_0,\tau_1,t}(z_0, z_1) = z_0^{\tau_0}\mathrm{e}^{\beta_0 z_0 t}\mathrm{e}^{(\tau_1+t\beta_1)z_1}.$$

Schneider:

$$g_{\sigma_0,\sigma_1,s}(z_0, z_1) = z_0^{\sigma_0} z_1^{\sigma_1}(\mathrm{e}^{z_0}\alpha_1^{z_1})^s.$$

Duality:

$$\frac{\partial^{\sigma_0+\sigma_1}}{(\partial z_0)^{\sigma_0}(\partial z_1)^{\sigma_1}} f_{\tau_0,\tau_1,t}(s, s\log\alpha_1) =$$

$$\left(\frac{\partial}{\partial z_0}\right)^{\sigma_0} (z_0^{\tau_0}\mathrm{e}^{\beta_0 z_0 t})(s)(\tau_1 + t\beta_1)^{\sigma_1}(\alpha_1^{\tau_1}\alpha_2^t)^s =$$

$$\left(\frac{\partial}{\partial z_0}\right)^{\tau_0} (z_0^{\sigma_0}\mathrm{e}^{sz_0})(t\beta_0)(\tau_1 + t\beta_1)^{\sigma_1}(\alpha_1^{\tau_1}\alpha_2^t)^s =$$

$$\left(\frac{\partial}{\partial z_0}\right)^{\tau_0} g_{\sigma_0,\sigma_1,s}(t\beta_0, \tau_1 + t\beta_1).$$

Dual proofs

Instead of an auxiliary function, one can construct an auxiliary functional. This introduces exponential polynomials for studying transcendence in algebraic groups. The zero estimate is replaced by an interpolation estimate (again due to Masser).

Masser, D. W.
Interpolation on group varieties. Approximations diophantiennes et nombres transcendants, Colloq. Luminy/Fr. 1982, Prog. Math. 31, 151-171 (1983). Zbl 0579.14038

Waldschmidt, M.
Dependance de logarithmes dans les groupes algébriques. Approximations diophantiennes et nombres transcendants, Colloq. Luminy/Fr. 1982, Prog. Math. 31, 289-328 (1983).
Zbl 0513.14028

Michel Waldschmidt
Fonctions auxiliaires et fonctionnelles analytiques. J. Analyse Math. 56 (1991), I: 231-254, II: 255-279.
Zbl 0742.11035 Zbl 0742.11036

Matrices with entries linear combinations of logarithms of algebraic numbers. D. Roy

A matrix M, the entries of which are linear combinations with algebraic coefficients of 1 and logarithms of algebraic numbers, has rank $\geqslant \mathrm{rstr}(M)/2$.

More generally:
If such a $d \times \ell$ matrix M has rank $< \ell d/(d+\ell)$, then there exist two regular matrices P and Q with algebraic coefficients such that

$$PMQ = \begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix}$$

where the 0 bloc is a $d' \times \ell'$ matrix with

$$\frac{d'}{d} + \frac{\ell'}{\ell} > 1.$$

Interpolation determinants – one variable

Let r and R be two real numbers with $0 < r \leqslant R$, $\varphi_1, \ldots, \varphi_L$ be functions of one complex variable, which are analytic in the disc $|z| \leqslant R$ of $\mathbb{C}$, and let $\zeta_1, \ldots, \zeta_L$ belong to the disc $|z| \leqslant r$. Then the absolute value of the determinant

$$\Delta = \begin{pmatrix} \varphi_1(\zeta_1) & \cdots & \varphi_L(\zeta_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(\zeta_L) & \cdots & \varphi_L(\zeta_L) \end{pmatrix}$$

is bounded from above by

$$|\Delta| \leq \left(\frac{R}{r}\right)^{-L(L-1)/2} L! \prod_{\lambda=1}^{L} |\varphi_\lambda|_R.$$

Interpolation determinants – one variable

`Hint:` The function

$$\Delta(z) = \begin{pmatrix} \varphi_1(\zeta_1 z) & \cdots & \varphi_L(\zeta_1 z) \\ \vdots & \ddots & \vdots \\ \varphi_1(\zeta_L z) & \cdots & \varphi_L(\zeta_L z) \end{pmatrix}$$

has a zero of multiplicity at least $L(L-1)/2$ at the origin.

Michel Laurent
Sur quelques résultats récents de transcendance. In *Journées arithmétiques.* Exp. Congr., Luminy France, 17-21 Juillet, 1989, *Astérisque* **198-200**, 209-230. Société Mathématique de France (SMF), Paris, 1991.
`Zbl 65929` `MR 1144324`
numdam.org: $AST\_1991\_\_198-199-200\_\_209\_0$

Interpolation determinants – several variables

Let $n \geqslant 2$ and let L be a sufficiently large integer. Let r and R be two real numbers with $0 < r \leqslant R$, $\varphi_1, \ldots, \varphi_L$ be functions of n variables, which are analytic in $|z| \leqslant R$ and let $\zeta_1, \ldots, \zeta_L$ belong to the disc $|z| \leqslant r$. Then the absolute value of the determinant

$$\Delta = \begin{pmatrix} \varphi_1(\zeta_1) & \cdots & \varphi_L(\zeta_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(\zeta_L) & \cdots & \varphi_L(\zeta_L) \end{pmatrix}$$

is bounded from above by

$$|\Delta| \leq \left(\frac{R}{r}\right)^{-(1/2)L^{1+(1/n)}} L! \prod_{\lambda=1}^{L} |\varphi_\lambda|_R.$$

Interpolation determinants – several variables

`Hint:` The function of a single variable z

$$\Delta(z) = \begin{pmatrix} \varphi_1(\zeta_1 z) & \cdots & \varphi_L(\zeta_1 z) \\ \vdots & \ddots & \vdots \\ \varphi_1(\zeta_L z) & \cdots & \varphi_L(\zeta_L z) \end{pmatrix}$$

has a zero of multiplicity at least $(1/2)L^{1+(1/n)}$ at the origin.

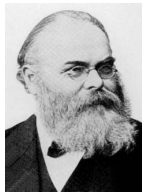
Waldschmidt, Michel
Linear independence of logarithms of algebraic numbers (with an appendix by M. Laurent).
IMSc Report. 116. Madras: Institute of Mathematical Sciences, 168 p. (1992).
Zbl 0809.11038

Slope inequalities Arakelov geometry


Bost, Jean-Benoît
Périodes et isogénies des variétés abéliennes sur les corps de nombres [d'après D. Masser et G. Wüstholz]. Séminaire Bourbaki. Volume 1994/95. Exposés 790-804. Paris: Société Mathématique de France, Astérisque. 237, 115-161, Exp. No. 795 (1996).
Zbl 0936.11042


Bost, Jean-Benoît; Chambert-Loir, Antoine
Analytic curves in algebraic varieties over number fields. Tschinkel, Yuri (ed.) et al., Algebra, arithmetic, and geometry. In honor of Yu. I. Manin on the occasion of his 70th birthday. Vol. I. Boston, MA: Birkhäuser Progress in Mathematics 269, 69-124 (2009).
Zbl 1201.14016


Mathilde Herblot
Algebraic points on meromorphic curves.
arXiv 1204.6336 [math.NT] 2012



Hermann Amandus Schwarz

1843 – 1921

Enrico Bombieri

Masayoshi Nagata

1927 – 2008

Conjeevaram Srirangachar Seshadri

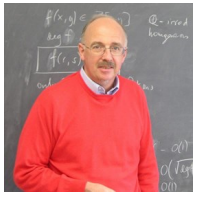
1932 – 2020

Joseph Fourier

1768 – 1830

Émile Borel

1871 – 1956



David Masser

Samit Dasgupta

Mahesh Kakde

Michel Laurent

Damien Roy

Jean–Benoit Bost

Antoine Chambert-Loir

Modular functions. Mahler's method

- Lindemann – Weierstrass, Siegel – Šhidlovskiĭ.
- Liouville's criterion for transcendence,
  Gel'fond's criterion for small transcendence degree.
  Criteria for large transcendence degree
  (Chudnovskiĭ, Philippon, Nesterenko).
- Chudnovskiĭ: algebraic independence of π and $\Gamma(1/4)$.
- Modular functions
  Schneider's Problem 2 on $j(\tau)$
  The Stéphanois Theorem on $J(q)$.
  Nesterenko: algebraic independence of π, e^{π} and $\Gamma(1/4)$.
- Mahler's method.
- Automata.

Algebraic independence of logarithms

Conjecture.

$\mathbb{Q}$–linearly independent logarithms of algebraic numbers are algebraically independent.

This is a consequence of Schanuel's Conjecture.

Damien Roy: this conjecture is equivalent to the equality $\mathrm{rank}(M) = \mathrm{rstr}(M)$ for a matrix M à coefficients in the $\mathbb{Q}$–space spanned by 1 and the logarithms of algebraic numbers.

Lemma.

Any polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$ is the determinant of some matrix with coefficients in $\mathbb{Q} + \mathbb{Q}X_1 + \cdots + \mathbb{Q}X_n$.

Open problem

Prove that the field generated by all logarithms of all nonzero algebraic numbers has transcendence degree $\geqslant 2$.

Damien Roy: this does not follow from the known lower bounds for the rank of matrices with coefficients in the $\mathbb{Q}$–space spanned by 1 and the logarithms of algebraic numbers.

Lindemann – Weierstrass

Theorem (Lindemann – Weierstrass (1882)).

Let $\beta_1, \ldots, \beta_m$ be $\mathbb{Q}$–linearly independent algebraic numbers. Then $\mathrm{e}^{\beta_1}, \ldots, \mathrm{e}^{\beta_m}$ are algebraically independent.

This is a result of linear independence:

Let $\gamma_1, \ldots, \gamma_n$ be distinct algebraic numbers. Then $\mathrm{e}^{\gamma_1}, \ldots, \mathrm{e}^{\gamma_m}$ are linearly independent (over $\mathbb{Q}$ or over $\overline{\mathbb{Q}}$).

Siegel (1929): E and G functions, A.B. Šhidlovskiĭ.

Fel'dman, N. I. & Nesterenko, Yu. V.
Number theory IV: Transcendental numbers. Encyclopaedia of Mathematical Sciences 44. Berlin: Springer 345 p. (1998).
Zbl 0885.11004

Liouville's Criterion

Let $\mathrm{H}(f)$ denote the maximum modulus of the coefficients of a polynomial $f \in \mathbb{C}[X_1, \ldots, X_m]$.

Proposition (Liouville's Criterion).

Let $\theta_1, \ldots, \theta_m$ be complex numbers. Assume that for any $\kappa > 0$ there exists a polynomial $f \in \mathbb{Z}[X_1, \ldots, X_m]$ and a positive integer T with

$$\deg f + \log \mathrm{H}(f) \leqslant T$$

and

$$0 < |f(\theta_1, \ldots, \theta_m)| \leqslant e^{-\kappa T}.$$

Then one at least of the numbers $\theta_1, \ldots, \theta_m$ is transcendental.

Gel'fond 's Criterion

In 1949 A.O. Gel'fond proved algebraic independence results for values of the exponential function. One of his main tools was the following.

Proposition (Gel'fond 's Criterion).

Let $\theta_1, \ldots, \theta_m$ be complex numbers. Assume that for any positive integer T there exists a polynomial $f_T \in \mathbb{Z}[X_1, \ldots, X_m]$ with

$$\deg f_T + \log \mathrm{H}(f_T) \leqslant T$$

and

$$0 < |f_T(\theta_1, \ldots, \theta_m)| \leqslant e^{-6T^2}.$$

Then two at least of the numbers $\theta_1, \ldots, \theta_m$ are algebraically independent.

Sketch of proof of Gel'fond 's Criterion

By Liouville's criterion the transcendence degree of the field $K := \mathbb{Q}(\theta_1, \ldots, \theta_m)$ is at least 1. Assume that it is 1; let $\theta \in K$ be transcendental. Using a norm one deduces that for each sufficiently large N there exists a polynomial $Q_N \in \mathbb{Z}[X]$ with $Q_N \neq 0$, $\deg Q_N \leq N$, $\log H(Q_N) \leq N$ such that

$$|Q_N(\theta)| \leq \mathrm{e}^{-c_1 N^2}.$$

Let α_N be a root of Q_N at minimal distance of θ. Then

$$|\theta - \alpha_N| \leq \mathrm{e}^{-c_2 N^2}.$$

Liouville's inequality yields $\alpha_N = \alpha_{N+1}$ for sufficiently large N, and the result follows.

Gel'fond : algebraic independence result

Let α be a nonzero algebraic number with $\log \alpha \neq 0$ and β an algebraic number of degree $d \geqslant 3$. Then among the numbers

$$\alpha^{\beta},\ \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}},$$

at least two are algebraically independent.

Example: the two numbers

$$2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{4}}$$

are algebraically independent.

Sketch of proof of Gel'fond's result on α^{β^j}

Select parameters.

Show the existence of a nonzero polynomial $P \in \mathbb{Z}[X_0, \ldots, X_{d-1}]$ such that the auxiliary function

$$F(z) = P(\mathrm{e}^z, \mathrm{e}^{\beta z}, \mathrm{e}^{\beta^2 z}, \ldots, \mathrm{e}^{\beta^{d-1} z})$$

has a zero of high multiplicity at many points of the form

$$z \in (\mathbb{Z} + \mathbb{Z}\beta + \cdots + \mathbb{Z}\beta^{d-1}) \log \alpha.$$

Using a zero estimate, show the existence of another point of this form and of a derivative producing a nonzero value.

Conclude by means of Gel'fond's Criterion.

Cassels's example

Let $\varphi\colon \mathbb{N} \to \mathbb{R}_{>0}$ be a positive valued function. There exist algebraically independent numbers θ_1, θ_2 with the following property: for any positive integer T, there is a linear forms in three variables

$$L_T(X_0, X_1, X_2) = a_T X_0 + b_T X_1 + c_T X_2 \in \mathbb{Z}[X_0, X_1, X_2]$$

with integer coefficients of absolute values bounded by T such that $c_T \neq 0$ and

$$|L_T(1, \theta_1, \theta_2)| \leqslant \varphi(T).$$

Cassels, J. W. S.
An introduction to Diophantine approximation.
Cambridge Tracts in Mathematics and Mathematical Physics. No. 45. Cambridge: At the University Press, 166 p. (1957).
Zbl 0077.04801

Philippon's algebraic independence criterion

Let $\theta_1, \ldots, \theta_m$ be complex numbers, t a nonnegative integer, σ, λ, R unbounded increasing functions $\mathbb{N} \to \mathbb{R}$, such that λ/σ^t is increasing and

$$\liminf_{T\to\infty} \lambda(T)/R(T+1) > 0, \qquad \liminf_{T\to\infty} \lambda(T)/\sigma(T+1)^t = \infty.$$

Assume that there exists a sequence $(f_T)_{T\geqslant 0}$ of nonzero polynomials $\mathbb{Z}[X_1, \ldots, X_m]$ such that, for all sufficiently large T,

$$\deg f_T \leq \sigma(T), \qquad \log H(f_T) \leq \sigma(T),$$

and

$$|f_T(\theta_1, \ldots, \theta_m)| \leqslant e^{-\lambda(T)}.$$

Assume f_T does not vanish in the ball $\max_{1\leqslant i\leq m} |z_i - \theta_i| \leqslant e^{-R(T)}$ of $\mathbb{C}^m$. Then the field $\mathbb{Q}(\theta_1, \ldots, \theta_m)$ has transcendence degree $\geqslant t$.

Large transcendence degree

Chudnovskiĭ (1976): first lower bounds for the transcendence degree t of the field

$$\mathbb{Q}(\alpha^{\beta},\ \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}).$$

He introduced the first criterion of algebraic independence providing large transcendence degree.
Variants by Nesterenko and Philippon.
Best known lower bound: for $d \geqslant 3$,

$$t \geqslant \left\lfloor \frac{d+1}{2} \right\rfloor.$$

For $d = 3$, $t \geqslant 2$.
Conjecture: $t = d - 1$ (follows from Schanuel).

Diaz, Guy
Grands degrés de transcendance pour des familles d'exponentielles.
J. Number Theory 31, No. 1, 1–23 (1989).
Zbl 0661.10047

The technical hypothesis

Lower bounds for the transcendence degree of the fields (in one variable)

$$\mathbb{Q}(\mathrm{e}^{x_i y_j} \mid 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant \ell),$$
$$\mathbb{Q}(x_i, \mathrm{e}^{x_i y_j} \mid 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant \ell),$$
$$\mathbb{Q}(x_i, y_j, \mathrm{e}^{x_i y_j} \mid 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant \ell)$$

are known so far only when one assumes a measure of linear independence of $x_1, \ldots, x_d$ and also a measure of linear independence of $y_1, \ldots, y_\ell$.
To remove such an assumption is an open problem.

Weierstrass functions

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ and $\mathcal{E}$ the elliptic curve $\mathbb{C}/\Omega$. Consider the following Weierstrass functions:

- The canonical product of Weierstrass associated with Ω:

$$\sigma(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

- Weierstrass zeta function $\zeta = \sigma'/\sigma$.
- Weierstrass elliptic function $\wp = -\zeta'$.

The $\wp$–function is periodic with respect to Ω:

$$\wp(z + \omega) = \wp(z).$$

and satisfies a differential equation $\wp'^2 = 4\wp^3 - g_2\wp - g_3$.
The zeta function is quasi–periodic with respect to Ω:

$$\zeta(z + \omega) = \zeta(z) + \eta.$$

Algebraic independence and elliptic functions

Theorem (Chudnovskĭ (1976)).

Let $\wp$ be a Weierstrass elliptic function with invariants g_2 and g_3 and let ω_1, ω_2 be a pair of fundamental periods. Denote by η_1 and η_2 the associated quasi–periods of the Weierstrass zeta function. Then at least two of the numbers $g_2, g_3, \omega_1, \omega_2, \eta_1$ and η_2 are algebraically independent.

Corollary.

The two numbers π and $\Gamma(1/4)$ are algebraically independent.

Proof of the corollary.

$$y^2 = 4x^3 - 4x, \quad g_2 = 4, \quad g_3 = 0, \quad j = 1728, \quad \tau = \mathrm{i},$$
$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}}, \quad \omega_2 = \mathrm{i}\omega_1, \quad \eta_1 = \mathrm{i}\eta_2 = \frac{\pi}{\omega_1}.$$

Chudnovskiĭ: sketch of proof

We first select parameters which will be used as upper bounds for the partial degrees of the auxiliary polynomials and as lower bounds for the multiplicities and the number of points that we will consider.

Step 1. There exists a non-zero polynomial $P(X,Y,Z) \in \mathbb{Z}[g_2,g_3][X,Y,Z]$ such that the function $F(z) = P\big(z,\wp(z),\zeta(z)\big)$ has zeroes of high multiplicity at points of the form $(\omega_1/2) + m\omega_1 + n\omega_2$ for some set of integers m and n.

Step 2. zero estimate There exists a nonzero value $\gamma = (\mathrm{d}^t F/\mathrm{d}z^t)(z_0)$ of a derivative of F at a point z_0 of the form $(\omega_1/2) + m\omega_1 + n\omega_2$.

Step 3. Using Schwarz's Lemma and Cauchy's inequalities, we deduce an upper bound for $|\gamma|$.

Step 4. The conclusion follows from Gel'fond's Criterion

Another result of Chudnovskiĭ

Theorem (Chudnovskiĭ (1976)).

Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2 and g_3. Let Ω be the group of periods of $\wp$, ω a nonzero period and η the associated quasi–period of the Weierstrass zeta function. Let $u \in \mathbb{C} \setminus \Omega$ Then the two numbers

$$\zeta(u) - \frac{\eta}{\omega}u, \ \frac{\eta}{\omega}$$

are algebraically independent.

This implies the algebraic independence of two at least of the numbers $\omega_1, \omega_2, \eta_1$ and η_2 when g_2, g_3 are algebraic.

In this last result of Chudnovskiĭ one would like to remove the assumption that g_2 and g_3 are algebraic, replacing the conclusion with the fact that two at least of the numbers

$$g_2, \ g_3\ \zeta(u) - \frac{\eta}{\omega}u, \ \frac{\eta}{\omega}$$

are algebraically independent.

Chudnovskiĭ: sketch of proof

• Using the Thue–Siegel Lemma, one shows the existence for each sufficiently large integer N of a nonzero polynomial P with coefficients in $\mathbb{Z}[\eta/\omega]$ such that the meromorphic function

$$F(z) := P(\wp(z), \zeta(z) - (\eta/\omega)z)$$

has a zero of high multiplicity at many points hu, $h \in \mathbb{Z}$.

• A zero estimate enables one to find a nonzero value $F^{(t)}(hu)$.

• One main new remark is that the function F is periodic of period ω. This produces a strong improvement on the upper bound arising from Schwarz's Lemma.

Philibert's measure of algebraic independence

Assume g_2, g_3 are algebraic. Let $\omega \in \Omega \setminus \{0\}$ and $\eta = \zeta(z+\omega) - \zeta(z)$.
For any $\epsilon > 0$ there exists $C = C(\epsilon, g_2, g_3, \omega) > 0$ such that, for any $B \in \mathbb{Z}[X, Y] \setminus \{0\}$, then

$$|B(\omega/\pi, \eta/\pi)| > \exp\{-C(\deg B + \log \mathrm{H}(B))^{3+\epsilon}\}.$$

Up to ϵ, the exponent is optimal. A measure of algebraic independence of t numbers has an exponent at least $t+1$ by Dirichlet's box principle.

Further references


Waldschmidt, Michel
Les travaux de G. V. Chudnovskiĭ sur les nombres transcendants. Sémin. Bourbaki 1975/76, Lect. Notes Math. 567, Exp. No. 488, 19 p. (1977).
Zbl 0345.10020


Waldschmidt, Michel
Transcendence methods. Queen's Pap. Pure Appl. Math. 52, 132 p. (1979).
Zbl 0432.10016


Chudnovskiĭ, Gregory V.
Contributions to the theory of transcendental numbers. Mathematical Surveys and Monographs, No. 19. Providence, R.I.: American Mathematical Society (AMS). 450 p (1984).
Zbl 0594.10024

Recall: Schneider's result on $j(\tau)$

Let $\tau \in \mathfrak{H}$. Then τ and $j(\tau)$ are both algebraic if and only if τ is quadratic (τ is a *special point*).

Schneider Problem 2: prove this result using properties of the modular function, avoiding the use of elliptic functions.

Mahler–Manin Conjecture: let

$$J(\mathrm{e}^{2\mathrm{i}\pi\tau}) = j(\tau).$$

For algebraic $q \in \mathbb{C}$ satisfying $0 < |q| < 1$, the number $J(q)$ is transcendental.

The modular function J

Consider Eisenstein series of weight 2, 4 and 6 (Ramanujan notations)

$$P(z) = 1 - 24 \sum_{n=1}^{\infty} \frac{n z^n}{1 - z^n},$$

$$Q(z) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 z^n}{1 - z^n},$$

$$R(z) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 z^n}{1 - z^n}.$$

Let

$$\Delta = 12^{-3}(Q^3 - R^2) \quad \text{and} \quad J = Q^3/\Delta$$

so that $J(\mathrm{e}^{2\mathrm{i}\pi\tau}) = j(\tau)$ and

$$J(z) = \frac{1}{z} + 744 + 196\,884 z + 21\,493\,760 z^2 + 864\,299\,970 z^3 + \cdots$$

The Stéphanois Theorem *(Saint Étienne)*

Theorem (Stéphanois Theorem).

Let $q \in \mathbb{C}$ satisfy $0 < |q| < 1$. Then one at least of q, $J(q)$ is transcendental.

Barré-Sirieix, Katia; Diaz, Guy; Gramain, François; Philibert, Georges
Une preuve de la conjecture de Mahler–Manin.
Invent. Math. 124, No. 1-3, 1-9 (1996).
Zbl 0853.11059

Sketch of proof 1

• *First step.* There exists a nonzero polynomial $A \in \mathbb{Z}[X, Y]$, of degree $< N$ with respect to each variable such that $F(z) = \Delta(z)^{2N} A\big(z, J(z)\big)$ has a zero at $z = 0$ of multiplicity $\geqslant N^2/2$.

• *Second step.* The functions z and $J(z)$ are algebraically independent, hence F is not the zero function. Let $M = \mathrm{ord}_0 F$ denote the multiplicity of zero of F at $z = 0$.

• *Third step.* Schwarz Lemma.

$$|F(z)| < C^N M^{30N} \frac{|z|^M}{(1-|z|)^{12N+1}} \qquad \text{for } |z| < 1.$$

• *Fourth step.* Let $q \in \mathbb{C}$ satisfy $0 < |q| < 1$. Let S denote the least integer $\geqslant 1$ such that $F(q^S) \neq 0$. Then (Schwarz Lemma): $S^2 \leqslant \gamma N \log M$, with $\gamma = 63\big(\log(1/|q|)\big)^{-1}$.

Sketch of proof 2

- *Fifth step.* Assume q and $J(q)$ are algebraic. Then

$$|F(q^S)| \geqslant \exp\{-CNS(S+\log N)\log\log(3S)\}.$$

- *Sixth step.* Conclusion. From

$$|F(q^S)| \leqslant |q|^{MS} M^{31N}$$

one deduces

$$M \leqslant C_{14} N(S+\log M)\log\log(3S).$$

which is not compatible with

$$M \geqslant L = [N^2/2], \qquad S^2 \leqslant \gamma N \log M.$$

Serre's suggestion (1971)

Ramanujan's functions P, Q, R satisfy the system of differential equations

$$\begin{cases} DP = \dfrac{1}{12}(P^2 - Q), \\ DQ = \dfrac{1}{3}(PQ - R), \\ DR = \dfrac{1}{2}(PR - Q^2) \end{cases}$$

with $D = z(\mathrm{d}/\mathrm{d}z)$.

Compare with the Theorem of Schneider–Lang:

the derivatives DP, DQ, DR belong to $\mathbb{Q}[P, Q, R]$

Bertrand, Philippon, Nesterenko.

Nesterenko's Theorem

Theorem.

Let $q \in \mathbb{C}$ satisfy $0 < |q| < 1$. Then the transcendence degree of the field

$$\mathbb{Q}\big(q, P(q), Q(q), R(q)\big)$$

is at least 3.

Corollary.

Let $q \in \mathbb{C}$ satisfy $0 < |q| < 1$. Assume $J(q)$ is algebraic. Then the three numbers q, $P(q)$ et $\Delta(q)$ are algebraically independent.

For $q = \mathrm{e}^{-2\pi}$ one deduces the algebraic independence of the three numbers π, e^{π}, $\Gamma(1/4)$.


Nesterenko, Yu. V.
Modular functions and transcendence questions. Sb. Math. 187, No. 9, 1319-1348 (1996); translation from Mat. Sb. 187, No. 9, 65-96 (1996).
Zbl 0898.11031

Lemniscate: $y^2 = 4x^3 - 4x$

$$g_2 = 4, \quad g_3 = 0, \quad j = 1728, \quad \tau = \mathrm{i},$$

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}}, \quad \omega_2 = \mathrm{i}\omega_1, \quad \eta_1 = \mathrm{i}\eta_2 = \frac{\pi}{\omega_1},$$

$$P(\mathrm{e}^{-2\pi}) = \frac{3}{\pi}, \quad Q(\mathrm{e}^{-2\pi}) = 3\left(\frac{\omega_1}{\pi}\right)^4, \quad R(\mathrm{e}^{-2\pi}) = 0,$$

$$\Delta(\mathrm{e}^{-2\pi}) = \frac{1}{2^6}\left(\frac{\omega_1}{\pi}\right)^{12}.$$

Nesterenko's transcendence argument

Let $q \in \mathbb{C}$, $0 < |q| < 1$. There exists a sequence $(A_N)_{N \geqslant 2}$ of nonzero polynomials in $\mathbb{Z}[z, X_1, X_2, X_3]$, which satisfy, for each $N \geqslant 2$,

$$\deg A_N \leqslant cN \log N, \qquad \log H(A_N) \leqslant cN(\log N)^2,$$

and

$$\exp\{-\kappa_2 N^4\} \leqslant \left|A_N\big(q, P(q), Q(q), R(q)\big)\right| \leq \exp\{-\kappa_1 N^4\}.$$

Nesterenko's Theorem follows from Philippon's criterion for algebraic independence (or a variant of it, some of them being due to Nesterenko) with $t = 3$.

Proof of Nesterenko's Theorem 1

First step: Choice of parameters.
Let N be a sufficiently large integer, $L = [N^4/2]$.

Second step. There is a nonzero polynomial $A \in \mathbb{Z}[z, X_1, X_2, X_3]$, of degree $\leqslant N$ with respect to each of the four variable, such that $F(z) = A\big(z, P(z), Q(z), R(z)\big)$ has a zero at the origin of multiplicity $\geqslant L$ and

$$\mathrm{H}(A) \leqslant N^{85N}.$$

Proof of Nesterenko's Theorem 2

Third step. Let $M = \mathrm{ord}_0 F$ so that $M \geqslant L$.

Nesterenko zero estimate: $M \leqslant cN^4$ with an absolute constant c.

Fourth step. For $0 < r < 1$ and N sufficiently large,

$$|F(z)| \leqslant |z|^M M^{48N} \qquad \text{for } |z| \leqslant r.$$

Fifth step. Let $T := \mathrm{ord}_q F$. Then

$$T \leqslant \gamma N \log M \qquad \text{with} \qquad \gamma = 48\big(\log(r/|q|)\big)^{-1}.$$

Proof: The function

$$H(z) = \frac{F(z)}{z^M} \cdot \left(\frac{r^2 - \overline{q}z}{r(z-q)}\right)^T.$$

satisfies

$$|H|_r = r^{-M}|F|_r \leqslant M^{48N}.$$

Since $(1/M!)F^{(M)}(0) \in \mathbb{Z} \setminus \{0\}$, we have

$$|H(0)| \geqslant (r/|q|)^T.$$

Proof of Nesterenko's Theorem 3

Introduce the derivative operator

$$D = z\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{12}(X_1^2 - X_2)\frac{\partial}{\partial X_1} + \frac{1}{3}(X_1X_2 - X_3)\frac{\partial}{\partial X_2} + \frac{1}{2}(X_1X_3 - X_2^2)\frac{\partial}{\partial X_3}$$

on the ring $\mathbb{C}[z, X_1, X_2, X_3]$, so that for $F(z) = A(z, P(z), Q(z), R(z))$,

$$z\frac{\mathrm{d}}{\mathrm{d}z}F(z) = (DA)\big(z, P(z), Q(z), R(z)\big).$$

Set

$$A_N(z, X_1, X_2, X_3) = (12z)^T(z^{-1}D)^T A(z, X_1, X_2, X_3).$$

Proof of Nesterenko's Theorem 4

$$\deg A_N \leqslant 4N + T \leqslant (\gamma + 1)N \log M,$$
$$\mathrm{H}(A_N) \leqslant N^{85N} 5^{4N} (48N + 24T)^T \leqslant \exp\{2\gamma N (\log M)^2\}$$

and

$$|A_N\big(q, P(q), Q(q), R(q)\big)| \leqslant 12^T T! (r - |q|)^{-T} r^M M^{48N}$$
$$\leq e^{-\kappa M}$$

which is the upper bound needed for the criterion for large transcendence degree.

The lower bound for $|A_N\big(q, P(q), Q(q), R(q)\big)|$ rests on Nesterenko zero estimate: $M \leqslant cN^4$.

Proof of Nesterenko's Corollary

Recall the corollary:
Let $q \in \mathbb{C}$ satisfying $0 < |q| < 1$. Assume $J(q)$ is algebraic. Then q, $P(q)$ and $\Delta(q)$ are algebraically independent.

Philippon: a direct proof of this corollary avoids Nesterenko zero estimate and the criterion for large transcendence degree: instead one uses Philibert's measure of algebraic independence. Each of the four numbers q, $P(q)$, $Q(q)$, $R(q)$ is root of a polynomial $f_i(X_i, \omega/\pi, \eta/\pi)$, $(i = 0, 1, 2, 3)$, with $f_i \in \mathbb{Z}[X_i, Y_1, Y_2]$. In $\mathbb{Z}[X_0, X_1, X_2, X_3, Y_1, Y_2]$, eliminate the 4 variables X_0, X_1, X_2, X_3 using the 5 polynomials A, f_0, f_1, f_2, f_3.

Nesterenko's zero estimate

Let L_0 and L be positive integers, $A \in \mathbb{C}[z, X_1, X_2, X_3]$ a nonzero polynomial of degree $\leqslant L_0$ in z and $\leqslant L$ in each of the other three variables X_1, X_2, X_3. Then the multiplicity of zero at the origin of the function

$$A\big(z, P(z), Q(z), R(z)\big)$$

is at most cL_0L^3, with $c = 2 \cdot 10^{45}$.

Further references

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Transcendance et indépendance algébrique de valeurs de fonctions modulaires.
Fifth conference of the Canadian Number Theory Association, Ottawa, Ontario, Canada, August 17–22, 1996. American Mathematical Society. CRM Proc. Lect. Notes. 19, 353-375 (1999).
Zbl 0930.11055

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Zbl 0908.11029

Murty, M. Ram; Rath, Purusottam
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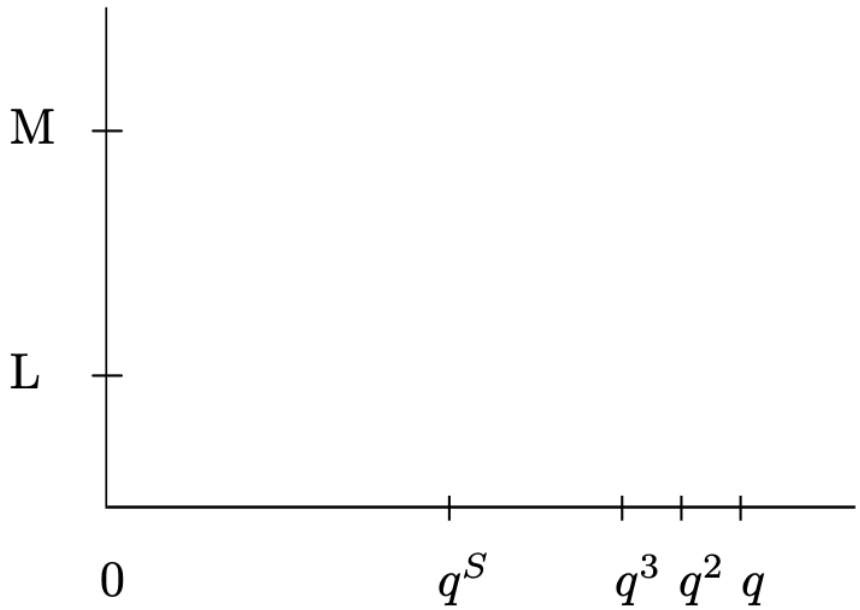
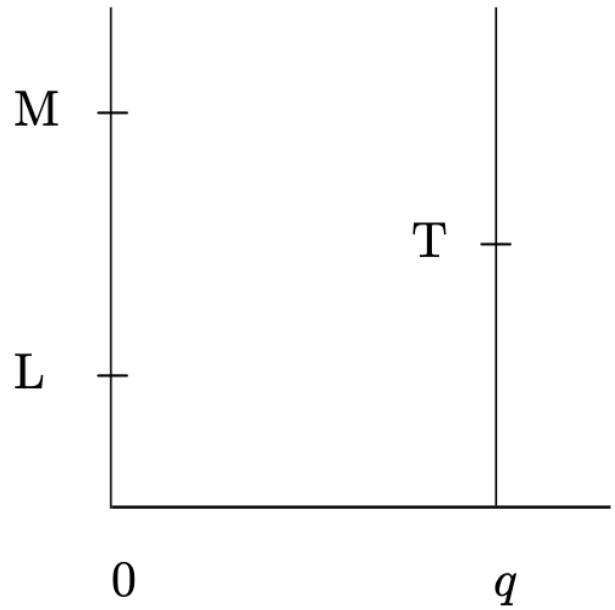
New York, NY: Springer 217 p. (2014).
Zbl 1297.11001

Stephanois's proof vs Nesterenko's proof





Mahler's method for the transcendence of $\sum_{n\geqslant 0} 2^{-2^n}$

- Mahler (1930, 1969): the function $f(z) = \sum_{n\geqslant 0} z^{-2^n}$ satisfies $f(z^2) + z = f(z)$ for $|z| < 1$.

- J.H. Loxton and A.J. van der Poorten (1982–1988).

- K. Nishioka (1991): algebraic independence measures for the values of Mahler's functions.

Nishioka, Kumiko
Mahler functions and transcendence. Lecture Notes in Mathematics. 1631. Berlin: Springer. 185 p. (1996).
Zbl 0876.11034

Mahler's method : sketch of proof 1

Let

$$f(z) = \sum_{n \geqslant 0} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots$$

Assume $f(1/2)$ is algebraic. Let K be a number field containing this number.

The functions z and $f(z)$ are algebraically independent.

Since $f(z^2) = f(z) - z$, these functions take values in K for $z = 2^{-2^k}$ for $k \geqslant 0$.

• Using **linear algebra only**, one proves the existence of a nonzero polynomial $P \in \mathbb{Z}[X, Y]$ such that the auxiliary function

$$F(z) = P(z, f(z))$$

has a zero of high multiplicity at the origin.

Mahler's method : sketch of proof 2

• Let H be the multiplicity of the zero $z = 0$ of F and let $b_H = F^{(H)}(0)/H!$.

From the previous step we know a lower bound for H. Since $b_H \in \mathbb{Z}$ we have $|b_H| \geqslant 1$.

• Let k be sufficiently large. Then $F(2^{-2^k})$ is a nonzero algebraic number for the absolute value of which we have an upper bound and a lower bound, contradicting Liouville's inequality.

Algebraic independence

Let $d \geqslant 2$ and let α be an algebraic number, $0 < |\alpha| < 1$. Let

$$F(x) := \sum_{n \geqslant 0} \alpha^{d^n} x^n.$$

Then the numbers

$$F^{(\ell)}(a), \quad a \in \overline{\mathbb{Q}}^{\times}, \quad \ell \geqslant 0$$

are algebraically independent.
Also *for $d \geqslant 2$, α algebraic with $0 < |\alpha| < 1$ and*

$$f(z) = \sum_{n \geqslant 0} z^{d^n} = z + z^d + z^{d^2} + z^{d^3} + \cdots$$

the values of the derivatives of f at α,

$$f^{(\ell)}(\alpha), \quad \ell \geqslant 0$$

are algebraically independent.

Automata

An example: *Let*

$$x = [1, 2, 2, 1, 2, 1, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 2, 1, 1, 2, 1, 2 \dots]$$

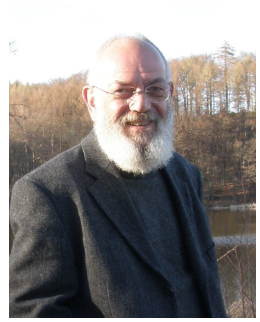
be the real number in the interval $(1, 2)$ *whose partial quotients* $t_1, t_2, \dots$ *are given by the Thue – Morse sequence on* $\{1, 2\}$ *(fixed point of the morphism* $1 \mapsto 12$, $2 \mapsto 21$*). Then the number* x *is transcendental.*

Open problem. *Is the number* $\sum_{n \geqslant 1} t_n/n!$ *transcendental?*

See Chapter 13 *Automatic real numbers* of

Allouche, Jean-Paul; Shallit, Jeffrey
Automatic sequences. Theory, applications, generalizations.
Cambridge: Cambridge University Press 571 p. (2003).
Zbl 1086.11015

Gregory V. Chudnovskii



François Gramain

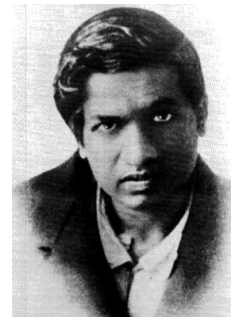
Yuri Nesterenko

Kurt Mahler

1903 – 1988

Gotthold Eisenstein

1823 – 1852



Srinivasa Ramanujan

1887 – 1920

Daniel Bertrand

Patrice Philippon

- Statement
- Consequences
- Algebraic independence of logarithms.
- Rank of matrices. Structural rank.
- Almost all
- Roy's Conjecture
- Consequences of Schanuel's Conjecture;
  the fields L and E.
- Grothendieck – André
  Elliptico toric Conjecture of Bertolin
- Two open problems

Schanuel's Conjecture

Schanuel's conjecture asserts that for linearly independent complex numbers $x_1, ..., x_n$, there are at least n algebraically independent numbers among the $2n$ numbers

$$x_1, \ldots, x_n, \ \exp(x_1), \ldots, \exp(x_n).$$

Course given by Serge Lang (1927–2005) at Columbia in the 60's, also attended by M. Nagata (1927–2008)
(14*th* Problem of Hilbert).

S. Lang
Introduction to transcendental numbers, Addison-Wesley 1966.

Algebraic independence results

• Lindemann– Weierstrass: if $x_1, \ldots, x_n$ are algebraic and $\mathbb{Q}$–linearly independent, then $\mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}$ are algebraically independent.

• A.O. Gel'fond: if β is a cubic irrationality and α a nonzero algebraic number with $\log \alpha \neq 0$, then α^{β} and α^{β^2} are algebraically independent.

• G V. Chudnovskiĭ: the two numbers π and $\Gamma(1/4)$ are algebraically independent.

Yu V. Nesterenko: the three numbers π, e^{π} and $\Gamma(1/4)$ are algebraically independent.

Statement by Gel'fond (1934)

Let $\beta_1, \ldots, \beta_n$ be $\mathbb{Q}$-linearly independent algebraic numbers and let $\log \alpha_1, \ldots, \log \alpha_m$ be $\mathbb{Q}$-linearly independent logarithms of algebraic numbers. Then the numbers

$$\mathrm{e}^{\beta_1}, \ldots, \mathrm{e}^{\beta_n}, \ \log \alpha_1, \ldots, \log \alpha_m$$

are algebraically independent over $\mathbb{Q}$.

Further statement by Gel'fond

Let $\beta_1, \ldots, \beta_n$ be algebraic numbers with $\beta_1 \neq 0$ and let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers with $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_i \neq 0$. Then the numbers

$$e^{\beta_1 e^{\beta_2 e^{\cdot^{\cdot^{\cdot^{\beta_{n-1} e^{\beta_n}}}}}}} \quad \text{and} \quad \alpha_1^{\alpha_2^{\cdot^{\cdot^{\cdot^{\alpha_m}}}}}$$

are transcendental, and there is no nontrivial algebraic relation between such numbers.

`Remark`: The condition on the α_i should be that they are irrational.

Easy consequence of Schanuel's Conjecture

According to Schanuel's Conjecture, the following numbers are algebraically independent:

$$e+\pi,\ e\pi,\ \pi^e,\ \mathrm{e}^{\mathrm{e}},\mathrm{e}^{\mathrm{e}^2},\ldots,\ \mathrm{e}^{\mathrm{e}^{\mathrm{e}}},\ldots,\ \pi^{\pi},\pi^{\pi^2},\ldots\ \pi^{\pi^{\pi}}\ldots$$

$$\log\pi,\ \log(\log 2),\ \pi\log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3}\ldots$$

Proof: Use Schanuel's Conjecture several times.

Question by Tung T. Nguyen

While working on a cryptography project, we came upon the following problem. Let x be the real solution of $x^{x^x}=100$. Is x a transcendental number?

Conditional answer assuming Schanuel's Conjecture: if $x>0$, $x\notin\mathbb{Z}$, is such that x^{x^x} is an integer, then x is transcendental.

Fixed points of the exponential function

Open problem (Federico Pellarin)
Let $z_1, \ldots, z_n$ be distinct fixed points of the exponential function:

$$\mathrm{e}^{z_i} = z_i \quad i = 1, \ldots, n.$$

Prove that $z_1, \ldots, z_n$ are $\mathbb{Q}$–linearly independent.

Schanuel's Conjecture implies that $z_1, \ldots, z_n$ *are algebraically independent.*

Further consequences of Schanuel's Conjecture

Transcendental values of class group L–functions, Gamma values, log Gamma values, digamma function, generalized Euler-Briggs constants, derivatives of L–series and generalized Stieltjes constants ...

Papers by Purusottam Rath, Ram Murty, Sanoli Gun, Kumar Murty, Ekata Saha, Siddhi Pathak, ...

Murty, M. Ram; Rath, Purusottam
Transcendental numbers.
New York, NY: Springer 217 p. (2014).
Zbl 1297.11001

Towards Schanuel's Conjecture

Let $x_1, \ldots, x_n$ be $\mathbb{Q}$–linearly independent complex number.

Let s, u be positive integers satisfying $1 < s < u < 5/4$.

From the construction (in the second course) of an auxiliary function of a single variable with first Taylor coefficients at the origin having small absolute values, we deduce the existence of a sequence of nonzero polynomials $P_N(X_0, X_1)$, of partial degree $\leqslant N$ in X_0, partial degree $\leqslant \sqrt{N}$ in X_1 and height $\leqslant e^N$, such that the function $F_N(z) = P_N(z, \mathrm{e}^z)$ satisfies

$$|F^{(k)}(m_1x_1 + \cdots + m_nx_n)| \leqslant \exp(-N^u)$$

for any non-negative integers k, $m_1, \ldots, m_n$ with $k \leqslant N^s$ and $\max\{m_1, \ldots, m_n\} \leqslant N^{s/2}$.

Roy's Conjecture

From the existence of such a sequence, one wishes to deduce that the transcendence degree of

$$\mathbb{Q}(x_1, \ldots, x_n, \mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n})$$

is at least n.

D. Roy (Journées Arithmétiques, Roma, 1999):
no information is lost!

Conjecture equivalent to Schanuel's Conjecture.

Damien Roy
An arithmetic criterion for the values of the exponential function.
Acta Arith. **97**, No. 2, 183–194 (2001).
Zbl 0981.11025 MR 1824984

Roy's approach to Schanuel's Conjecture (1999)

Let $\mathcal{D}$ denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

over the ring $\mathbb{C}[X_0, X_1]$, so that

$$\text{for } F(z) = P(z, \mathrm{e}^z), \quad F'(z) = \mathcal{D}P(z, \mathrm{e}^z).$$

Let n be a positive integer, $y_1, \ldots, y_n$ complex numbers which are linearly independent over $\mathbb{Q}$, $\alpha_1, \ldots, \alpha_n$ non-zero complex numbers and s_0, s_1, t_0, t_1, u positive real numbers satisfying

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$$

and

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Roy's Conjecture

Assume that, for any sufficiently large positive integer N, *there exists a non-zero polynomial* $P_N \in \mathbb{Z}[X_0, X_1]$ *with partial degree* $\leqslant N^{t_0}$ *in* X_0, *partial degree* $\leqslant N^{t_1}$ *in* X_1 *and height* $\leqslant e^N$ *which satisfies*

$$\left|(\mathcal{D}^k P_N)\Big(\sum_{j=1}^{n} m_j y_j, \prod_{j=1}^{n} {\alpha_j}^{m_j}\Big)\right| \leqslant \exp(-N^u)$$

for any non-negative integers k, $m_1, \ldots, m_n$ *with* $k \leqslant N^{s_0}$ *and* $\max\{m_1, \ldots, m_n\} \leqslant N^{s_1}$.
Then

$$\operatorname{tr\,deg}_{\mathbb{Q}}(y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_n) \geqslant n.$$

Equivalence between Schanuel and Roy

Let $(y, \alpha) \in \mathbb{C} \times \mathbb{C}^\times$, and let s_0, s_1, t_0, t_1, u be positive real numbers satisfying the inequalities of Roy's Conjecture.
Then the following conditions are equivalent:

(a) The number αe^{-y} is a root of unity.

(b) For any sufficiently large positive integer N, there exists a nonzero polynomial $Q_N \in \mathbb{Z}[X_0, X_1]$ with partial degree $\leqslant N^{t_0}$ in X_0, partial degree $\leq N^{t_1}$ in X_1 and height $\mathrm{H}(Q_N) \leqslant e^N$ such that

$$\left|(\mathcal{D}^k Q_N)(my, \alpha^m)\right| \leqslant \exp(-N^u)$$

for any $k, m \in \mathbb{N}$ with $k \leqslant N^{s_0}$ and $m \leqslant N^{s_1}$.

On the parameters

Roy's Conjecture deals with polynomials vanishing on some subsets of $\mathbb{C} \times \mathbb{C}^{\times}$ with multiplicity along the space associated with the derivation $\partial/\partial X + Y\partial/\partial Y$.

D. Roy conjecture depends on parameters s_0, s_1, t_0, t_1, u in a certain range.

D. Roy proved that if his conjecture is true for one choice of values of these parameters in the given range, then Schanuel's Conjecture is true, and that conversely, if Schanuel's Conjecture is true, then his conjecture is true for all choices of parameters in the same range.

Special case

$n = 1$: the conjecture is true (Hermite–Lindemann Theorem).

$n = 2$: the conjecture implies the algebraic independence of e and π, also of $\log 2$ and $\log 3$, for instance.

The conclusion when $n = 2$ is that the transcendence degree is at least 2, the same conclusion as Gel'fond's criterion.

Here the numbers are not small enough for Gel'fond's arguments, but there are many of them, not all of which are zero.

A proof should combine ideas from zero estimate and criteria of algebraic independence (elimination).

Progress by D. Roy: $\mathbb{G}_a$, $\mathbb{G}_m$, $\mathbb{G}_a \times \mathbb{G}_m$.

Roy, Damien
Small value estimates for the additive group.
Int. J. Number Theory 6, No. 4, 919–956 (2010).
Zbl 1200.11056

Roy, Damien
Small value estimates for the multiplicative group.
Acta Arith. 135, No. 4, 357–393 (2008).
Zbl 1214.11087

Roy, Damien
A small value estimate for $\mathbb{G}_a \times \mathbb{G}_m$.
Mathematika 59, No. 2, 333–363 (2013).
Zbl 1335.11060

Nguyen, Ngoc Ai Van; Roy, Damien
A small value estimate in dimension two involving translations by rational points.
Int. J. Number Theory 12, No. 5, 1273–1293 (2016).
Zbl 1369.11056

Zarankiewicz problem

Given integers m_1, n_1, m, n with $2 \leqslant m_1 \leqslant m$ and $2 \leqslant n_1 \leqslant n$, denote by $k(m_1, n_1; m, n)$ the smallest integer k such that any $m \times n$ matrix with coefficients in $\{0, 1\}$ containing at least k ones admits a sub-matrix of size $m_1 \times n_1$ consisting only of ones.

D. Roy

$$k(2, n_1; m, n) \leqslant 1 + n + (n_1 - 1)m(m-1)/2.$$

P. Erdős and J. Spencer
Probabilistic methods in combinatorics. Budapest: Akademiai Kiado (1974).
Zbl 0308.05001

Roy, Damien
Small value estimates for the additive group.
Int. J. Number Theory 6, No. 4, 919–956 (2010).
Zbl 1200.11056

Almost all

The set of tuples $(x_1, \ldots, x_n)$ in $\mathbb{C}^n$ such that the $2n$ numbers $x_1, \ldots, x_n, \ \mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}$ are algebraically independent

- is a G_δ set (countable intersection of dense open sets) in Baire's classification (a *generic set* for dynamical systems)

- and has full Lebesgue measure.

True for any transcendental function in place of the exponential function.

Algebraic independence of values of algebraically independent functions

Theorem. *Let K be a finitely generated extension of $\mathbb{Q}$. Let $f_1, \ldots, f_t$ be meromorphic functions in $\mathbb{C}$ which are algebraically independent over K. Then for almost all tuples $(z_1, \ldots, z_n)$ of complex numbers, the nt numbers*

$$f_j(z_i) \qquad (i = 1, \ldots, n, \quad j = 1, \ldots, t)$$

are algebraically independent over K.

Proof

- If F is a nonzero meromorphic function in $\mathbb{C}^n$, then the set $Z(F)$ of zeroes of F in $\mathbb{C}^n$ has Lebesgue measure zero.
- A countable union of sets of Lebesgue measure zero has Lebesgue measure zero.
- The set of polynomials in nt variables with coefficients in K is countable.

For P a nonzero polynomial in nt variables with coefficients in K, define a nonzero meromorphic function F in $\mathbb{C}^n$ by

$$F(z_1, \ldots, z_n) = P\big((f_j(z_i))_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t}}\big)$$

and let $Z(F) \subset \mathbb{C}^n$ be the set of zeroes of F. The set of tuples $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that the nt numbers

$$f_j(z_i) \qquad (i = 1, \ldots, n, \quad j = 1, \ldots, t)$$

are algebraically dependent over K is the union of all $Z(F)$ with $P \in K[(X_{ij})_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t}}] \setminus \{0\}$. Hence the result.

With Senthil Kumar and Thangadurai

Given two integers m and n with $1 \leqslant n$, there exist uncountably many tuples $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $x_1, \ldots, x_n$ and $\mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}$ are all Liouville numbers and the transcendence degree of the field

$$\mathbb{Q}(x_1, \ldots, x_n, \mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n})$$

is $n + m$.

We do not know whether there are Liouville numbers x such that e^x is also a Liouville number and the two numbers x and e^x are algebraically dependent.


Kumar, K. Senthil; Thangadurai, R.; Waldschmidt, M.
Liouville numbers, Liouville sets and Liouville fields. Proc. Am. Math. Soc. 143, No. 8, 3215–3229 (2015). Zbl 1320.11065


Kumar, K. Senthil; Thangadurai, R.; Waldschmidt, M.
Liouville numbers and Schanuel's conjecture.
Arch. Math. 102, No. 1, 59–70 (2014). Zbl 1294.11120

Lang's exercise

Define $E_0 = \mathbb{Q}$. Inductively, for $n \geq 1$, define E_n as the algebraic closure of the field generated over E_{n-1} by the numbers $\exp(x) = \mathrm{e}^x$, where x ranges over E_{n-1}. Let E be the union of E_n, $n \geq 0$.

Then Schanuel's Conjecture implies that the number π does not belong to E.

More precisely: Schanuel's Conjecture implies that the numbers $\pi, \log \pi, \log \log \pi, \log \log \log \pi, \ldots$ are algebraically independent over E.


Lang, Serge
Introduction to transcendental numbers. Addison-Wesley Publishing Company. VI, 105 p., 1966
Zbl 3232021 MR 0214547 Collected Papers Vol. 1 1952–1970


Lang, Serge
Algebra. 3rd revised ed. Graduate Texts in Mathematics. 211. New York, Springer. 914 p. (2002). Zbl 0984.00001
Appendix 1: The transcendence of e and π.

A variant of Lang's exercise

Define $L_0 = \mathbb{Q}$. Inductively, for $n \geq 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y, where y ranges over the set of complex numbers such that $\mathrm{e}^y \in L_{n-1}$. Let L be the union of L_n, $n \geq 0$. Then Schanuel's Conjecture implies that the number e does not belong to L.

More precisely: Schanuel's Conjecture implies that the numbers $\mathrm{e}, \mathrm{e}^{\mathrm{e}}, \mathrm{e}^{\mathrm{e}^{\mathrm{e}}}, \mathrm{e}^{\mathrm{e}^{\mathrm{e}^{\mathrm{e}}}} \ldots$ are algebraically independent over L.

Theorem *Schanuel's Conjecture implies that the fields E and L are linearly disjoint over $\overline{\mathbb{Q}}$.*

Definition Given a field extension F/K and two subextensions $F_1, F_2 \subseteq F$, we say F_1, F_2 are linearly disjoint over K when the following holds: *any set $\{x_1, \ldots, x_n\} \subseteq F_1$ of K– linearly independent elements is linearly independent over F_2.*

Cheng, Chuangxun; Dietel, Brian; Herblot, Mathilde; Huang, Jingjing; Krieger, Holly; Marques, Diego; Mason, Jonathan; Mereb, Martin; Wilson, S. Robert
Some consequences of Schanuel's conjecture.
J. Number Theory 129, No. 6, 1464-1467 (2009).
Zbl 1242.11052 arXiv 0804.3550 [math.NT] 2008

Philippon et al

Philippon, Patrice; Saha, Biswajyoti; Saha, Ekata
An abelian analogue of Schanuel's Conjecture and applications, Ramanujan J., 52, 2, 381-392, 2020.
Zbl 1496.11099

Bertolin, Cristiana; Philippon, Patrice; Saha, Biswajyoti; Saha, Ekata.
Semi-abelian analogues of Schanuel conjecture and applications.
J. Algebra 596, 250-288 (2022).
Zbl 1495.11085

Conjectures by A. Grothendieck and Y. André

Generalized Conjecture on Periods by Grothendieck:
Dimension of the Mumford–Tate group of a smooth projective variety.

Generalization by Y. André to motives.

Consequence of Grothendieck's Conjecture:

The numbers π, $\log 2$, $\zeta(3)$ *are algebraically independent.*

Kumar Murty's lecture at the International Conference on Number Theory and Related Topics ICNTRT-2024.

The Institute of Mathematical Sciences (IMSc), Mathematics and Indian Institute of Technology IIT Madras on December 19, 2024.

Payman Eskandari; Kumar Murty; Yusuke Nemoto
Mixed motives and linear forms in the Catalan constant.
Preprint, arXiv:2510.20648 [math.NT] (2026).

Bertolin's elliptico- toric Conjecture

C. Bertolin.
Périodes de 1-motifs et transcendence.
J. Number Theory 97 no. 2, 204-221 (2002).
doi 10.1016/S0022-314X(02)00002-1
Zbl 1067.11041 MR 1942957

C. Bertolin.
Third kind elliptic integrals and 1-motives. With a letter of Y. André and an appendix by M. Waldschmidt.
J. Pure Appl. Algebra 224, no.10, Article ID 106396 (2020).
doi 10.1016/j.jpaa.2020.106396 Zbl 1450.11077 MR 4093081

Elliptic and quasi elliptic functions


Waldschmidt, Michel
Schanuel property for elliptic and quasi-elliptic functions.
Hardy-Ramanujan J. 48, Paper No. 15601, 25 p. (2025).
Zbl 08204912


Bertolin, Cristiana; Waldschmidt, Michel
Variations on Schanuel's conjecture for elliptic and quasi-elliptic functions. I: The split case.
Hardy-Ramanujan J. 48, Paper No. 15602, 20 p. (2025).
Zbl 08204908


Bertolin, Cristiana; Waldschmidt, Michel
Variations on Schanuel's conjecture for elliptic and quasi-elliptic functions. II: The general case.
Work in progress.

Weierstrass functions and Serre functions

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ and $\mathcal{E}$ the elliptic curve $\mathbb{C}/\Omega$. Consider the following Weierstrass functions:

- The canonical product of Weierstrass associated with Ω:

$$\sigma(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

- Weierstrass zeta function $\zeta = \sigma'/\sigma$.
- Weierstrass elliptic function $\wp = -\zeta'$.

Further, for $q \in \mathbb{C} \setminus \Omega$, we introduce Serre function

$$f_q(z) = \frac{\sigma(z+q)}{\sigma(z)\sigma(q)} \mathrm{e}^{-\zeta(q)z}.$$

Elliptic integrals

• The periods of the Weierstrass elliptic function $\wp$ are elliptic integrals of the first kind.

• The Weierstrass zeta function ζ has quasi periods $\eta = \zeta(z+\omega) - \zeta(z)$ which are given by elliptic integrals of the second kind.

• Serre functions f_q were introduced in order to study elliptic integrals of the third kind.

For $\omega \in \Omega$,

• $\wp(z+\omega) = \wp(z)$,

• $\zeta(z+\omega) = \zeta(z) + \eta(\omega)$,

• $\sigma(z+\omega) = \epsilon(\omega)\sigma(z)\exp\left(\eta(\omega)\left(z+\frac{\omega}{2}\right)\right)$

with $\epsilon(\omega) = 1$ if $\omega/2 \in \Omega$, $\epsilon(\omega) = -1$ if $\omega/2 \notin \Omega$,

• $f_q(z+\omega) = f_q(z)\exp\left(\omega\zeta(q) - q\eta(\omega)\right).$

Algebraic groups

Consider the algebraic group G which is an extension of the elliptic curve $\mathcal{E}$ by the multiplicative group $\mathbb{G}_m$ parametrized by the point $Q = \exp_{\mathcal{E}^*}(q)$ of the dual elliptic curve $\mathcal{E}^*$, that we identify with $\mathcal{E}$.

The semi-elliptic exponential function of G (composed with a projective embedding) is

$$\begin{aligned}\exp_G : \mathrm{Lie} G_{\mathbb{C}} &\longrightarrow G_{\mathbb{C}}(\mathbb{C}) \subset \mathbb{P}^4(\mathbb{C}) \\ (z,t) &\longmapsto \sigma(z)^3 \Big[\wp(z) : \wp'(z) : 1 : \mathrm{e}^t f_q(z) : \\ &\qquad \mathrm{e}^t f_q(z) \Big(\wp(z) + \frac{\wp'(z) - \wp'(q)}{\wp(z) - \wp(q)} \Big) \Big].\end{aligned}$$

Masser, David
Auxiliary polynomials in number theory. Cambridge Tracts in Mathematics 207. Cambridge: Cambridge University Press 348 p. (2016). Zbl 1354.11002
Chapter 20, Exercise 20.104

Algebraic independence of elliptic functions

Theorem. *Let $t_1, \dots, t_r$ be complex numbers linearly independent over $\mathbb{Q}$. Let $q_1, \dots, q_s$ be complex numbers such that $\omega_1, \omega_2, q_1, \dots, q_s$ are linearly independent over $\mathbb{Q}$. Then the $s + r + 4$ functions*

$$z, \wp(z), \zeta(z), \sigma(z), \mathrm{e}^{t_1 z}, \dots, \mathrm{e}^{t_r z}, f_{q_1}(z), \dots, f_{q_s}(z)$$

are algebraically independent.

Almost all tuples

Theorem.

Let K be a finitely generated extension of $\mathbb{Q}$. Let $t_1, \ldots, t_r$ be complex numbers linearly independent over $\mathbb{Q}$. Let $q_1, \ldots, q_s$ be complex numbers such that $\omega_1, \omega_2, q_1, \ldots, q_s$ are linearly independent over $\mathbb{Q}$.

Then for almost all n–tuples $(z_1, \ldots, z_n)$ of complex numbers, the $n(s+r+4)$ numbers

$$z_i, \wp(z_i), \zeta(z_i), \sigma(z_i), \mathrm{e}^{t_1 z_i}, \ldots, \mathrm{e}^{t_r z_i}, f_{q_1}(z_i), \ldots, f_{q_s}(z_i)$$

$(i = 1, \ldots, n)$ are algebraically independent over $K(g_2, g_3)$.

A conjecture for all tuples

Let $s \geqslant 0$ and $n \geqslant 0$ be two integers. Let $t_1, \ldots, t_s$, $p_1, \ldots, p_n$ be complex numbers. We assume $p_i \notin \Omega$.
Let K be the field generated by the $2s + 6 + 3n + n(n+1)/2$ numbers

$$t_\ell, e^{t_\ell}, g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, p_i, \wp(p_i), \zeta(p_i), \ 1 \leqslant \ell \leqslant s, \ 1 \leqslant i \leqslant n,$$
$$f_{p_j}(p_i), \quad 1 \leqslant i \leqslant j \leqslant n.$$

Assume

- $2\pi\mathrm{i}, t_1, \ldots, t_s$ are $\mathbb{Q}$–linearly independent.
- $\omega_1, \omega_2, p_1, \ldots, p_n$ are $\mathbb{Q}$–linearly independent in the non–CM case
- $\omega_1, p_1, \ldots, p_n$ are k–linearly independent in the CM case.

Then

$$\operatorname{tran.deg} K \geqslant s + \frac{4}{[k:\mathbb{Q}]} + 2n + \frac{n(n+1)}{2}.$$

Two conjectures

Conjecture.

Let $\wp$ be a Weierstrass elliptic function with invariants g_2 and g_3 and without complex multiplication. Let $\{\omega_1, \omega_2\}$ be a basis of the lattice of periods of $\wp$ and let η_1 and η_2 be the corresponding quasi–periods. Then the transcendence degree of the field

$$\mathbb{Q}(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2)$$

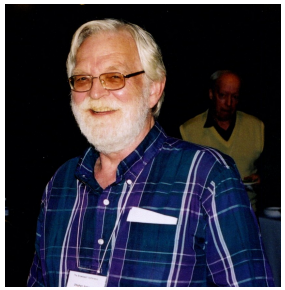
is $\geqslant 4$.

Conjecture.

The transcendence degree is 6 for a set of g_2, g_3 of full Lebesgue measure in $\mathbb{C}^2$.

Karl Weierstrass

1815 – 1897



Stephen Schanuel

1933 – 2014

Alexandre Grothendieck

1928 – 2014

Henri Lebesgue

1875 – 1941

Yves André

Cristiana Bertolin

Jean-Pierre Serre

With Senthil Kumar & R. Thangadurai