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Strategies of transcendence proofs

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2 *Several variables*

◇ Rank of matrices.

◇ Schwarz's Lemma: one variable;
Several variables; Cartesian products;
Bombieri; Hypersurfaces; Commutative algebra;
Seshadri's constant. Nagata's conjectures.

◇ Construction of auxiliary functions.
Masser's zero estimate. Exponentials in several variables.
The matrix coefficient conjecture.

◇ Duality. Fourier–Borel transform

◇ Dual of Baker's proof.
Matrices with entries linear combinations of logarithms.

◇ Interpolation determinants.
Slope inequalities - Bost, Chambert-Loir

Regulators

Let ℓ_{ij} ($1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant \ell$) be complex numbers such that $\alpha_{ij} := \mathrm{e}^{\ell_{ij}}$ are algebraic numbers. Write

$$\ell_{ij} := \log \alpha_{ij}.$$

Consider the $d \times \ell$ matrix

$$M := \bigl(\ell_{ij}\bigr)_{\substack{1 \leqslant i \leqslant d \\ 1 \leqslant j \leq \ell}}.$$

Assume that the rows of M are $\mathbb{Q}$–linearly independent, and that the columns of M are also $\mathbb{Q}$–linearly independent.

The conclusion of the Six Exponentials Theorem is that the rank of the matrix M is at least 2 if $\ell d > \ell + d$, the conclusion of the Four Exponentials Conjecture is the same under the assumption $\ell d \geqslant \ell + d$.

What should one expect?

A consequence of Schanuel's Conjecture is that $\mathbb{Q}$–*linearly independent logarithms of algebraic numbers are algebraically independent.*

Let $\lambda_1, \ldots, \lambda_k$ be a basis of the $\mathbb{Q}$–vector space spanned by the entries ℓ_{ij} of the matrix and let b_{ijh} be the rational numbers such that

$$\ell_{ij} = \sum_{h=1}^{k} b_{ijh}\lambda_h, \quad 1 \leqslant i \leqslant d,\ 1 \leqslant j \leq \ell.$$

Damien Roy defines the *structural rank* $\mathrm{rstr}(M)$ of the matrix M as the rank of the matrix

$$\left(\sum_{h=1}^{k} b_{ijh}X_h\right)_{\substack{1\leqslant i\leqslant d\\ 1\leqslant j\leq \ell}} \in \mathrm{Mat}_{d\times\ell}(\mathbb{Q}(X_1, \ldots, X_k)).$$

Hence the conjecture: *the rank of M should be* $\mathrm{rstr}(M)$.

Rank of matrices

A $d \times \ell$ matrix M has rank $\leqslant n$ if and only if there exists $x_1, \dots, x_d$ and $y_1, \dots, y_\ell$ in $\mathbb{C}^n$ such that

$$M = \bigl(x_i y_j\bigr)_{\substack{1 \leqslant i \leqslant d \\ 1 \leqslant j \leqslant \ell}}$$

where for u and v in $\mathbb{C}^n$, uv denotes the usual scalar product

$$(u_1, \dots, u_n)(v_1, \dots, v_n) = u_1 v_1 + \cdots + u_n v_n.$$

Assume that the numbers $\mathrm{e}^{x_i y_j}$ are algebraic. We wish to prove a lower bound for n (under suitable assumptions on linear independence on $x_1, \dots, x_d$ and $y_1, \dots, y_\ell$). We will consider a non–zero polynomial $P \in \mathbb{Z}[X_1, \dots, X_d]$ and introduce the entire function of n variables

$$F(z) = P(\mathrm{e}^{x_1 z}, \dots, \mathrm{e}^{x_d z})$$

at the points $z \in \mathbb{Z} y_1 + \cdots + \mathbb{Z} y_\ell$.

Rank of matrices

The first assumptions are that $x_1, \ldots, x_d$ are $\mathbb{Q}$–linearly independent (so that F is not the zero function) and that $y_1, \ldots, y_\ell$ are $\mathbb{Q}$–linearly independent (so that the points in $\mathbb{Z}y_1 + \cdots + \mathbb{Z}y_\ell$ are pairwise distinct).
Comparing with the proof of the six exponentials Theorem (the case $n = 1$), a natural approach would be to start with the existence of a polynomial P such that many numbers $F(s_1y_1 + \cdots + s_\ell y_\ell)$ vanish.
Then, by induction, we would wish to prove that all these numbers, with $(s_1, \ldots, s_\ell) \in \mathbb{Z}^\ell$, vanish.
A lower bound for n would easily follow.
So far we are not able to complete the proof following these lines, because of lack of a suitable Schwarz Lemma in several variables.

Schwarz's Lemma in one variable

Let f be an analytic function in a disc $|z| \leqslant R$ of $\mathbb{C}$, with at least N zeroes in a disc $|z| \leqslant r$ with $r < R$. Then

$$|f|_r \leqslant \left(\frac{3r}{R}\right)^N |f|_R$$

where

$$|f|_r = \sup_{|z|=r} |f(z)|.$$

When $R > 3r$, this improves the maximum modulus bound $|f|_r \leqslant |f|_R$.

Schwarz Lemma in several variables

The zeroes of an analytic function in several variables are not isolated. The number of zeroes is not relevant.

Three solutions:

- Cartesian products
- Lelong number
- Hypersufaces.

Cartesian products (grids)

Let f be an analytic function in a ball $|z| \leqslant R$ of $\mathbb{C}^n$. Assume f vanishes with multiplicity at least t on a set $\mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ where each $\mathcal{S}_i$ is contained in a disc $|z| \leqslant r$ with $r < R$ and has at least s elements.
Then

$$|f|_r \leqslant \left(\frac{3r}{R}\right)^{st} |f|_R.$$

Schwarz Lemma for Cartesian products can be proved by induction.

§4.3 of M.W. *Diophantine Approximation on Linear Algebraic Groups.* Grund. Math. Wiss. **326** Springer-Verlag (2000).

Cartesian products

Another proof, based on integral formulae, yields a weaker result : for $R > 3r$,

$$|f|_r \leqslant \left(\frac{R-3r}{2r}\right)^n \left(\frac{3r}{R}\right)^{st} |f|_R.$$

The conclusion follows from a homogeneity argument: replace f by f^N (and t by Nt) and let $N \to \infty$. (Landau's trick — book by Pólya and Szegő).

Chap. 7 of M.W. *Nombres transcendants et groupes algébriques.* Astérisque, **69–70** (1979).

Landau's trick

Pólya and Szegő mention that a rough estimate may sometimes be transformed into a sharper estimate by making appropriate use of the generality for which the original estimate is valid.

Pólya, George; Szegő, Gabor
Problems and theorems in analysis II. Theory of functions, zeros, polynomials, determinants, number theory, geometry. Transl. from the German by C. E. Billigheimer. Reprint of the 1976 English translation. Classics in Mathematics. Berlin: Springer. 392 p. (1998). Zbl 1024.00003

Schneider – Lang Theorem in several variables: Cartesian products (1941, 1966)

Let $f_1, \ldots, f_m$ *be meromorphic functions in* $\mathbb{C}^n$ *with* $m \geqslant n+1$*. Assume* $f_1, \ldots, f_{n+1}$ *are algebraically independent of finite order. Let* $\mathbb{K}$ *be a number field. Assume* $(\partial/\partial z_i) f_j$ *belongs to* $\mathbb{K}[f_1, \ldots, f_m]$ *for* $j = 1, \ldots, m$ *and* $i = 1, \ldots, n$*. Let*

$$\mathcal{S} = \{w \in \mathbb{C}^n \mid w \text{ not pole of } f_j,\ f_j(w) \in \mathbb{K}\ (j = 1, \ldots, m)\}$$

and let $e_1, \ldots, e_n$ *be a basis of* $\mathbb{C}^n$*. Then* $\mathcal{S}$ *does not contain the Cartesian product*

$$\{s_1 e_1 + \cdots + s_n e_n \mid (s_1, \ldots, s_n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n\}$$

where each $\mathcal{S}_i$ *is infinite.*

Schneider–Lang for Cartesian products

Schneider's Theorem on Euler's Beta function *Let a, b be rational numbers, not integers. Then the number*

$$\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental.

Th. Schneider and S. Lang: abelian functions and algebraic groups.

Generalizations of Baker's Theorem to algebraic groups.

Transcendence and linear independence of *periods* (as defined by Kontsevich – Zagier).

Baker, Alan; Wüstholz, Gisbert
Logarithmic forms and Diophantine geometry. New Mathematical Monographs 9. Cambridge: Cambridge University Press 198 p. (2007). `Zbl 1145.11004`

Schneider–Lang for Cartesian products

Bertrand – Masser: The Schneider – Lang Theorem for Cartesian products implies Baker's Theorem.
Assume

$$\ell_1 + \sqrt[3]{2}\ell_2 + \sqrt[3]{4}\ell_3 = 0,$$

where ℓ_1, ℓ_2, ℓ_3 are nonzero logarithms of algebraic numbers
By Gel'fond – Schneider Theorem, the three numbers ℓ_1, ℓ_2, ℓ_3 are linearly independent over $\mathbb{Q}$.
We multiply the given relation by $\sqrt[3]{2}$ and by $\sqrt[3]{4}$:

$$2\ell_3 + \sqrt[3]{2}\ell_1 + \sqrt[3]{4}\ell_2 = 0 \quad \text{and} \quad 2\ell_2 + 2\sqrt[3]{2}\ell_3 + \sqrt[3]{4}\ell_1 = 0.$$

Therefore the three functions e^{z_1}, e^{z_2} and $e^{\sqrt[3]{2}z_1 + \sqrt[3]{4}z_2}$ take algebraic values at the points $\mathbf{y}_1 = (\ell_2, \ell_3)$, $\mathbf{y}_2 = (\ell_1, \ell_2)$ and $\mathbf{y}_3 = (2\ell_3, \ell_1)$. By the Six Exponentials Theorem, $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ contains a basis of $\mathbb{C}^2$ over $\mathbb{C}$.

Bertrand, Daniel; Masser, David
Linear forms in elliptic integrals. Invent. Math. 58, 283-288 (1980).
Zbl 0425.10041

Schwarz Lemma in several variables

In order to deal with sets of zeroes which do not contain a cartesian product, Bombieri and Lang used the Lelong number.

In the p–adic case an analog is due to Serre, in connexion with the characters of the idèles class groups

Bombieri, Enrico; Lang, Serge
Analytic subgroups of group varieties. Invent. Math. 11, 1-14 (1970).
Zbl 0237.14015

Serre, Jean-Pierre
Dépendance d'exponentielles p–adiques. Théorie des Nombres, Sém. Delange-Pisot-Poitou 7 (1965/66), No. 15, 14 pp. (1967).
Zbl 0254.10033

Bombieri – Lang (1970)

Let f be an analytic function in a ball $|z| \leqslant R$ of $\mathbb{C}^n$. Assume f vanishes at N points z_i (counting multiplicities) in a ball $|z| \leqslant r$ with $r < R$. Assume $\min_{z_i \neq z_k} |z_i - z_k| \geqslant \delta$.

Then

$$|f|_r \leqslant \left(\frac{3r}{R}\right)^M |f|_R$$

with

$$M = N\left(\frac{\delta}{6r}\right)^{2n-2}.$$

Nagata's suggestion (1966)

Masayoshi Nagata (1927 – 2008)
In the conclusion of the Schneider – Lang Theorem, replace the fact that $\mathcal{S}$ does not contain a Cartesian product $\mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ where each $\mathcal{S}_i$ is infinite by the fact that $\mathcal{S}$ is contained in an algebraic hypersurface.

Bombieri's Theorem (1970)

Theorem.
Let $f_1, \dots, f_m$ be meromorphic functions in $\mathbb{C}^n$ with $m \geqslant n+1$. Assume $f_1, \dots, f_{n+1}$ are algebraically independent and of finite order. Let $\mathbb{K}$ be a number field. Assume $(\partial/\partial z_i) f_j$ belongs to $\mathbb{K}[f_1, \dots, f_m]$ for $j = 1, \dots, m$ and $i = 1, \dots, n$. Then the set

$$\mathcal{S} := \{w \in \mathbb{C}^n \mid f_j(w) \in \mathbb{K} \ (j = 1, \dots, m)\}$$

is contained in an algebraic hypersurface.

Bombieri, E.
Algebraic values of meromorphic maps.
Invent. Math. 10, 267-287 (1970); addendum ibid. 11, 163-166 (1970).
Zbl 0214.33702

L^2 – estimates of Hörmander

Existence theorem for the $\overline{\partial}$ operator.

Result of E. Bombieri, improved by H. Skoda.
Let φ be a plurisubharmonic function in $\mathbb{C}^n$ not identically $-\infty$. For any $\epsilon > 0$ there exists a nonzero entire function F such that

$$\int_{\mathbb{C}^n} |F(z)|^2 e^{-\varphi(z)}(1+|z|^2)^{-n-\epsilon} d\lambda(z) < \infty.$$

Example of a plurisubharmonic function: $c \log |f|$ where f is entire $\neq 0$ and $c \in \mathbb{R}$ a constant.

Bombieri's method yields a Schwarz lemma in several variables.

Schwarz lemma in several variables

Let $\mathcal{S}$ be a finite set of $\mathbb{C}^n$ and t a positive integer.

Denote by $\omega_t(\mathcal{S})$ the smallest degree of a polynomial vanishing at each point of $\mathcal{S}$ with multiplicity $\geqslant t$.

Let t be a positive integer. There exists a real number r such that for $R > r$, if f is an analytic function in the ball $|z| \leqslant R$ of $\mathbb{C}^n$ which vanishes with multiplicity at least t at each point of $\mathcal{S}$, then

$$|f|_r \leqslant \left(\frac{e^n r}{R}\right)^{\omega_t(\mathcal{S})} |f|_R.$$

The exponent $\omega_t(\mathcal{S})$ cannot be improved: take for f a non–zero polynomial of degree $\omega_t(\mathcal{S})$.

Upper bound for $\omega_t(\mathcal{S})$: linear algebra

Given a finite subset $\mathcal{S}$ of $\mathbb{C}^n$ and a positive integer t, if D is a positive integer such that

$$|\mathcal{S}|\binom{t+n-1}{n} < \binom{D+n}{n},$$

then

$$\omega_t(\mathcal{S}) \leqslant D.$$

Consequence:

$$\omega_t(\mathcal{S}) \leqslant (t+n-1)|\mathcal{S}|^{1/n}.$$

For instance $\omega_1(\mathcal{S}) \leqslant n|\mathcal{S}|^{1/n}$. Also $\omega_t(\mathcal{S}) \leqslant t\omega_1(\mathcal{S})$.

The invariant $\Omega(\mathcal{S})$

Since

$$\omega_{t_1+t_2}(\mathcal{S}) \leqslant \omega_{t_1}(\mathcal{S}) + \omega_{t_2}(\mathcal{S}),$$

by Fekete's lemma the sequence

$$\left(\frac{1}{t}\omega_t(\mathcal{S})\right)_{t \geqslant 1}$$

has a limit $\Omega(\mathcal{S})$ as $t \to \infty$ and $\Omega(\mathcal{S}) = \inf_{t \geqslant 1} \omega_t(S)/t$.

Consequence of the L^2 – estimate for the $\overline{\partial}$ operator:

$$\frac{1}{n}\omega_1(\mathcal{S}) \leqslant \Omega(\mathcal{S}) \leqslant \omega_1(\mathcal{S}).$$

The invariant $\Omega(\mathcal{S})$

Connections with
• A Conjecture of Nagata related with his solution of Hilbert's Problem 14
• Seshadri's Constant.

Further works by
G.V. Chudnovskiĭ,
H. Skoda, J-P. Demailly, A. Azhari,
H. Esnault and E. Viehweg, A. Hirschowitz,
B. Harbourne, C. Bocci, L. Ein, R. Lazarfeld, K.E. Smith,
M. Hochster, C. Huneke, M. Dumnicki, T. Szemberg,
H. Tutaj-Gasińska, T. Bauer, M. Baczyńska, A. Habura,
G. Malara, J. Szpond, S.M. Cooper, E. Guardo, L. Farnik.

Zero estimate for $Y(S)$

Let $Y = \mathbb{Z}y_1 + \cdots + \mathbb{Z}y_\ell$ be a finitely generated subgroup of $\mathbb{C}^n$. For $S > 0$, let

$$Y(S) = \{s_1 y_1 + \cdots + s_\ell y_\ell \mid 0 \leqslant s_j \leqslant S\}.$$

Easy: $\omega_1(Y(S)) \leqslant n(S+1)^{\ell/n}$.

The *Generalized Dirichlet exponent* of Y is defined as

$$\mu(Y) = \min_{V \subset \mathbb{C}^n} \frac{\mathrm{rang}_{\mathbb{Z}}(s_V(Y))}{\dim(\mathbb{C}^n/V)}$$

where V runs over the vector subspaces of $\mathbb{C}^n$ distinct form $\mathbb{C}^n$ and $s_V : \mathbb{C}^n \to \mathbb{C}^n/V$ is the canonical surjective map.

Hence

$$\omega_1(Y(S)) \leqslant c(Y) S^{\mu(Y)}.$$

Zero estimate for $Y(S)$

Theorem (D.W. Masser).

$$\omega_1(Y(S)) \geqslant (S/n)^{\mu(Y)}.$$

Masser, D. W.
On polynomials and exponential polynomials in several complex variables. Invent. Math. 63, 81-95 (1981).
Zbl 0436.32005

Schwarz's Lemma for $Y(S)$

Corollary.

There exists a constant c *real number* r **depending on** S *such that for* $R > r$, *if* f *is an analytic function in the ball* $|z| \leqslant R$ *of* $\mathbb{C}^n$ *which vanishes at each point of* $Y(S)$, *then*

$$|f|_r \leqslant \left(\frac{cr}{R}\right)^{(S/n)^{\mu(Y)}} |f|_R.$$

Question: does there exists r independent of S?
A consequence would be

$$\mu(Y) \leqslant \frac{d}{d-n},$$

hence, (under suitable assumptions on the x_i and y_j),

$$n \geqslant \frac{\ell d}{\ell + d}.$$

Schwarz's Lemma for $Y(S)$

My Conjecture 7.1.10 in § 7 of the reference below, according to which there exists r independent of S, was disproved by Damien Roy.

Waldschmidt, Michel
Nombres transcendants et groupes algébriques. Complété par deux appendices de Daniel Bertrand et Jean-Pierre Serre. 2e éd. Astérisque, 69/70 (1987).
Zbl 0621.10022

Roy., Damien
Interpolation formulas and auxiliary functions. J. Number Theory 94, No. 2, 248-285 (2002).
Zbl 1010.11039

Another strategy

Waldschmidt, Michel
Transcendance et exponentielles en plusieurs variables. Invent. Math. 63, 97-127 (1981). Zbl 0454.10020

Michel Waldschmidt
Diophantine approximation on linear algebraic groups. Transcendence properties of the exponential function in several variables, *Grundlehren Math. Wiss.* **326** Berlin: Springer, 2000.
Zbl 1467843 MR 1756786 doi 10.1007/978-3-662-11569-5

Auxiliary function

Proposition.

Let L and n be positive integers, N, U, V, R, r positive real numbers and $\varphi_1, \ldots, \varphi_L$ entire functions in $\mathbb{C}^n$. Define $W = N + U + V$ and assume

$$W \geqslant 12n^2, \quad e \leqslant \frac{R}{r} \leqslant e^{W/6}, \quad \sum_{\lambda=1}^{L} |\varphi_\lambda|_R \leqslant e^U$$

and

$$(2W)^{n+1} \leqslant LN\big(\log(R/r)\big)^n.$$

Then there exist rational integers $p_1, \ldots, p_L$, with

$$0 < \max_{1 \leq \lambda \leqslant L} |p_\lambda| \leqslant e^N,$$

such that the function $F = p_1\varphi_1 + \cdots + p_L\varphi_L$ satisfies

$$|F|_r \leqslant e^{-V}.$$

Thue–Siegel Lemma

Let X be a positive integer, U, V be positive real numbers and u_{ij} $(1 \leqslant i \leqslant \nu,\ 1 \leqslant j \leq \mu)$ be complex numbers. Assume

$$\sum_{i=1}^{\nu} |u_{ij}| \leqslant e^{U}, \qquad (1 \leqslant j \leq \mu)$$

and

$$\left(\sqrt{2}Xe^{U+V}+1\right)^{2\mu} \leqslant (X+1)^{\nu}.$$

Then there exists $(\xi_1, \ldots, \xi_\nu) \in \mathbb{Z}^{\nu}$ satisfying

$$0 < \max_{1 \leqslant i \leq \nu} |\xi_i| \leqslant X$$

and

$$\max_{1 \leqslant j \leqslant \mu} \left| \sum_{i=1}^{\nu} u_{ij}\xi_i \right| \leqslant e^{-V}.$$

Interpolation Formula

Let r and R be real numbers satisfying $0 < r < R$, T a positive integer, and F an entire function in $\mathbb{C}^n$. Then

$$|F|_r \leq (1+\sqrt{T}) \left(\frac{r}{R}\right)^T |F|_R + \sum_{\|\underline{\tau}\|<T} |D^{\underline{\tau}} F(0)| \frac{r^{\|\underline{\tau}\|}}{\underline{\tau}!}.$$

Consequence
If the the numbers $|D^{\underline{\tau}} F(0)|$ for all $\|\underline{\tau}\| < T$ are small, then $|F|_r$ is small.

Auxiliary function: sketch of proof

Let T be sufficiently large. Consider the linear system of $\binom{T+n-1}{n} \leqslant T^n$ inequalities:

$$\left| \sum_{\lambda=1}^{L} p_\lambda D^{\underline{\tau}} \varphi_\lambda(0) \right| \leqslant \mathrm{e}^{-V_1}, \qquad (\underline{\tau} \in \mathbb{N}^n, \ \|\underline{\tau}\| < T).$$

From the Thue–Siegel Lemma we deduce that there exists a nontrivial solution $(p_1, \ldots, p_L) \in \mathbb{Z}^L$ with $\max_{1 \leq \lambda \leqslant L} |p_\lambda| \leqslant e^N$. The function $F = p_1\varphi_1 + \cdots + p_L\varphi_L$ then satisfies

$$\sum_{\|\underline{\tau}\| < T} |D^{\underline{\tau}} F(0)| \frac{r^{\|\underline{\tau}\|}}{\underline{\tau}!} \leqslant \mathrm{e}^{-V_2}.$$

The Interpolation Formula provides the conclusion

$$|F|_r \leqslant \mathrm{e}^{-V}.$$

Consequence

Given any $x_1, \ldots, x_d$ in $\mathbb{C}^n$, for sufficiently large T, there exists a nonzero polynomial P such that the function

$$F(z) = P(\mathrm{e}^{x_1 z}, \ldots, \mathrm{e}^{x_d z})$$

satisfies (for some r and c_1)

$$|F|_r \leqslant \mathrm{e}^{-c_1 T^{d/n}}.$$

Assume $\mathrm{e}^{x_i y_j}$ are algebraic $(1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant \ell)$. If

$$|F(s_1 y_1 + \cdots + s_\ell y_\ell)| \leqslant \mathrm{e}^{-c_2 TS},$$

then $F(s_1 y_1 + \cdots + s_\ell y_\ell) = 0$.
One needs more equations than unknowns: $S^\ell > c_3 T^d$. This gives the requirement

$$c_3 T^{d/\ell} < S < c_4 T^{(d/n)-1}, \qquad \frac{d}{\ell} < \frac{d}{n} - 1,$$

i.e. $d\ell > n(d+\ell)$.

Masser's Zero Estimate

Course in Paris, spring 1980

The previous construction shows the existence of a nonzero polynomial such that $P(\mathrm{e}^{x_1 z}, \ldots, \mathrm{e}^{x_d z})$ vanishes at $Y(S)$.

The existence of an auxiliary function gives an upper bound for the degree, and Masser's Zero Estimate for exponential polynomials gives a lower bound for this degree.

Masser, D. W.
On polynomials and exponential polynomials in several complex variables. Invent. Math. 63, 81-95 (1981).
Zbl 0436.32005

Consequence

Square case:
A matrix M with entries logarithms of algebraic numbers has rank $\geqslant \mathrm{rstr}(M)/2$.

Non square case:
If a $d \times \ell$ matrix M with entries logarithms of algebraic numbers has rank $< \ell d/(d+\ell)$, then there exist two regular matrices P and Q with rational coefficients such that

$$PMQ = \begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix}$$

where the 0 bloc is a $d' \times \ell'$ matrix with

$$\frac{d'}{d} + \frac{\ell'}{\ell} > 1.$$

The matrix coefficient conjecture

Conjecture.

Let M be a square matrix with entries logarithms of algebraic numbers and zero determinant. Then there exist u and v in $\mathbb{Q}^n \smallsetminus \{0\}$ such that $uMv = 0$.

Samit Dasgupta
Introduction to Transcendence Theory. Course Math 790, March 2021, Duke University,

Dasgupta, Samit
Ranks of matrices of logarithms of algebraic numbers. I: The theorems of Baker and Waldschmidt-Masser. Essent. Number Theory 2, No. 1, 93-138 (2023).
Zbl 1543.11059

Dasgupta, Samit; Kakde, Mahesh
Ranks of matrices of logarithms of algebraic numbers. II: The matrix coefficient conjecture. Adv. Math. 487, Article ID 110753, 19 p. (2026).
Zbl 08155756

The matrix coefficient conjecture

The matrix coefficient conjecture is a consequence of the conjecture that the rank of M is equal to its structural rank, and it implies a number of conjectures on nonvanishing of regulators (Leopoldt, Gross–Kuz'min).

New tool: Instead of the degree, Samit Dasgupta and Mahesh Kakde introduce the number of nonzero coefficients of the polynomial. They replace Masser's Zero estimate with a result involving a theorem of Bernstein–Kushnirenko.

Fourier–Borel transform

Let $H(\mathbb{C})$ denote the space of entire functions in $\mathbb{C}$ and $H'(\mathbb{C})$ the space of *analytic functionals*, namely linear maps $\eta : H(\mathbb{C}) \to \mathbb{C}$ such that there exists a constant $C = C(\eta)$ such that, for any entire function F,

$$\eta(F) \leqslant C|F|_R.$$

For $F(z) = \sum_{n\geqslant 0} b_n z^n$, we have $\eta(F) = \sum_{n\geqslant 0} b_n \eta(z^n)$.

The Fourier – Borel transform of η is the entire function of finite exponential type

$$\mathcal{F}_\eta(u) = \eta(\mathrm{e}^z) = \sum_{n\geqslant 0} \eta(z^n)\frac{u^n}{n!}\cdot$$

Duality: Fourier – Borel

Conversely, given an entire function

$$\Phi(u) = \sum_{n \geqslant 0} a_n \frac{u^n}{n!}$$

of finite exponential type, i.e. such that there exists C and R positive with

$$|\Phi(u)| \leqslant C \mathrm{e}^{Ru}$$

for all $u \in \mathbb{C}$, one defines an analytic functional $\mathcal{H}_\Phi \in H'(\mathbb{C})$ by

$$\mathcal{H}_\Phi(F) = \sum_{n \geqslant 0} a_n b_n \quad \text{for } F(z) = \sum_{n \geqslant 0} b_n z^n.$$

The maps $\eta \mapsto \mathcal{F}_\eta$ and $\Phi \to \mathcal{H}_\Phi$ are inverse bijective maps from $H'(\mathbb{C})$ to the set of entire function of finite exponential type.

Duality: Fourier – Borel

- For $v \in \mathbb{C}$, the Fourier – Borel transform of $F \mapsto F(v)$ is $\Phi(u) = \mathrm{e}^{vu}$.

- For $\eta \in H'(\mathbb{C})$, the Fourier – Borel transform of $\eta \circ (\mathrm{d}/\mathrm{d}z) : F \mapsto \eta(\mathrm{d}F/\mathrm{d}z)$ is $u\mathcal{F}_\eta(u)$.

- For $\eta \in H'(\mathbb{C})$, the Fourier – Borel transform of $\eta \circ z : F \mapsto \eta(zF)$ is $(\mathrm{d}/\mathrm{d}u)\mathcal{F}_\eta(u)$.

Michel Waldschmidt
La transformation de Fourier-Borel: une dualité en transcendance.
Troisième Rencontre Internationale Mathématique de Delphes, 29 septembre 1989.
Groupe d'études sur les problèmes diophantiens 1988-1989,
Publ. Math. Univ. P. et M. Curie, 90 N°8.

https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/DelphesFourierBorel.pdf

Duality: Fourier – Borel — Gel'fond – Schneider

For s and t non negative integers and $v \in \mathbb{C}$, the Fourier – Borel transform of the analytic functional

$$\eta : F \mapsto \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\sigma} (z^{\tau} F)_{z=v}$$

is

$$\mathcal{F}_{\eta}(u) = \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{\tau} (u^{\sigma} \mathrm{e}^{vu}) \, .$$

In other words

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\sigma} (z^{\tau} \mathrm{e}^{tz})(s) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\tau} (z^{\sigma} \mathrm{e}^{sz})(t).$$

This common value is

$$\left(\frac{\partial^{\sigma+\tau}}{(\partial z_1)^{\sigma} (\partial z_2)^{\tau}} \mathrm{e}^{(z_1+s)(z_2+t)}\right)_{(z_1,z_2)=(0,0)} .$$

Baker's Theorem by Schneider's method

Assume

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n.$$

Recall Baker's method: consider the functions

$$z_0, \mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_{n-1}}, \mathrm{e}^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$$

and their derivatives at the points in

$$\mathbb{Z}(1, \log \alpha_1, \ldots, \log \alpha_{n-1}).$$

The *dual* proof, generalizing Schneider's method to several variables, deals with $n+1$ functions of n variables

$$z_0, z_1, \ldots, z_{n-1}, \mathrm{e}^{z_0} \alpha_1^{z_1} \cdots \alpha_{n-1}^{z_{n-1}}.$$

The auxiliary function is evaluated at the points

$$(t\beta_0, \tau_1 + t\beta_1, \ldots, \tau_{n-1} + t\beta_{n-1}), \; (\tau_1, \ldots, \tau_{n-1}, t) \in \mathbb{Z}^n$$

and only the powers of one derivative, $\partial / \partial z_0$, is used.

Example 1: $\beta_0 = 0$, $n = 3$, $\alpha_1^{\beta_1}\alpha_2^{\beta_2} = \alpha_3$ (no z_0)

Baker

$$f_{\tau_1,\tau_2,t}(z_1, z_2) = \mathrm{e}^{(\tau_1+t\beta_1)z_1}\mathrm{e}^{(\tau_2+t\beta_2)z_2}$$

Schneider

$$g_{\sigma_1,\sigma_2,s}(z_1, z_2) = z_1^{\sigma_1} z_2^{\sigma_2}(\alpha_1^{z_1}\alpha_2^{z_2})^s$$

Duality

$$\left(\frac{\partial}{\partial z_1}\right)^{\sigma_1}\left(\frac{\partial}{\partial z_2}\right)^{\sigma_2} f_{\tau_1,\tau_2,t}(s\log\alpha_1, s\log\alpha_2)$$

$$=$$

$$(\tau_1 + t\beta_1)^{\sigma_1}(\tau_2 + t\beta_2)^{\sigma_2}(\alpha_1^{\tau_1}\alpha_2^{\tau_2}\alpha_3^{t})^s$$

$$=$$

$$g_{\sigma_1,\sigma_2,s}(\tau_1 + t\beta_1, \tau_2 + t\beta_2).$$

Example 2: $\beta_0 \neq 0$, $n = 2$, $\mathrm{e}^{\beta_0}\alpha_1^{\beta_1} = \alpha_2$

Baker:

$$f_{\tau_0,\tau_1,t}(z_0, z_1) = z_0^{\tau_0}\mathrm{e}^{\beta_0 z_0 t}\mathrm{e}^{(\tau_1+t\beta_1)z_1}.$$

Schneider:

$$g_{\sigma_0,\sigma_1,s}(z_0, z_1) = z_0^{\sigma_0} z_1^{\sigma_1}(\mathrm{e}^{z_0}\alpha_1^{z_1})^s.$$

Duality:

$$\frac{\partial^{\sigma_0+\sigma_1}}{(\partial z_0)^{\sigma_0}(\partial z_1)^{\sigma_1}} f_{\tau_0,\tau_1,t}(s, s\log\alpha_1) =$$

$$\left(\frac{\partial}{\partial z_0}\right)^{\sigma_0} (z_0^{\tau_0}\mathrm{e}^{\beta_0 z_0 t})(s)(\tau_1 + t\beta_1)^{\sigma_1}(\alpha_1^{\tau_1}\alpha_2^{t})^s =$$

$$\left(\frac{\partial}{\partial z_0}\right)^{\tau_0} (z_0^{\sigma_0}\mathrm{e}^{s z_0})(t\beta_0)(\tau_1 + t\beta_1)^{\sigma_1}(\alpha_1^{\tau_1}\alpha_2^{t})^s =$$

$$\left(\frac{\partial}{\partial z_0}\right)^{\tau_0} g_{\sigma_0,\sigma_1,s}(t\beta_0, \tau_1 + t\beta_1).$$

Dual proofs

Instead of an auxiliary function, one can construct an auxiliary functional. This introduces exponential polynomials for studying transcendence in algebraic groups. The zero estimate is replaced by an interpolation estimate (again due to Masser).

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Zbl 0742.11035 Zbl 0742.11036

https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/FAFA2JAnalyseMath.pdf

Matrices with entries linear combinations of logarithms of algebraic numbers. D. Roy

A matrix M, the entries of which are linear combinations with algebraic coefficients of 1 and logarithms of algebraic numbers, has rank $\geqslant \mathrm{rstr}(M)/2$.

More generally:

If such a $d \times \ell$ matrix M has rank $< \ell d/(d+\ell)$, then there exist two regular matrices P and Q with algebraic coefficients such that

$$PMQ = \begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix}$$

where the 0 bloc is a $d' \times \ell'$ matrix with

$$\frac{d'}{d} + \frac{\ell'}{\ell} > 1.$$

Interpolation determinants – one variable

Let r and R be two real numbers with $0 < r \leqslant R$, $\varphi_1, \ldots, \varphi_L$ be functions of one complex variable, which are analytic in the disc $|z| \leqslant R$ of $\mathbb{C}$, and let $\zeta_1, \ldots, \zeta_L$ belong to the disc $|z| \leqslant r$. Then the absolute value of the determinant

$$\Delta = \begin{pmatrix} \varphi_1(\zeta_1) & \ldots & \varphi_L(\zeta_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(\zeta_L) & \ldots & \varphi_L(\zeta_L) \end{pmatrix}$$

is bounded from above by

$$|\Delta| \leq \left(\frac{R}{r}\right)^{-L(L-1)/2} L! \prod_{\lambda=1}^{L} |\varphi_\lambda|_R.$$

Interpolation determinants – one variable

`Hint:` The function

$$\Delta(z) = \begin{pmatrix} \varphi_1(\zeta_1 z) & \dots & \varphi_L(\zeta_1 z) \\ \vdots & \ddots & \vdots \\ \varphi_1(\zeta_L z) & \dots & \varphi_L(\zeta_L z) \end{pmatrix}$$

has a zero of multiplicity at least $L(L-1)/2$ at the origin.

Michel Laurent
Sur quelques résultats récents de transcendance. In *Journées arithmétiques.* Exp. Congr., Luminy France, 17-21 Juillet, 1989, *Astérisque* **198-200**, 209-230. Société Mathématique de France (SMF), Paris, 1991.
`Zbl 65929` `MR 1144324`
numdam.org: $AST\_1991\_\_198-199-200\_\_209\_0$

Interpolation determinants – several variables

Let $n \geqslant 2$ and let L be a sufficiently large integer. Let r and R be two real numbers with $0 < r \leqslant R$, $\varphi_1, \ldots, \varphi_L$ be functions of n variables, which are analytic in $|z| \leqslant R$ and let $\zeta_1, \ldots, \zeta_L$ belong to the disc $|z| \leqslant r$. Then the absolute value of the determinant

$$\Delta = \begin{pmatrix} \varphi_1(\zeta_1) & \ldots & \varphi_L(\zeta_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(\zeta_L) & \ldots & \varphi_L(\zeta_L) \end{pmatrix}$$

is bounded from above by

$$|\Delta| \leq \left(\frac{R}{r}\right)^{-(1/2)L^{1+(1/n)}} L! \prod_{\lambda=1}^{L} |\varphi_\lambda|_R.$$

Interpolation determinants – several variables

`Hint:` The function of a single variable z

$$\Delta(z) = \begin{pmatrix} \varphi_1(\zeta_1 z) & \dots & \varphi_L(\zeta_1 z) \\ \vdots & \ddots & \vdots \\ \varphi_1(\zeta_L z) & \dots & \varphi_L(\zeta_L z) \end{pmatrix}$$

has a zero of multiplicity at least $(1/2)L^{1+(1/n)}$ at the origin.

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Zbl 0809.11038

Slope inequalities Arakelov geometry

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Périodes et isogénies des variétés abéliennes sur les corps de nombres [d'après D. Masser et G. Wüstholz]. Séminaire Bourbaki. Volume 1994/95. Exposés 790-804. Paris: Société Mathématique de France, Astérisque. 237, 115-161, Exp. No. 795 (1996).
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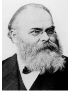


Hermann Amandus Schwarz

1843 – 1921

Enrico Bombieri

Masayoshi Nagata

1927 – 2008

Conjeevaram Srirangachar Seshadri

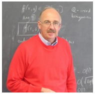
1932 – 2020

Joseph Fourier

1768 – 1830

Émile Borel

1871 – 1956



David Masser

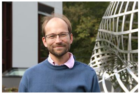
Samit Dasgupta

Mahesh Kakde

Michel Laurent

Damien Roy

Jean–Benoit Bost



Antoine Chambert-Loir