Mathematics Faculty of TIFR, Mumbai October, 22, 2007

Mathematics Colloquium lecture

Density of rational points on Abelian varieties

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Abstract

In a recent paper on the rank of elliptic curves in cubic extension, H. Kisilevsky asks under which conditions the set of multiples of a rational point on an elliptic curve defined over the rational number field is dense in the space of complex points. The answer is easy in the CM case, while in the non-CM case a complete answer relies on a conjecture from transcendental number theory. This topic is closely related to a question raised by B. Mazur in 1994 which suggests the following conjecture: *Let A be a simple Abelian variety defined over a real number field K. Denote by $A(\mathbf{R})$ the Lie group of its real points and by $A(\mathbf{R})^0$ the connected component of the origin. Then the group $\mathbf{Z}P$ generated by any point P of infinite order in $A(K) \cap A(\mathbf{R})^0$ is dense in $A(\mathbf{R})^0$.*

Transcendence methods yield weaker statements like the following: *Let A be a simple Abelian variety of dimension d defined over a number field K embedded in $\mathbf{R}$. Let Γ be a subgroup of $A(K) \cap A(\mathbf{R})^0$ of rank $\ge d^2 - d + 1$. Then Γ is dense in $A(\mathbf{R})^0$.*

Some results can also be obtained on the density in $A(\mathbf{C})$ of subgroups of $A(K)$, when K is any number field embedded into the field $\mathbf{C}$ of complex numbers.

Related questions have been considered by Dipendra Prasad in 2004, by Gopal Prasad and Andrei Rapinchuk in 2005.

A complete answer to such questions would follow from a special case of Schanuel's conjecture: it would suffice to prove that linearly independent logarithms of algebraic numbers are algebraically independent. We shall explain the state of the art on this question.

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