Integer–valued functions, Hurwitz functions
and related topics: a survey

by

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Abstract

An integer–valued function is an entire function which maps the nonnegative integers \( \mathbb{N} \) to the integers. An example is \( 2^z \). A Hurwitz function is an entire function having all derivatives taking integer values at 0. An example is \( e^z \).

Lower bound for the growth order of such functions have a rich history. Many variants have been considered: for instance, assuming that the first \( k \) derivatives at the integers are integers, or assuming that the derivatives at \( k \) points are integers. These as well as and many other variants have been considered. We survey some of them.

Contents

1 On the sequence of values \( (f(0), f(1), f(2), \ldots) \) 2
  1.1 Order and type of an entire function 2
  1.2 Functions vanishing at 0, 1, 2, \ldots 2
  1.3 Integer–valued functions 3
  1.4 Completely integer–valued functions 4
  1.5 Further results 6

2 On the sequence of values \( (f(0), f'(0), f''(0), \ldots) \) 6
  2.1 Hurwitz functions 6
  2.2 Refined estimates by Sato and Straus 7

3 Several points and / or several derivatives 7
  3.1 Introduction 7
  3.2 \( k \)–times integer–valued functions 9
  3.3 \( k \)–point Hurwitz functions 9
  3.4 Utterly integer–valued functions 10
  3.5 Abel interpolation 11

4 Variants 12

5 Connection with transcendental number theory 12
  5.1 From \( \mathbb{Z} \) to \( \mathbb{Z}[i] \) 12
  5.2 Transcendence 13
1 On the sequence of values \((f(0), f(1), f(2), \ldots)\)

The topic of integer-valued entire functions was initiated by Pólya’s fundamental result on transcendental entire functions taking their values in \(\mathbb{Z}\) at each point in \(\{0, 1, 2, \ldots\}\).

1.1 Order and type of an entire function

Let \(f\) be an entire function. For \(r \geq 0\) we define \(|f|_r = \sup_{|z| \leq r} |f(z)|\). From the maximum modulus principle we deduce \(|f|_r = \sup_{|z|=r} |f(z)|\).

The order of an entire function \(f\) is

\[
\varrho(f) = \limsup_{r \to \infty} \frac{\log \log |f|_r}{\log r},
\]

while the exponential type of an entire function is

\[
\tau(f) = \limsup_{r \to \infty} \frac{\log |f|_r}{r} = \limsup_{n \to \infty} \left| f^{(n)}(z_0) \right|^{1/n} \quad (z_0 \in \mathbb{C}).
\]

We use the notation:

\[
f^{(n)}(z) = \frac{d^n}{dz^n} f(z).
\]

If the exponential type is finite, then \(f\) has order \(\leq 1\). If \(f\) has order \(< 1\), then the exponential type is \(0\). For \(\tau \in \mathbb{C} \setminus \{0\}\), the function \(e^{\tau z}\) has order 1 and exponential type \(|\tau|\).

For \(\varrho > 0\), we define

\[
\tau_{\varrho}(f) = \limsup_{r \to \infty} \frac{\log |f|_r}{r^{\varrho}},
\]

so that \(\tau_1(f) = \tau(f)\). If \(f\) has order \(< \varrho\), then \(\tau_{\varrho}(f) = 0\). If \(f\) has order \(> \varrho\), then \(\tau_{\varrho}(f) = +\infty\).

1.2 Functions vanishing at 0, 1, 2, \ldots

Let us consider the entire functions which vanish at each point in \(\mathbb{N} = \{0, 1, 2, \ldots\}\). An example of such a function is the Weierstrass canonical product for \(\mathbb{N}\), namely (Titchmarsh, 1939, § 4.41 and § 8.4.(vi)), (Shorey, 2019, Chap. 6)

\[
\frac{1}{\Gamma(-z)} = -z e^{-\gamma z} \prod_{m=1}^{\infty} \left(1 - \frac{z}{m}\right) e^{z/m}
\]
(Hadamard product for the Gamma function). This is an entire function of order 1 and infinite type \cite{Gelfond1952} (1.3 (17)). The “smallest” entire functions vanishing at each point in \( \mathbb{N} \) is
\[
\sin(\pi z) = \pi z \prod_{m \in \mathbb{Z}, \{0\}} \left( 1 - \frac{z}{m} \right) e^{z/m}
\]
which in fact vanishes at each point in \( \mathbb{Z} \) (Weierstrass canonical product for \( \mathbb{Z} \), Hadamard product for the sine function), which has order 1 and exponential type \( \pi \). See \cite{Titchmarsh1939,§3.23 and §8.4.(v)}, and \cite{Boas1954,§9.4}. Indeed, a well–known theorem from the thesis of F. Carlson in 1914 using the Phragmén–Lindelöf principle states that there is no nonzero entire function \( f \) of exponential type \( <\pi \) satisfying \( f(\mathbb{N}) = \{0\} \) – see for instance \cite{Titchmarsh1939,§5.81}, \cite{Gelfond1952,§3 Th. VII and Chap. 3, §3.1}, \cite{Boas1954, Th. 9.2.1}, \cite{BoasBuck1964, Chap. IV §19}.

1.3 Integer–valued functions

An \textit{integer–valued function} is an entire function \( f \) such that \( f(n) \in \mathbb{Z} \) for all \( n \geq 0 \). A basis of the \( \mathbb{Z} \)–module of polynomials in \( \mathbb{C}[z] \) which map \( \mathbb{N} \) to \( \mathbb{Z} \) is given by the polynomials
\[
1, z, \frac{z(z-1)}{2}, \ldots, \frac{z(z-1) \cdots (z-m+1)}{m!}, \ldots
\]
\cite{Narkiewicz1995}, \cite{CahenChabert1997}, which are sometimes called (in papers from transcendental number theory) the Feldman’s polynomials, after N.I. Feldman introduced them for improving Baker’s lower bounds for linear forms in logarithms of algebraic numbers – see for instance \cite{Bugeaud2018}.

The simplest example of a transcendental integer–valued function is \( 2^z \). In 1915, in his seminal paper \cite{Polya1915}, G. Pólya proved that if \( f \) is an entire function such that
\[
\lim_{r \to \infty} |f(r)| 2^{-r} \sqrt{r} = 0,
\]
then \( f \) is a polynomial.

The method of Pólya is to expand \( f \) as an interpolation series
\[
f(z) = a_0 + a_1 z + a_2 \frac{z(z-1)}{2} + \cdots + a_m \frac{z(z-1) \cdots (z-m+1)}{m!} + \cdots
\]
and to estimate the coefficients \( a_m \). See for instance \cite[Th. 9.12.1]{Boas1954}.

Pólya conjectured that the conclusion of his theorem should be true under the weaker assumption
\[
\lim_{r \to \infty} |f(r)| 2^{-r} = 0.
\]
This refinement was achieved by [Hardy, 1917]. See also [Landau, 1920] and [Whittaker, 1935, § 11 Th. 11]. In [Pólya, 1920], the same conclusion is obtained under the weaker assumption
\[
\limsup_{r \to \infty} |f| r^{-2} < 1;
\]
further, if \( |f| r = O(2^r r^k) \) for some \( k > 0 \), then \( f(z) \) is of the form \( P(z) 2^z + Q(z) \) where \( P \) and \( Q \) are polynomials with rational coefficients. It is interesting to note that there are only countably many such functions.

Twenty years later, [Selberg, 1941b] proved that if an integer–valued function \( f \) satisfies
\[
\tau(f) \leq \log 2 + \frac{1}{1500},
\]
then \( f(z) \) is of the form \( P(z) 2^z + Q(z) \). [Pisot, 1942] went one step further. If an integer–valued function \( f \) has exponential type \( \leq 0.8 \), then \( f \) is of the form
\[
P_0(z) + 2^z P_1(z) + \gamma^z P_2(z) + \bar{\gamma}^z P_3(z),
\]
where \( P_0, P_1, P_2, P_3 \) are polynomials and
\[
\gamma = \frac{3 + i\sqrt{3}}{2}, \quad \bar{\gamma} = \frac{3 - i\sqrt{3}}{2}
\]
are the roots of the polynomial \( z^2 - 3z + 3 \). See [Boas, 1954, Th. 9.12.2], [Buck, 1948a]. This contains the result of Selberg, since \( |\log \gamma| = 0.758 \cdots > \log 2 = 0.693 \cdots \). Pisot [Pisot, 1946b] obtained more general results for functions of exponential type \( < 0.9934 \cdots \) with additional terms; he also investigated the growth of transcendental entire functions \( f \) with values \( f(n) \) close to integers: \( f(n) = u_n + \epsilon_n \) with \( u_n \in \mathbb{Z}, |\epsilon_n| < \kappa^n \) for sufficiently large \( n \) and \( 0 < \kappa < 1 \). Functions which are almost integer-valued were already considered by [Gelfond, 1929c] and by [Buck, 1946, Th. 6.3]. Besides, [Buck, 1946, Th. 7.1] considered the problem of transcendental entire functions taking prime values at the integers.

### 1.4 Completely integer–valued functions

A complete integer–valued function is an entire function which takes values in \( \mathbb{Z} \) at all points in \( \mathbb{Z} \).

Let \( u > 1 \) be a quadratic unit, root of a polynomial \( X^2 + aX + 1 \) for some \( a \in \mathbb{Z} \). Then the functions
\[
u^z + u^{-z} \quad \text{and} \quad \frac{u^z - u^{-z}}{u - u^{-1}}
\]
are completely integer–valued functions of exponential type \( \log u \).
Examples of such quadratic units are the roots of the polynomial $X^2 - 3X + 1$:

$$\theta = \frac{3 + \sqrt{5}}{2}, \quad \theta^{-1} = \frac{3 - \sqrt{5}}{2}.$$ 

Hence, examples of completely integer–valued functions are

$$\theta^z + \theta^{-z} \quad \text{and} \quad \frac{1}{\sqrt{5}}(\theta^z - \theta^{-z}),$$

both of exponential type $\log \theta = 0.962\,423\ldots$

Notice that $\theta = \phi^2$ where $\phi$ is the Golden ratio $\frac{1 + \sqrt{5}}{2}$. Let $\tilde{\phi} = -\phi^{-1}$, so that

$$X^2 - X - 1 = (X - \phi)(X - \tilde{\phi}).$$

For any $m \in \mathbb{Z}$ we have

$$\phi^m + \tilde{\phi}^m \in \mathbb{Z}.$$ 

The function $\phi^z = \exp(z \log \phi)$ has exponential type $\log \phi < \log 2$ and we have $\log |\tilde{\phi}| = -\log \phi$. However, $\phi^z + \tilde{\phi}^z$ not a counterexample to Pólya’s result on the growth of transcendental integer-valued entire functions: indeed, while $\phi^z = \exp(z \log \phi)$ is well defined since $\phi > 0$, the definition of $\tilde{\phi}^z$ requires to choose a logarithm of the negative number $\tilde{\phi} = -\phi^{-1}$. With $\log \tilde{\phi} = -\log \phi + i\pi$, the function $\tilde{\phi}^z = \exp(z \log \tilde{\phi})$ has exponential type $((\log \phi)^2 + \pi^2)^{1/2} = 3.178\,23\ldots > \log 2$.

According to (Pólya, 1915), a completely integer–valued function $f$ which satisfies

$$\lim_{r \to \infty} |f|, r^{-3/2} = 0$$

is a polynomial. In (Pólya, 1920), it is proved that a completely integer–valued function $f$ which satisfies

$$\limsup_{r \to \infty} |f|, r^{-k} < \infty$$

for some $k > 0$ is of the form

$$P_0(z) + P_1(z)\theta^z + P_2(z)\theta^{-z}$$

where $P_0, P_1, P_2$ are polynomials.

(Carlson, 1921) gave refined results by means of Laplace transform – see (Pólya, 1929), (Whittaker, 1935 § 10), (Buck, 1948a Th. 5.2, Cor. 2), (Gel’fond, 1952 Chap. 3, § 2), (Boas, 1954 § 9.12), (Robinson, 1971). According to (Selberg, 1941b), if a completely integer–valued function satisfies

$$\tau(f) \leq \log \theta + 2 \cdot 10^{-6},$$

then $f$ is of the form

$$P_0(z) + P_1(z)\theta^z + P_2(z)\theta^{-z}$$

where $P_0, P_1, P_2$ are polynomials.
1.5 Further results

Among the surveys on these topics, let us quote (Gramain, 1978), (Gramain, 1986), (Gramain and Schnitzer, 1989) and (Gramain, 1990).

Further results have been proved on the growth of entire functions satisfying $f(N) \subset \mathbb{K}$ or $f(\mathbb{Z}) \subset \mathbb{K}$, where $\mathbb{K}$ is the field of algebraic numbers. Assuming suitable growth conditions on $|f|_r$, on the algebraic numbers $f(m)$ and all of their conjugates, one deduces that $f$ is a polynomial. Interpolation formulæ yield such results, and also methods from transcendental number theory. We come back to this topic in § 5.2.

2 On the sequence of values $(f(0), f'(0), f''(0), \ldots)$

In place of the values of $f$ at the integers, we now consider the derivatives of $f$ at 0. The sequence polynomials (1) is replaced by the polynomials $z^n/n!$, while the interpolation series (2) is replaced by Taylor expansion.

2.1 Hurwitz functions

A Hurwitz function is an entire function $f$ such that $f^{(n)}(0) \in \mathbb{Z}$ for all $n \geq 0$. Polynomials in $\mathbb{Q}[z]$ of the form

$$\sum_{n=0}^{N} a_n \frac{z^n}{n!}$$

with $a_0, a_1, \ldots, a_N$ in $\mathbb{Z}$ are Hurwitz functions, while the exponential function

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots$$

is a transcendental Hurwitz function. A basis of the $\mathbb{Z}$-module of polynomials in $\mathbb{C}[z]$ which are Hurwitz functions is given by $1, z, z^2/2, \ldots, z^n/n!, \ldots$.

The first lower bound for the growth of a transcendental Hurwitz function is due to (Kakeya, 1916), who proved that a Hurwitz function satisfying

$$\limsup_{r \to \infty} |f|_r e^{-r\sqrt{r}} = 0$$

is a polynomial. This was refined by (Pólya, 1921) – see also (Pólya and Szegő, 1998, Part VIII, Chap. 3, § 6, n°187): a Hurwitz function satisfying

$$\limsup_{r \to \infty} |f|_r e^{-r\sqrt{r}} < \frac{1}{\sqrt{2\pi}}$$
is a polynomial. This is best possible for uncountably many functions, as shown by the functions
\[ f(z) = \sum_{n \geq 0} e_n \frac{z^n}{n!} 2^n \]
with \( e_n \in \{1, -1\} \) which satisfy
\[ \limsup_{r \to \infty} |f|r^{-r} e^{-r} \sqrt{r} = \frac{1}{\sqrt{2\pi}}. \]

### 2.2 Refined estimates by Sato and Straus

More precise results are achieved by D. Sato and E.G. Straus thanks to a careful study of the function
\[ \phi(r) = \max_{n \geq 0} r^n \frac{n!}{n!}. \]

In (Sato and Straus, 1964, Corollary 1 p. 304) and (Sato and Straus, 1965, Corollary p. 20) (see also (Sato, 1971)), they proved that for every \( \epsilon > 0 \), there exists a transcendental Hurwitz function with
\[ \limsup_{r \to \infty} |f|r^{-r} e^{-r} \sqrt{2\pi r} \left(1 + \frac{1 - \epsilon}{24r}\right)^{-1} < 1, \]
while every Hurwitz function for which
\[ \limsup_{r \to \infty} |f|r^{-r} e^{-r} \sqrt{2\pi r} \left(1 + \frac{1 + \epsilon}{24r}\right)^{-1} \leq 1 \]
is a polynomial.

### 3 Several points and / or several derivatives

#### 3.1 Introduction

There are several natural ways to mix integer–valued functions and Hurwitz functions: one may include finitely may derivatives in the study of integer–valued functions, yielding to the study of \( k \)–times integer–valued functions. One may consider entire functions having all their derivatives at \( k \) points taking integer values, yielding to the study of \( k \)–point Hurwitz functions. One may consider entire functions \( f \) which satisfy \( f^{(n)}(m) \in \mathbb{Z} \) for all \( n \geq 0 \) and \( m \in \mathbb{Z} \), which are the so–called utterly integer–valued functions. Finally, the study of entire functions \( f \) such that \( f^{(n)}(n) \in \mathbb{Z} \) is related with Abel’s interpolation.

Let us display horizontally the rational integers and vertically the derivatives.
Integer–valued functions: horizontal

\[ f \circ \cdots \circ f \circ \cdots \circ 0 \circ 1 \circ 2 \circ \cdots \circ m \circ \cdots \]

Hurwitz functions: vertical

\[
\begin{array}{c}
\vdots \\
\vdots \\
f^{(n)} \circ \\
\vdots \\
f' \circ \\
f \circ \\
0
\end{array}
\]

2–times integer–valued functions

\[
\begin{array}{c}
\vdots \\
\vdots \\
f'^{n} \circ \cdots \circ f' \circ \cdots \circ f \circ \cdots \circ 0 \circ 1 \circ 2 \circ \cdots \circ m \circ \cdots 
\end{array}
\]

2–point Hurwitz functions

\[
\begin{array}{c}
\vdots \\
\vdots \\
f^{(n)} \circ \\
\vdots \\
f' \circ \\
f \circ \\
0 \ 1
\end{array}
\]

Utterly integer–valued 
entire functions

\[
\begin{array}{c}
\vdots \\
\vdots \\
f^{(n)} \circ \cdots \circ f' \circ \cdots \circ f \circ \cdots \circ 0 \circ 1 \circ \cdots \circ m \circ \cdots 
\end{array}
\]

8
Abel interpolation

\[ f^{(n)} \}
\[ f' \}
\[ f \}

0 1 \ldots n \ldots

3.2 \ k\text{-}times integer\text{-}valued functions

The first natural way to mix integer\text{-}valued functions and Hurwitz functions is horizontal, including finitely many derivatives in the study of integer\text{-}valued functions, like in (Gel'fond, 1929a), (Selberg, 1941a). Let us call \( k\text{-}times integer\text{-}valued function \) an entire function \( f \) such that \( f^{(n)}(m) \in \mathbb{Z} \) for all \( m \geq 0 \) and \( n = 0, 1, \ldots, k - 1 \). According to (Gel'fond, 1929b), a \( k\text{-}times integer\text{-}valued function of exponential type \( < k \log \left( 1 + e^{-k-1} \right) \) is a polynomial. A proof is given in (Fridman, 1968). Improvements are due to (Selberg, 1941b), and (Bundschuh and Zudilin, 2004) who also investigated what could be the best possible results. The best known results so far are due to (Welter, 2005b).

3.3 \ k\text{-}point Hurwitz functions

The second solution is vertical, considering \( k\text{-}point Hurwitz functions \), namely entire functions having all their derivatives at 0, 1, \ldots, \( k - 1 \) taking integer\text{-}values. This question was first considered by (Gel'fond, 1934). It has been proved by (Straus, 1950) that the order of such a function is \( \geq k \), and this is best possible, as shown by the function \( e^{z(z-1)\cdots(z-k+1)} \).

Precise results are known for \( k = 2 \). (Sato, 1971) proved that there exist transcendental two point Hurwitz entire functions with

\[ |f|_r \leq \exp \left( r^2 + r - \log r + O(1) \right), \]

while every two point Hurwitz entire functions with

\[ |f|_r \leq C \exp \left( r^2 - r - \log r \right) \]

for some positive constant \( C \) must be a polynomial.

Another example is (Straus, 1950) Th. 3): if \( f \) is a transcendental entire function and \( f^{(n)}(z) \) is in \( \mathbb{Z} \) for \( z = 0 \) and \( z = p/q \) with \( \gcd(p, q) = 1 \), then \( f \) is at least of order 2 and type \( \tau_2(f) \geq q/p \). A similar estimate from (Straus, 1950) Th. 2) holds when \( f^{(n)}(z) \) is in \( \mathbb{Z} \) for \( z = 0, p_1/q_1, \ldots, p_{k-1}/q_{k-1} \).
For \( k \geq 3 \) our knowledge is more limited. Bieberbach (1953) stated that if a transcendental entire function \( f \) of order \( \varrho \) is a \( k \)-point Hurwitz entire function, then either \( \varrho > k \), or \( \varrho = k \) and the type \( \tau_k(f) \) of \( f \) satisfies \( \tau_k(f) \geq 1 \). However, as noted by Fridman (1968) and Sato (1971, p. 2), this result is not true. Indeed, the polynomial

\[
a(z) = \frac{1}{2}z(z - 1)(z - 2)(z - 3)
\]

can be written

\[
a(z) = \frac{1}{2}z^4 - 3z^3 - \frac{11}{2}z^2 - 3z,
\]

hence it satisfies \( a'(z) \in \mathbb{Z}[z] \); it follows that the function \( e^{a(z)} \) is a transcendental 4-point Hurwitz function of order \( \varrho = 4 \) and \( \tau_4(f) = 1/2 \).

### 3.4 Utterly integer–valued functions

An utterly integer–valued function is an entire function \( f \) which satisfies \( f^{(n)}(m) \in \mathbb{Z} \) for all \( n \geq 0 \) and \( m \in \mathbb{Z} \). From either the results on \( k \)-point Hurwitz functions or the results on \( k \)-times integer–valued functions, it follows that a transcendental utterly integer–valued function must be of infinite order. One also deduces the irrationality of any nonzero period of a nonconstant Hurwitz function of finite order – compare with §5.2.

Straus (1951) suggested that transcendental utterly integer–valued functions may not exist. Fridman (1968) showed that there exists transcendental utterly integer–valued function \( f \) with

\[
\limsup_{r \to \infty} \frac{\log \log |f_r|}{r} \leq \pi
\]

and proved that a transcendental utterly integer–valued function \( f \) satisfying

\[
\limsup_{r \to \infty} \frac{\log \log |f_r|}{r} \geq \log(1 + 1/e)
\]

is a polynomial. The bound \( \log(1 + 1/e) \) was improved by Welter (2005a) to \( \log 2 \). Hence a transcendental utterly integer–valued function grows at least like the double exponential \( e^{2z} \).

Sato (1985) constructed a nondenumerable set of utterly integer–valued functions. He selected inductively the coefficients \( a_n \) with

\[
\frac{1}{n!(2\pi)^n} \leq |a_n| \leq \frac{3}{n!(2\pi)^n}
\]

and defined

\[
f(z) = \sum_{n \geq 0} a_n \sin^n(2\pi z).
\]
3.5 Abel interpolation

There is also a diagonal way of mixing the questions of integer-valued functions and Hurwitz functions by considering entire functions \( f \) such that \( f^{(n)}(n) \in \mathbb{Z} \).

The source of this question goes back to [Abel, 1881a], [Abel, 1881b]. The related interpolation problem was studied by [Halphen, 1882a], [Halphen, 1882b]. See also [Pareto, 1892], [Gontcharoff, 1930], [Buck, 1946] and [Buck, 1948b], § 7 and § 10 Corollary 5). Further references are [Whittaker, 1935] Chap. III, [Gelfond, 1952] Chap. III, § 3.2), [Boas, 1954], § 9.10], [Bézivin, 1992].

The sequence of polynomials \((P_n)_{n \geq 0}\) defined by
\[
P_0 = 1, \quad P_n(z) = \frac{1}{n!}z(z - n)^{n-1} \quad (n \geq 1)
\]
was introduced by [Abel, 1881b]. These polynomials satisfy
\[
P_n'(z) = P_{n-1}(z - 1) \quad (n \geq 1).
\]
Therefore \(P_n^{(k)}(k) = \delta_{kn}\) for \( k \) and \( n \geq 0 \). One deduces that any polynomial \( f \) has a finite expansion
\[
f(z) = \sum_{n \geq 0} f^{(n)}(n)P_n(z).
\]

It was proved by [Halphen, 1882b] (see also [Gontcharoff, 1930] p. 31) that such an expansion (with a series in the right hand side which is absolutely and uniformly convergent on any compact of \( \mathbb{C} \)) holds also for any entire function \( f \) of finite exponential type \( < \omega \), where \( \omega = 0.278464542\ldots \) is the positive real number defined by \( \omega e^{\omega+1} = 1 \). The proof rests on Laplace transform – see [Gelfond, 1952] Chap. 3 § 2 p.209).

Lower bounds for the growth of entire functions satisfying \( f^{(n)}(n) \in \mathbb{Z} \) were investigated by [Bertrandias, 1958]. See also [Wallisser, 1969]. The method arises from [Pisot, 1946b] and [Pisot, 1946a].

For \( t \in \mathbb{C} \), the function \( f_t(z) := e^{tz} \) satisfies the functional differential equation
\[
f'(z) = te^zf(z - 1)
\]
with the initial condition \( f_t(0) = 1 \), hence \( f_t^{(n)}(n) = (te^n)^n \) for all \( n \geq 0 \). Let \( \tau_0 = 0.567143290\ldots \) be the positive real number defined by \( \tau_0 e^{\tau_0} = 1 \). We have \( f_t^{(n)}(n) = 1 \) for all \( n \geq 0 \).

Following [Bertrandias, 1958], an entire function \( f \) of exponential type \( < \tau_0 \) such that \( f^{(n)}(n) \in \mathbb{Z} \) for all sufficiently large integers \( n \geq 0 \) is a polynomial. More precisely, let \( \tau_1 \) be the complex number defined by \( \tau_1 e^{-\tau_1} = (1 + i\sqrt{3})/2 \); its modulus is \( |\tau_1| = 0.616\ldots \). Then an entire function \( f \) of exponential type \( < |\tau_1| \) such that \( f^{(n)}(n) \in \mathbb{Z} \) for all sufficiently large integers \( n \geq 0 \) is of the form \( P(z) + Q(z)e^{\tau_0 z} \), where \( P \) and \( Q \) are polynomials.
4 Variants

We do claim to quote all variations and related works around this theme; we only give a few examples among those which would deserve further surveys.

- \( q \) analogues and multiplicative versions (geometric progressions): (Gel’fond, 1933), (Gel’fond, 1952, Chap. 2, § 3.4, Th. VIII), (Gel’fond, 1967), (Kaz’min, 1973), (Bézivin, 1984), (Wallisser, 1984), (Gramain, 1990), (Bundschuh, 1992), (Bézivin, 1990), (Bézivin, 1994a), (Bézivin, 1994b), (Bundschuh and Shiokawa, 1995), (Welter, 2000), (Welter, 2005a), (Welter, 2005b), (Bézivin, 2014) . . .
- extensions of the additive and multiplicative versions considered in (Pila and Rodriguez Villagas, 1999) who introduce the notion of concordant sequences; further developments along these lines are in (Perelli and Zannier, 1981), (Bézivin, 1998), (Pila, 2002), (Pila, 2003), (Pila, 2005), (Pila, 2008), and (Pila, 2009).
- definable functions and minimal models: (Jones et al., 2012).

5 Connection with transcendental number theory

5.1 From \( \mathbb{Z} \) to \( \mathbb{Z}[i] \)

Historically, the very first steps towards a solution of Hilbert’s seventh on the transcendence of \( a^b \) came from the extension by (Fukasawa, 1928) and (Gel’fond, 1929a) of Pólya’s result on integer–valued functions on \( \mathbb{N} \) to integer–valued functions on \( \mathbb{Z}[i] \).

According to A.O. Gel’fond, an entire function \( f \) which is not a polynomial and satisfies \( f(a + ib) \in \mathbb{Z}[i] \) for all \( a + ib \in \mathbb{Z}[i] \) satisfies

\[
\limsup_{R \to \infty} \frac{1}{R^2} \log |f|_R \geq \gamma.
\]

For the proof, Gel’fond expands \( f(z) \) into a Newton interpolation series at the Gaussian integers. He obtained \( \gamma \geq 10^{-45} \).

Since the canonical product associated with the lattice \( \mathbb{Z}[i] \), namely the Weierstrass sigma function

\[
\sigma(z) = z \prod_{\omega \in \mathbb{Z}[i] \setminus \{0\}} \left( 1 - \frac{z}{\omega} \right) \exp \left( \frac{z}{\omega} + \frac{z^2}{2\omega^2} \right),
\]

is an entire function vanishing on \( \mathbb{Z}[i] \) of order 2 with \( \tau_2(\sigma) = \pi/2 \) (Hurwitz and Courant, 1944, Part 2, Chap. I, § 13), (Pólya and Szegő, 1998, Part IV, Chap. 1, § 3, n°49), the best (largest)
admissible value for $\gamma$ satisfies
\[ 10^{-45} \leq \gamma \leq \frac{\pi}{2}. \]
The study of entire functions $f$ satisfying $f(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$ was pursued by (Cayford, 1969) and (Graman, 1980). The exact value is $\gamma = \frac{\pi}{2e}$; (Masser, 1980) proved the upper bound and (Gramain, 1981) the lower bound.

A side effect of these works is the introduction of the so-called Masser–Gramain–Weber constant (Masser, 1980), (Gramain and Weber, 1985), an analog of Euler’s constant for $\mathbb{Z}[i]$, which arises in a 2–dimensional analogue of Stirling’s formula:
\[
\delta = \lim_{n \to \infty} \left( \sum_{k=2}^{n} (\pi r_k^2)^{-1} - \log n \right),
\]
where $r_k$ is the radius of the smallest disc in $\mathbb{R}^2$ that contains at least $k$ integer lattice points inside it or on its boundary. In (Melquiond et al., 2013) the first four digits are computed:
\[
1.819776 < \delta < 1.819833,
\]
disproving a conjecture of (Gramain, 1982).

5.2 Transcendence
One of the main motivations of studying this kind of problems arises from transcendental number theory. See for instance the application to the Hermite–Lindemann Theorem in (Gel’fond, 1952, Chap. 2, § 3.4, Th. IX).

(Straus, 1950) developed the subject of integer–valued functions in connection with transcendental number theory; he deduces the Hermite–Lindemann Theorem from his (Straus, 1950, Th. 4). However, as he pointed out in a footnote, since his paper was written, there has appeared the paper (Schneider, 1949) which contains an approach very similar, which is more profound, and the results are much more complete. Schneider’s work gave rise to the so–called Schneider–Lang Criterion (Lang, 1962).

One may notice that a main assumption in the final version of the Schneider–Lang criterion is that the functions satisfy a differential equation; hence this criterion does not contain (Schneider, 1949, Satz III) nor (Straus, 1950, Th. 4) where no such condition occurs. For instance, one easily deduces from either of these two results the transcendence of any nonzero period of a nonconstant Hurwitz function of finite order. The main example of course is the transcendence of $\pi$.

When there is no assumption on the derivatives, one may apply the transcendence method introduced by Th. Schneider for the solution of Hilbert’s seventh problem – see for instance (Waldschmidt, 1978). Examples are given in (Gramain and Mignotte, 1983), (Gramain et al., 1986), (Rochev, 2007), (Rochev, 2011) and (Ably, 2011). A connection with the six exponentials Theorem is introduced in (Pila, 2008).
6 Lidstone interpolation and generalizations

We conclude by stating some new results related with integer–valued functions. The proofs are given in (Waldschmidt, 2020a) and (Waldschmidt, 2020b).

6.1 Arithmetic result for Poritsky and Lidstone interpolation

Theorem 1. Let \(s_0, s_1, \ldots, s_{m-1}\) be distinct complex numbers and \(f\) an entire function of sufficiently small exponential type. If

\[ f^{(mn)}(s_j) \in \mathbb{Z} \]

for all sufficiently large \(n\) and for \(0 \leq j \leq m - 1\), then \(f\) is a polynomial.

For \(m = 2\) (Lidstone interpolation), with \(f^{(2n)}(s_0) \in \mathbb{Z}\) and \(f^{(2n)}(s_1) \in \mathbb{Z}\), the assumption on the exponential type \(\tau(f)\) of \(f\) is

\[ \tau(f) < \min\{1, \pi/|s_1 - s_0|\}, \]

and this is best possible, as shown by the functions

\[ f(z) = \frac{\sinh(z - s_1)}{\sinh(s_0 - s_1)} \quad \text{and} \quad f(z) = \sin \left( \frac{\pi}{2} \cdot \frac{z - s_0}{s_1 - s_0} \right), \]

which have exponential type 1 and \(\pi/|s_1 - s_0|\) respectively.

When \(|s_1 - s_0| \leq 2\), there is a nonenumerable set of entire functions \(f\) of exponential type \(\leq 1\) satisfying \(f^{(2n)}(s_0) = 0\) and \(f^{(2n)}(s_1) \in \{-1, 0, 1\}\) for all \(n \geq 0\).

6.2 Arithmetic result for Gontcharoff and Whittaker interpolation

Theorem 2. Let \(s_0, s_1, \ldots, s_{m-1}\) be distinct complex numbers and \(f\) an entire function of sufficiently small exponential type. Assume that for each sufficiently large \(n\), one at least of the numbers

\[ f^{(n)}(s_j) \quad j = 0, 1, \ldots, m - 1 \]

is in \(\mathbb{Z}\). Then \(f\) is a polynomial.

In the case \(m = 2\) with \(f^{(2n+1)}(s_0) \in \mathbb{Z}\) and \(f^{(2n)}(s_1) \in \mathbb{Z}\) (Whittaker interpolation), the assumption on the exponential type of \(f\) is

\[ \tau(f) < \min\left\{1, \frac{\pi}{2|s_1 - s_0|}\right\}, \]

and this is best possible, as shown by the functions

\[ f(z) = \frac{\cosh(z - s_1)}{\cosh(s_0 - s_1)} \quad \text{and} \quad f(z) = \cos \left( \frac{\pi}{2} \cdot \frac{z - s_0}{s_1 - s_0} \right), \]
which have exponential type 1 and \( \frac{\pi}{2|s_1 - s_0|} \) respectively.

When \(|s_1 - s_0| < \log(2 + \sqrt{3}) = 1.316\ldots\), there is a nondenumerable set of entire functions \( f \) of exponential type \( \leq 1 \) satisfying \( f^{(2n+1)}(s_0) = 0 \) and \( f^{(2n)}(s_1) \in \{-1, 0, 1\} \) for all \( n \geq 0 \).

As before, let us display horizontally the points (they are no longer assumed to be the consecutive integers) and vertically the derivatives.
- \( \bullet \) interpolation values
- \( \circ \) no condition

**Lidstone interpolation**

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
 f^{(2n+1)} & \circ & \circ & \\
 f^{(2n)} & \bullet & \bullet & \\
 \vdots & \vdots & \vdots & \\
 f'' & \bullet & \bullet & \\
 f' & \circ & \circ & \\
 f & \bullet & \bullet & \circ \circ s_0 \ s_1
\end{array}
\]

**Whittaker interpolation**

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
 f^{(2n+1)} & \bullet & \circ & \\
 f^{(2n)} & \circ & \bullet & \\
 \vdots & \vdots & \vdots & \\
 f'' & \circ & \bullet & \\
 f' & \bullet & \circ & \\
 f & \circ & \bullet & \circ \circ s_0 \ s_1
\end{array}
\]

**Poritsky interpolation (3 points)**

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
 f^{(2n+1)} & \bullet & \circ & \\
 f^{(2n)} & \circ & \bullet & \\
 \vdots & \vdots & \vdots & \\
 f'' & \circ & \bullet & \\
 f' & \bullet & \circ & \\
 f & \circ & \bullet & \circ \circ s_0 \ s_1
\end{array}
\]

15
Gontcharoff interpolation (3 points)
An example with a period

References


(Pólya, 1974), p. 131–140.


Waldschmidt, M. (2020a). On entire transcendental functions with infinitely many derivatives taking integer values at two points. [arXiv: 1912.00173 [math.NT]]


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