Abstract

This lecture will be devoted to a survey of transcendental number theory, including some history, the state of the art and some of the main conjectures, the limits of the current methods and the obstacles which are preventing from going further.

Extended abstract

An algebraic number is a complex number which is a root of a polynomial with rational coefficients. For instance $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1 + \sqrt{5})/2$, the roots of unity $e^{2\pi n/b}$, the roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers. A transcendental number is a complex number which is not algebraic.

Extended abstract (continued)

The existence of transcendental numbers was proved in 1844 by J. Liouville who gave explicit ad-hoc examples. The transcendence of constants from analysis is harder; the first result was achieved in 1873 by Ch. Hermite who proved the transcendence of $e$. In 1882, the proof by F. Lindemann of the transcendence of $\pi$ gave the final (and negative) answer to the Greek problem of squaring the circle. The transcendence of $2\sqrt{2}$ and $e^\pi$, which was included in Hilbert's seventh problem in 1900, was proved by Gel'fond and Schneider in 1934. During the last century, this theory has been extensively developed, and these developments gave rise to a number of deep applications. In spite of that, most questions are still open. In this lecture we survey the state of the art on known results and open problems.
Rational, algebraic irrational, transcendental

**Goal**: decide upon the arithmetic nature of “given” numbers: rational, algebraic irrational, transcendental.

Rational integers: \( \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots \} \).

Rational numbers: \( \mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1\} \).

Algebraic number: root of a polynomial with rational coefficients.

A **transcendental number** is complex number which is not algebraic.

Algebraic irrational numbers

Examples of algebraic irrational numbers:

- \( \sqrt{2}, i = \sqrt{-1} \), the Golden Ratio \((1 + \sqrt{5})/2\),
- \( \sqrt{d} \) for \( d \in \mathbb{Z} \) not the square of an integer (hence not the square of a rational number),
- the roots of unity \( e^{2\pi i/a} \), for \( a/b \in \mathbb{Q} \),
- and, of course, any root of an irreducible polynomial with rational coefficients of degree > 1.

Rule and compass; squaring the circle

Construct a square with the same area as a given circle by using only a finite number of steps with compass and straightedge.

Any constructible length is an algebraic number, though not every algebraic number is constructible (for example \( \sqrt{2} \) is not constructible).

Pierre Laurent Wantzel (1814 – 1848)

*Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas.* Journal de Mathématiques Pures et Appliquées 1 (2), (1837), 366–372.
Quadrature of the circle

Marie Jacob

La quadrature du cercle
Un problème
à la mesure des Lumières
Fayard (2006).

Resolution of equations by radicals

The roots of the polynomial
\(X^3 - 6X + 3\) are algebraic numbers, and are not expressible by radicals.

Evariste Galois
(1811 – 1832)

Gottfried Wilhelm Leibniz

Introduction of the concept of the transcendental in mathematics by Gottfried Wilhelm Leibniz in 1684:
“Nova methodus pro maximis et minimis itemque tangentiibus, qua nec fractas, nec irrationales quantitates moratur, . . .”


§1 Irrationality

Given a basis \(b \geq 2\), a real number \(x\) is rational if and only if its expansion in basis \(b\) is ultimately periodic.

\(b = 2\) : binary expansion.

\(b = 10\) : decimal expansion.

For instance the decimal number

\[0.123456789012345678901234567890\ldots\]

is rational:

\[
\begin{align*}
1234567890 & = 137174210 \\
999999999 & = 1111111111
\end{align*}
\]
First decimal digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.4142135623730950488016887242096980785696718753769480731766793
79907324784621073085038753432764157273301384623091229702492483
6055850737216441214970999358141322266592505592755795995050115
27820605714701095997160597027453496862014728157141864088919860
9523292304830871432145038976260362799525140798968275339664633
1808829640260615238352395054745750287759617298355752203375185
701135437460340498474160386899709699048150305440277903164424
7823066849293618621580578463111506687131031015618656898723723258
85092648612494977154218342042856860601468247207714358548741556
570667765372022648544701558801620758479422572260020855844665
21458398939443709265918003113882464681570826301005948587040031
864803421948972782996410450726368113137985526117322042450912
2770022694112757362728049573810896575041018369863684507257993647
2906762996941380475654823728971803268024742062962916485095021
81004459824150591120249441341728531478105803603371077309182693
14710171116839165817268894197587165821522829251848847 \ldots

Computation of decimals of $\sqrt{2}$

1542 decimals computed by hand by Horace Uhler in 1951

14 000 decimals computed in 1967

1 000 000 decimals in 1971

137 - $10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

• Motivation : computation of $\pi$.

Square root of 2 on the web

The first decimal digits of $\sqrt{2}$ are available on the web

1, 4, 1, 4, 2, 1, 3, 5, 6, 2, 3, 7, 3, 0, 9, 5, 0, 4, 8, 8, 0, 1,

6, 8, 8, 7, 2, 4, 2, 0, 9, 6, 9, 8, 0, 7, 8, 5, 6, 9, 6, 7, 1, 8, \ldots

http://oeis.org/A002193

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

http://oeis.org/
Pythagoras of Samos ~ 569 BC – ~ 475 BC

\[ a^2 + b^2 = c^2 = (a + b)^2 - 2ab. \]

Irrationality in Greek antiquity

Platon, La République: *incommensurable lines, irrational diagonals.*

Theodorus of Cyrene (about 370 BC.) irrationality of \( \sqrt{3}, \ldots, \sqrt{17} \).

Theetetes: if an integer \( n > 0 \) is the square of a rational number, then it is the square of an integer.

Irrationality of \( \sqrt{2} \)

Pythagoreas school

Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, ~800-500 BC.

Émile Borel: 1950

The sequence of decimal digits of \( \sqrt{2} \) should behave like a random sequence, each digit should be occurring with the same frequency \( 1/10 \), each sequence of 2 digits occurring with the same frequency \( 1/100 \ldots \)
Complexity of the $b$–ary expansion of an irrational algebraic real number

Let $b \geq 2$ be an integer.

- É. Borel (1909 and 1950): the $b$–ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (with respect to Lebesgue’s measure).

- Remark: no number satisfies all the laws which are shared by all numbers outside a set of measure zero, because the intersection of all these sets of full measure is empty!

\[ \bigcap_{x \in \mathbb{R}} \mathbb{R} \setminus \{x\} = \emptyset. \]

- More precise statements by B. Adamczewski and Y. Bugeaud.

The state of the art

There is no explicitly known example of a triple $(b, a, x)$, where $b \geq 3$ is an integer, $a$ is a digit in $\{0, \ldots, b - 1\}$ and $x$ is an algebraic irrational number, for which one can claim that the digit $a$ occurs infinitely often in the $b$–ary expansion of $x$.

A stronger conjecture, also due to Borel, is that algebraic irrational real numbers are normal: each sequence of $n$ digits in basis $b$ should occur with the frequency $1/b^n$, for all $b$ and all $n$.

Conjecture of Émile Borel

Conjecture (É. Borel). Let $x$ be an irrational algebraic real number, $b \geq 3$ a positive integer and $a$ an integer in the range $0 \leq a \leq b - 1$. Then the digit $a$ occurs at least once in the $b$–ary expansion of $x$.

Corollary. Each given sequence of digits should occur infinitely often in the $b$–ary expansion of any real irrational algebraic number.

(consider powers of $b$).

- An irrational number with a regular expansion in some basis $b$ should be transcendental.
What is known on the decimal expansion of $\sqrt{2}$?

The sequence of digits (in any basis) of $\sqrt{2}$ is not ultimately periodic.

Among the decimal digits

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

at least two of them occur infinitely often. Almost nothing else is known.

Complexity of the expansion in basis $b$ of a real irrational algebraic number

Theorem (B. Adamczewski, Y. Bugeaud 2005; conjecture of A. Cobham 1968).

If the sequence of digits of a real number $x$ is produced by a finite automaton, then $x$ is either rational or else transcendental.

§2 Irrationality of transcendental numbers

- The number $e$
- The number $\pi$
- Open problems

Introductio in analysin infinitorum

Leonhard Euler (1737)
(1707 – 1783)
Introductio in analysin infinitorum

Continued fraction of $e$:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}}}$$

$e$ is irrational.
Joseph Fourier

Fourier (1815) : proof by means of the series expansion

\[ e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{N!} + r_N \]

with \( r_N > 0 \) and \( N!r_N \to 0 \) as \( N \to +\infty \).

Course of analysis at the École Polytechnique Paris, 1815.

Variant of Fourier’s proof : \( e^{-1} \) is irrational

C.L. Siegel : Alternating series

For odd \( N \),

\[ 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{N!} < e^{-1} < 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(N+1)!} \]

\[ \frac{a_N}{N!} < e^{-1} < \frac{a_N}{N!} + \frac{1}{(N+1)!}, \quad a_N \in \mathbb{Z} \]

Hence \( N! e^{-1} \) is not an integer.

Irrationality of \( \pi \)

Āryabhaṭa, born 476 AD : \( \pi \sim 3.1416 \).

Nīlakaṇṭha Somayājī, born 1444 AD : Why then has an approximate value been mentioned here leaving behind the actual value? Because it (exact value) cannot be expressed.


Johann Heinrich Lambert (1728 – 1777)

Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,

Mémoires de l’Académie des Sciences de Berlin, 17 (1761), p. 265-322 ; lu en 1767 ; Math. Werke, t. II.

\( \tan(\nu) \) is irrational when \( \nu \neq 0 \) is rational.

As a consequence, \( \pi \) is irrational, since \( \tan(\pi/4) = 1 \).
Lambert and Frederick II, King of Prussia

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez-vous ?
— De moi-même !

Known and unknown transcendence results

Known:

\( e, \pi, \log 2, e^{\sqrt{2}}, e^\pi, 2^{\sqrt{2}}, \Gamma(1/4) \).

Not known:

\( e + \pi, e\pi, \log \pi, \pi^e, \Gamma(1/5), \zeta(3), \text{Euler constant} \).

Why is \( e^\pi \) known to be transcendental while \( \pi^e \) is not known to be irrational?
Answer: \( e^\pi = (-1)^{-i} \).

Catalan’s constant

Is Catalan’s constant
\[
\sum_{n \geq 1} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941772190150\ldots
\]
an irrational number?

Catalan’s constant, Dirichlet and Kronecker

Catalan’s constant is the value at \( s = 2 \) of the Dirichlet \( L \)-function \( L(s, \chi_{-4}) \) associated with the Kronecker character

\[
\chi_{-4}(n) = \left( \frac{n}{4} \right) = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
1 & \text{if } n \equiv 1 \pmod{4}, \\
-1 & \text{if } n \equiv -1 \pmod{4}.
\end{cases}
\]

Johann Peter Gustav Lejeune Dirichlet 1805 – 1859
Leopold Kronecker 1823 – 1891
Catalan’s constant, Dedekind and Riemann

The Dirichlet $L$-function $L(s, \chi_4)$ associated with the Kronecker character $\chi_4$ is the quotient of the Dedekind zeta function of $\mathbb{Q}(i)$ and the Riemann zeta function:

$$\zeta_{\mathbb{Q}(i)}(s) = L(s, \chi_4)\zeta(s)$$

Julius Wilhelm Richard Dedekind
1831 – 1916

Georg Friedrich Bernhard Riemann
1826 – 1866

Riemann zeta function

The function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

was studied by Euler (1707–1783) for integer values of $s$ and by Riemann (1859) for complex values of $s$.

Euler: for any even integer value of $s \geq 2$, the number $\zeta(s)$ is a rational multiple of $\pi^s$.

Examples: $\zeta(2) = \pi^2/6$,$\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450 \cdots$

Coefficients: Bernoulli numbers.

Riemann zeta function

The number

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202 056 903 159 594 285 399 738 161 511 \ldots$$

is irrational (Apéry 1978).

Recall that $\zeta(s)/\pi^s$ is rational for any even value of $s \geq 2$.

Open question: Is the number $\zeta(3)/\pi^3$ irrational?

T. Rivoal (2000): infinitely many $\zeta(2n + 1)$ are irrational.
Infinitely many odd zeta values are irrational

Tanguy Rivoal (2000)

Let $\epsilon > 0$. For any sufficiently large odd integer $a$, the dimension of the $\mathbb{Q}$–vector space spanned by the numbers $1$, $\zeta(3)$, $\zeta(5)$, $\ldots$, $\zeta(a)$ is at least

$$\frac{1 - \epsilon}{1 + \log 2} \log a.$$ 

Euler–Mascheroni constant

Euler’s Constant is

$$\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) = 0.577215664901532860606512090082\ldots$$

Is it a rational number?

$$\gamma = \sum_{k=1}^{\infty} \left( \frac{1}{k} - \log \left( 1 + \frac{1}{k} \right) \right) = \int_1^\infty \left( \frac{1}{x} - \frac{1}{x} \right) dx = \int_0^1 \int_0^1 \frac{(1-x)dx}{(1-xy)\log(xy)}$$

Euler's constant

Recent work by J. Sondow inspired by the work of F. Beukers on Apéry’s proof.

Jonathan Sondow

http://home.earthlink.net/~jsondow/

$$\gamma = \int_0^\infty \sum_{k=2}^{\infty} \frac{1}{k^2 (t+k)} dt$$

$$\gamma = \lim_{s \to 1^+} \sum_{n=1}^{\infty} \left( \frac{1}{n^s} - \frac{1}{s^n} \right)$$

$$\gamma = \int_1^\infty \frac{1}{2t(t+1)} F \left( 1, 2, 2; 3, t+2 \right) dt.$$
Euler Gamma function
Is the number \( \frac{1}{5} \) irrational?

\( \Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t} \)

Here is the set of rational values \( r \in (0, 1) \) for which the answer is known (and, for these arguments, the Gamma value \( \Gamma(r) \) is a transcendental number):

\( r \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 3, 5 \right\} \mod 1. \)

Georg Cantor (1845 - 1918)
The set of algebraic numbers is countable, not the set of real (or complex) numbers.

Cantor (1874 and 1891).

Henri Léon Lebesgue (1875 – 1941)
Almost all numbers for Lebesgue measure are transcendental numbers.

Most numbers are transcendental
Meta conjecture: any number given by some kind of limit, which is not obviously rational (resp. algebraic), is irrational (resp. transcendental).
Special values of hypergeometric series

Sum of values of a rational function

Let $P$ and $Q$ be non-zero polynomials having rational coefficients and $\deg Q \geq 2 + \deg P$. Consider

\[ \sum_{n \geq 0} \frac{P(n)}{Q(n)}. \]

Telescoping series

Examples

\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= 1, \\
\sum_{n=0}^{\infty} \frac{1}{n^2 - 1} &= \frac{3}{4}, \\
\sum_{n=0}^{\infty} \left( \frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) &= 0 \\
\sum_{n=0}^{\infty} \left( \frac{1}{5n+2} - \frac{3}{5n+7} + \frac{1}{5n-3} \right) &= \frac{5}{6}
\end{align*}
\]

Transcendental values

\[
\begin{align*}
\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} &= \log 2, \\
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}, \\
\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} &= \frac{\pi}{3}
\end{align*}
\]

are transcendental.
Transcendental values

\[\sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)} = \frac{1}{4320} (192 \log 2 - 81 \log 3 - 7\pi \sqrt{3})\]

\[\sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = \frac{\pi}{2} + \frac{\sinh^{-1}(1)}{1^2 - 1} = 2.0766740474 \ldots\]

\[\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + 1} = \frac{2\pi}{e^\pi - e^{-\pi}} = 0.272029054982 \ldots\]

Encyclopedia of integer sequences (again)

The Fibonacci sequence \((F_n)_{n \geq 0}\):

\[0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233 \ldots\]

is defined by

\[F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).\]

Series involving Fibonacci numbers

The number

\[\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1\]

is rational, while

\[\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7 - \sqrt{5}}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1 - \sqrt{5}}{2}\]

and

\[\sum_{n=1}^{\infty} \frac{1}{F_{2n-1} + 1} = \frac{\sqrt{5}}{2}\]

are irrational algebraic numbers.

Leonardo Pisano (Fibonacci)

Leonardo Pisano (Fibonacci)

(1170–1250)

http://oeis.org/A000045
Series involving Fibonacci numbers

The numbers
\[
\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{4n}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{6n}}
\]
\[
\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{2n}}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}}
\]
\[
\sum_{n=1}^{\infty} \frac{1}{F_{2n-1} + F_{2n+1}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n+1}}
\]
are all transcendental.

The Fibonacci zeta function

For \( \Re(s) > 0 \),
\[
\zeta_F(s) = \sum_{n \geq 1} \frac{1}{F_n^s}
\]
\( \zeta_F(2), \zeta_F(4), \zeta_F(6) \) are algebraically independent.

lekata Shiokawa, Carsten Elsner and Shun Shimomura (2006)

§3 Transcendental numbers

- Liouville (1844)
- Hermite (1873)
- Lindemann (1882)
- Hilbert’s Problem 7th (1900)
- Gel’fond–Schneider (1934)
- Baker (1968)
- Nesterenko (1995)
Existence of transcendental numbers (1844)

J. Liouville (1809 - 1882)
gave the first examples of
transcendental numbers.
For instance
\[ \sum_{n \geq 1} \frac{1}{10^n} = 0.110 001 000 000 0\ldots \]
is a transcendental number.

Charles Hermite and Ferdinand Lindemann

Hermite (1873) :
Transcendence of \( e \)
\[ e = 2.718 281 828 4\ldots \]
Lindemann (1882) :
Transcendence of \( \pi \)
\[ \pi = 3.141 592 653 5\ldots \]

Hermite–Lindemann Theorem

For any non-zero complex number \( z \), one at least of the two
numbers \( z \) and \( e^z \) is transcendental.

Corollaries : Transcendence of \( \log \alpha \) and of \( e^\beta \) for \( \alpha \) and \( \beta \)
non-zero algebraic complex numbers, provided \( \log \alpha \neq 0 \).

Transcendental functions

A complex function is called transcendental if it is
transcendental over the field \( \mathbb{C}(z) \), which means that the
functions \( z \) and \( f(z) \) are algebraically independent:
if \( P \in \mathbb{C}[X, Y] \) is a non-zero polynomial, then the function
\( P(z, f(z)) \) is not 0.

Exercise. An entire function (analytic in \( \mathbb{C} \)) is transcendental if
and only if it is not a polynomial.

Example. The transcendental entire function \( e^z \) takes an
algebraic value at an algebraic argument \( z \) only for \( z = 0 \).
Weierstrass question

Is it true that a transcendental entire function \( f \) takes usually transcendental values at algebraic arguments?

Examples: for \( f(z) = e^z \), there is a single exceptional point \( \alpha \) algebraic with \( e^\alpha \) also algebraic, namely \( \alpha = 0 \).

For \( f(z) = e^{P(z)} \) where \( P \in \mathbb{Z}[z] \) is a non–constant polynomial, there are finitely many exceptional points \( \alpha \), namely the roots of \( P \).

The exceptional set of \( e^z + e^{1+z} \) is empty (Lindemann–Weierstrass).

The exceptional set of functions like \( 2^z \) or \( e^{iz} \) is \( \mathbb{Q} \), (Gel'fond and Schneider).

Exceptional sets

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain...

If \( S \) is a countable subset of \( \mathbb{C} \) and \( T \) is a dense subset of \( \mathbb{C} \), there exist transcendental entire functions \( f \) mapping \( S \) into \( T \), as well as all its derivatives.

Any set of algebraic numbers is the exceptional set of some transcendental entire function.

Also multiplicities can be included.

van der Poorten: there are transcendental entire functions \( f \) such that \( D^k f(\alpha) \in \mathbb{Q}(\alpha) \) for all \( k \geq 0 \) and all algebraic \( \alpha \).

Integer valued entire functions

An integer valued entire function is a function \( f \), which is analytic in \( \mathbb{C} \), and maps \( \mathbb{N} \) into \( \mathbb{Z} \).

Example: \( 2^z \) is an integer valued entire function, not a polynomial.

Question: Are there integer valued entire function growing slower than \( 2^z \) without being a polynomial?

Let \( f \) be a transcendental entire function in \( \mathbb{C} \). For \( R > 0 \) set

\[
|f|_R = \sup_{|z|=R} |f(z)|.
\]

G. Pólya (1914):

If \( f \) is not a polynomial and \( f(n) \in \mathbb{Z} \) for \( n \in \mathbb{Z}_{>0} \), then

\[
\limsup_{R \to \infty} 2^{-R}|f|_R \geq 1.
\]

Further works on this topic by G.H. Hardy, G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross, ...
**Integer valued entire function on** $\mathbb{Z}[i]$

*A.O. Gel’fond (1929)*: growth of entire functions mapping the Gaussian integers into themselves.

Newton interpolation series at the points in $\mathbb{Z}[i]$.

An entire function $f$ which is not a polynomial and satisfies $f(a + ib) \in \mathbb{Z}[i]$ for all $a + ib \in \mathbb{Z}[i]$ satisfies

$$\limsup_{R \to \infty} \frac{1}{R^2} \log |f_R| \geq \delta.$$  

*F. Gramain (1981)*: $\delta = \pi/(2e) = 0.577 863 674 8\ldots$

This is best possible: *D.W. Masser (1980).*

---

**Transcendence of** $e^\pi$

*A.O. Gel’fond (1929).*

If

$$e^\pi = 23.140\,692\,632\,779\,269\,005\,729\,086\,367\ldots$$

is rational, then the function $e^{\pi z}$ takes values in $\mathbb{Q}(i)$ when the argument $z$ is in $\mathbb{Z}[i]$.

Expand $e^{\pi z}$ into an interpolation series at the Gaussian integers.

---

**Hilbert’s Problems**

August 8, 1900


Twin primes,

Goldbach’s Conjecture,

Riemann Hypothesis

David Hilbert (1862 - 1943)

Transcendence of $e^\pi$ and $2\sqrt{2}$

---

**A.O. Gel’fond and Th. Schneider**

Solution of Hilbert’s seventh problem (1934): *Transcendence of* $\alpha^\beta$ *and of* $(\log \alpha_1)/(\log \alpha_2)$ *for algebraic* $\alpha$, $\beta$, $\alpha_1$ *and* $\alpha_2$.
Transcendence of $\alpha^\beta$ and $\log \alpha_1 / \log \alpha_2$ : examples

The following numbers are transcendental:

$$2^{\sqrt{2}} = 2.6651441426 \ldots$$

$$\frac{\log 2}{\log 3} = 0.6309297535 \ldots$$

$$e^\pi = 23.1406926327 \ldots \quad (e^\pi = (-1)^{-i})$$

$$e^{\pi \sqrt{163}} = 262537412640768743999999999925 \ldots$$

$e^\pi = (-1)^{-i}$

Example: Transcendence of the number

$$e^{\pi \sqrt{163}} = 2625374126407687439999999999925 \ldots$$

Remark. For

$$\tau = \frac{1 + i \sqrt{163}}{2}, \quad q = e^{2i\pi \tau} = -e^{-\pi \sqrt{163}}$$

we have $j(\tau) = -640320^3$ and

$$|j(\tau) - \frac{1}{q} - 744| < 10^{-12}.$$  

Beta values : Th. Schneider 1948

Euler Gamma and Beta functions

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$  

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \frac{dt}{t}$$  

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Algebraic independence : A.O. Gel’fond 1948

The two numbers $2^{\sqrt{2}}$ and $2^{\sqrt{4}}$ are algebraically independent.

More generally, if $\alpha$ is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if $\beta$ is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^d-1}$$

are algebraically independent.
Alan Baker 1968

Transcendence of numbers like

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

or

$$e^{\beta_1 \alpha_1} \cdots e^{\beta_n \alpha_n}$$

for algebraic $\alpha_i$'s and $\beta_j$'s.

Example (Siegel):

$$\int_0^1 \frac{dx}{1 + x^3} = \frac{1}{3} \left( \log 2 + \frac{\pi}{\sqrt{3}} \right) = 0.835648848 \ldots$$

is transcendental.

Yuri V. Nesterenko

Yu.V.Nesterenko (1996)
Algebraic independence of $\Gamma(1/4)$, $\pi$ and $e^\pi$.
Also: Algebraic independence of $\Gamma(1/3)$, $\pi$ and $e^{\pi/\sqrt{3}}$.

Corollary: The numbers $\pi = 3.1415926535 \ldots$ and $e^\pi = 23.1406926327 \ldots$ are algebraically independent.

Transcendence of values of Dirichlet's $L$–functions:
Sanoli Gun, Ram Murty and Purusottam Rath (2009).

Gregory V. Chudnovsky

G.V. Chudnovsky (1976)
Algebraic independence of the numbers $\pi$ and $\Gamma(1/4)$.
Also: algebraic independence of the numbers $\pi$ and $\Gamma(1/3)$.

Corollaries: Transcendence of $\Gamma(1/4) = 3.6256099082 \ldots$ and $\Gamma(1/3) = 2.6789385347 \ldots$.

Weierstraß sigma function

Let $\Omega = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ be a lattice in $\mathbb{C}$. The canonical product attached to $\Omega$ is the **Weierstraß sigma function**

$$\sigma(z) = \sigma_{\Omega}(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left( 1 - \frac{z}{\omega} \right) e^{(z/\omega) + (z^2/2\omega^2)}.$$  

The number

$$\sigma_{\Omega}(1/2) = 2^{5/4} \pi^{1/2} e^{\pi/8 \Gamma(1/4)^{-2}}$$

is transcendental.
§4: Conjectures

Borel 1909, 1950
Schanuel 1964
Grothendieck 1960’s
Rohrlich and Lang 1970’s
André 1990’s

The number $\pi$

Basic example of a period:

\[ e^{z+2i\pi} = e^z \]

\[ 2i\pi = \int_{|z|=1} \frac{dz}{z} \]

\[ \pi = \int \int_{x^2+y^2\leq 1} \frac{dx dy}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{1-x^2}}. \]

Periods: Maxime Kontsevich and Don Zagier


A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Further examples of periods

\[ \sqrt{2} = \int_{2x^2 \leq 1} dx \]

and all algebraic numbers.

\[ \log 2 = \int_{1<\frac{x}{2}<2} \frac{dx}{x} \]

and all logarithms of algebraic numbers.

\[ \pi = \int_{x^2+y^2\leq 1} dx dy, \]

A product of periods is a period (subalgebra of $\mathbb{C}$), but $1/\pi$ is expected not to be a period.
Relations among periods

1. **Additivity**
   (in the integrand and in the domain of integration)

\[
\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,
\]

\[
\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.
\]

2. **Change of variables**:
   if \( y = f(x) \) is an invertible change of variables, then

\[
\int_{f(a)}^{f(b)} F(y)\,dy = \int_a^b F(f(x))f'(x)\,dx.
\]

Conjecture of Kontsevich and Zagier

A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following

**Conjecture** (Kontsevich–Zagier). If a period has two integral representations, then one can pass from one formula to another by using only rules 1, 2, 3 in which all functions and domains of integration are algebraic with algebraic coefficients.

Conjecture of Kontsevich and Zagier (continued)

In other words, we do not expect any miraculous coincidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.

This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

Advice: if you wish to prove a number is transcendental, first prove it is a period.
Conjectures by S. Schanuel and A. Grothendieck

- **Schanuel**: If \( x_1, \ldots, x_n \) are \( \mathbb{Q} \)-linearly independent complex numbers, then \( n \) at least of the \( 2n \) numbers \( x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n} \) are algebraically independent.

- **Periods conjecture by Grothendieck**: Dimension of the Mumford–Tate group of a smooth projective variety.

Consequences of Schanuel’s Conjecture

- **Ram Murty**
- **Kumar Murty**
- **N. Saradha**
- **Purusottam Rath, Ram Murty, Sanoli Gun**

Ram and Kumar Murty (2009)

Transcendental values of class group \( L \)-functions.

Motives

- **Y. André**: Generalization of Grothendieck’s conjecture to motives.

Case of \( 1 \)-motives: Elliptico-Toric Conjecture of C. Bertolin.
A simple geometric construction on the moduli spaces $\mathcal{M}_{0,n}$ of curves of genus 0 with $n$ ordered marked points is described which gives a common framework for many irrationality proofs for zeta values. This construction yields Apéry’s approximations to $\zeta(2)$ and $\zeta(3)$, and for larger $n$, an infinite family of small linear forms in multiple zeta values with an interesting algebraic structure. It also contains a generalisation of the linear forms used by Ball and Rivoal to prove that infinitely many odd zeta values are irrational.

For $k$, $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$, we set $\mathbf{s} = (s_1, \ldots, s_k)$ and

$$\zeta(\mathbf{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$ 

The $\mathbb{Q}$–vector space $\mathcal{Z}$ spanned by the numbers $\zeta(\mathbf{s})$ is also a $\mathbb{Q}$–algebra. For $n \geq 2$, denote by $\mathcal{Z}_n$ the $\mathbb{Q}$–subspace of $\mathcal{Z}$ spanned by the real numbers $\zeta(\mathbf{s})$ where $\mathbf{s}$ has weight $s_1 + \cdots + s_k = n$.

The numbers $\zeta(s_1, \ldots, s_k)$, $s_1 + \cdots + s_k = n$, where each $s_i$ is 2 or 3, span $\mathcal{Z}_n$ over $\mathbb{Q}$.