Transcendental Number Theory: recent results and open problems.

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Abstract

This lecture will be devoted to a survey of transcendental number theory, including some history, the state of the art and some of the main conjectures, the limits of the current methods and the obstacles which are preventing from going further.

http://www.imj-prg.fr/~michel.waldschmidt/
Extended abstract

An algebraic number is a complex number which is a root of a polynomial with rational coefficients. For instance $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1 + \sqrt{5})/2$, the roots of unity $e^{2i\pi a/b}$, the roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers. A **transcendental number** is a complex number which is not algebraic.
The existence of transcendental numbers was proved in 1844 by J. Liouville who gave explicit ad-hoc examples. The transcendence of constants from analysis is harder; the first result was achieved in 1873 by Ch. Hermite who proved the transcendence of $e$. In 1882, the proof by F. Lindemann of the transcendence of $\pi$ gave the final (and negative) answer to the Greek problem of squaring the circle. The transcendence of $2^{\sqrt{2}}$ and $e^{\pi}$, which was included in Hilbert’s seventh problem in 1900, was proved by Gel’fond and Schneider in 1934. During the last century, this theory has been extensively developed, and these developments gave rise to a number of deep applications. In spite of that, most questions are still open. In this lecture we survey the state of the art on known results and open problems.
Rational, algebraic irrational, transcendental

**Goal**: decide upon the arithmetic nature of “given” numbers: rational, algebraic irrational, transcendental.

Rational integers: \( \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots \} \).

Rational numbers: \( \mathbb{Q} = \{ p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1 \} \).

Algebraic number: root of a polynomial with rational coefficients.

A **transcendental number** is complex number which is not algebraic.
Rational, algebraic irrational, transcendental

**Goal**: decide whether a “given” real number is rational, algebraic irrational or else transcendental.

- **Question**: what means ”given”?

- **Criteria for irrationality**: development in a given basis (e.g.: decimal expansion, binary expansion), continued fraction.

- **Analytic formulae, limits, sums, integrals, infinite products, any limiting process.**
Algebraic irrational numbers

Examples of algebraic irrational numbers:

- $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1 + \sqrt{5})/2$,
- $\sqrt{d}$ for $d \in \mathbb{Z}$ not the square of an integer (hence not the square of a rational number),
- the roots of unity $e^{2i\pi a/b}$, for $a/b \in \mathbb{Q}$,
- and, of course, any root of an irreducible polynomial with rational coefficients of degree $> 1$. 
Rule and compass; squaring the circle

Construct a square with the same area as a given circle by using only a finite number of steps with compass and straightedge.

Any constructible length is an algebraic number, though not every algebraic number is constructible (for example $\sqrt[3]{2}$ is not constructible).

Pierre Laurent Wantzel (1814 – 1848)

*Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas.* Journal de Mathématiques Pures et Appliquées 1 (2), (1837), 366–372.
Quadrature of the circle

Marie Jacob

La quadrature du cercle

Un problème

à la mesure des Lumières

Fayard (2006).
Resolution of equations by radicals

The roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers, and are not expressible by radicals.

Evariste Galois
(1811 – 1832)
Gottfried Wilhelm Leibniz

Introduction of the concept of the transcendental in mathematics by Gottfried Wilhelm Leibniz in 1684:

“Nova methodus pro maximis et minimis itemque tangentibus, qua nec fractas, nec irrationales quantitates moratur, ...”


§1 Irrationality

Given a basis $b \geq 2$, a real number $x$ is rational if and only if its expansion in basis $b$ is ultimately periodic.

$b = 2$ : binary expansion.

$b = 10$ : decimal expansion.

For instance the decimal number

$$0.123456789012345678901234567890 \ldots$$

is rational:

$$\frac{1234567890}{9999999999} = \frac{137174210}{1111111111}.$$
First decimal digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

1.41421356237309504880168872420969807856967187537694807317667973
799073247846210703885038753432764157273501384623091229702492483
605585073721264412149709993583141322266592750559275579995050115
278206057147010955997160597027453459686201472851741864088919860
955232923048430871432145083976260362799525140798968725339654633
180882964062061525835239505474575028775996172983557522033753185
701135437460340849884716038689997069900481503054402779031645424
782306849293691862158057846311159666871301301561856898723723528
850926486124949771542183342042856860601468247207714358548741556
570696776537202264854470158580016207584749226572260020855844665
214583988939443709265918003113882464681570826301005948587040031
8648034219489727829064104507263688131373988552561173220402450912
277002269411275736272804957381089675040183698683684507257993647
290607629969413804756548237289971803268024744206292691248590521
810044598421505911202494413417285314781058036033710773091828693
1471017111168391658172688941975871658215212822951848847 ...
First binary digits of $\sqrt{2}$

1.01101010000010011111001100110011111101110111100110010010000
1000101100101111101100010011011001101100110010100110111110100
11111001110101111110110000010111101011001000101001111011011001
1001100111011010001011110101100100010110001100110011001111100
100010101010010101111110010000110000001000110101011100010100
01011000111010100010110001111111101101111110111001000011110
1101100111001000011110110110010101000101111000110111011001100
10010010100010000111100011011010101000100001110100111110001
11111101100100101001111000000100100011101101100110011001101
00010011101100100011101101000010111010001110111000111001101
11100011111101001110010100100001011011001100110011001101101
111000110011011111011100101001000011100110011110111001101100
1000100010000011011010000110010011101000011101100111001101
11011001110010011110001100111100110110101000011100010001111
00010001110110010000111010100001011101001110010001110001110
11100111111010011100101000001011011001100110011001101101
110000110011011111011100111001100101011100110011001101101
0001000100001110000011001000111011001001000111000111001101
1101111000100111000011001111001101101010001000111011001110
1011111000100111000110011011011001010010010100011110001100
1011011011010110110011001110011001001010010001111010001111
10011111001111100100001001101111101000101111000100011000111
00001101101101110110000101101110110101001010100101000110000
111001000001100001110011001110110011001101100110011011001
10011111001111100100001001101111101000101111000100011000111
00001101101101110110000101101110110101001010100101000110000
111001000001100001110011001110110011001101100110011011001
10011111001111100100001001101111101000101111000100011000111
00001101101101110110000101101110110101001010100101000110000
111001000001100001110011001110110011001101100110011011001
10011111001111100100001001101111101000101111000100011000111
00001101101101110110000101101110110101001010100101000110000
111001000001100001110011001110110011001101100110011011001
10011111001111100100001001101111101000101111000100011000111
00001101101101110110000101101110110101001010100101000110000
111001000001100001110011001110110011001101100110011011001
Computation of decimals of $\sqrt{2}$

1 542 decimals computed by hand by Horace Uhler in 1951

14 000 decimals computed in 1967

1 000 000 decimals in 1971

137 · $10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

- Motivation: computation of $\pi$. 
Square root of 2 on the web

The first decimal digits of $\sqrt{2}$ are available on the web

$$1, 4, 1, 4, 2, 1, 3, 5, 6, 2, 3, 7, 3, 0, 9, 5, 0, 4, 8, 8, 0, 1,$$
$$6, 8, 8, 7, 2, 4, 2, 0, 9, 6, 9, 8, 0, 7, 8, 5, 6, 9, 6, 7, 1, 8, \ldots$$

http://oeis.org/A002193

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

http://oeis.org/
Pythagoras of Samos ∼ 569 BC – ∼ 475 BC

\[ a^2 + b^2 = c^2 = (a + b)^2 - 2ab. \]

http://www-history.mcs.st-and.ac.uk/Mathematicians/Pythagoras.html
Irrationality in Greek antiquity

Platon, La République: *incommensurable lines, irrational diagonals.*

Theodorus of Cyrene (about 370 BC.) irrationality of $\sqrt{3}, \ldots, \sqrt{17}$.

Theetetes: if an integer $n > 0$ is the square of a rational number, then it is the square of an integer.
Irrationality of $\sqrt{2}$

Pythagorean school

Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, $\sim$800-500 BC.
The sequence of decimal digits of $\sqrt{2}$ should behave like a random sequence, each digit should be occurring with the same frequency $1/10$, each sequence of 2 digits occurring with the same frequency $1/100$ ...
Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques*,
  Palermo Rend. 27, 247-271 (1909).
  Jahrbuch Database JFM 40.0283.01
  http://www.emis.de/MATH/JFM/JFM.html

- *Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaînes*,
  Zbl 0035.08302
Complexity of the $b$–ary expansion of an irrational algebraic real number

Let $b \geq 2$ be an integer.

• É. Borel (1909 and 1950): the $b$–ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (with respect to Lebesgue’s measure).

• Remark: no number satisfies all the laws which are shared by all numbers outside a set of measure zero, because the intersection of all these sets of full measure is empty!

$$\bigcap_{x \in \mathbb{R}} \mathbb{R} \setminus \{x\} = \emptyset.$$ 

• More precise statements by B. Adamczewski and Y. Bugeaud.
**Conjecture of Émile Borel**

**Conjecture** (É. Borel). Let $x$ be an irrational algebraic real number, $b \geq 3$ a positive integer and $a$ an integer in the range $0 \leq a \leq b - 1$. Then the digit $a$ occurs at least once in the $b$–ary expansion of $x$.

**Corollary.** Each given sequence of digits should occur infinitely often in the $b$–ary expansion of any real irrational algebraic number. (consider powers of $b$).

- An irrational number with a *regular* expansion in some basis $b$ should be transcendental.
The state of the art

There is no explicitly known example of a triple \((b, a, x)\), where \(b \geq 3\) is an integer, \(a\) is a digit in \(\{0, \ldots, b - 1\}\) and \(x\) is an algebraic irrational number, for which one can claim that the digit \(a\) occurs infinitely often in the \(b\)-ary expansion of \(x\).

A stronger conjecture, also due to Borel, is that algebraic irrational real numbers are *normal*: each sequence of \(n\) digits in basis \(b\) should occur with the frequency \(1/b^n\), for all \(b\) and all \(n\).
What is known on the decimal expansion of $\sqrt{2}$?

The sequence of digits (in any basis) of $\sqrt{2}$ is not ultimately periodic.

Among the decimal digits

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

at least two of them occur infinitely often. Almost nothing else is known.
Complexity of the expansion in basis $b$ of a real irrational algebraic number

**Theorem** (B. Adamczewski, Y. Bugeaud 2005; conjecture of A. Cobham 1968).

*If the sequence of digits of a real number $x$ is produced by a finite automaton, then $x$ is either rational or else transcendental.*
§2 Irrationality of transcendental numbers

- The number $e$
- The number $\pi$
- Open problems
Introductio in analysin infinitorum

Leonhard Euler (1737)
(1707 – 1783)
Introductio in analysin infinitorum

Continued fraction of $e$:

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \ddots}}}}}$$

$e$ is irrational.
Joseph Fourier

Fourier (1815) : proof by means of the series expansion

\[ e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{N!} + r_N \]

with \( r_N > 0 \) and \( N!r_N \to 0 \) as \( N \to +\infty \).

Course of analysis at the École Polytechnique Paris, 1815.
Variant of Fourier’s proof: $e^{-1}$ is irrational

C.L. Siegel: Alternating series

For odd $N$,

$$1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{N!} < e^{-1} < 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(N+1)!}$$

$$\frac{a_N}{N!} < e^{-1} < \frac{a_N}{N!} + \frac{1}{(N+1)!}, \quad a_N \in \mathbb{Z}$$

$$a_N < N!e^{-1} < a_N + 1.$$ 

Hence $N!e^{-1}$ is not an integer.
Irrationality of $\pi$

Āryabhaṭa, born 476 AD: $\pi \sim 3.1416$.

Nīlakaṇṭha Somayājī, born 1444 AD: Why then has an approximate value been mentioned here leaving behind the actual value? Because it (exact value) cannot be expressed.

Irrationality of $\pi$

Johann Heinrich Lambert (1728 – 1777)
*Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,*
Mémoires de l’Académie des Sciences de Berlin, 17 (1761), p. 265-322; lu en 1767; Math. Werke, t. II.

$\tan(\nu)$ is irrational when $\nu \neq 0$ is rational.
As a consequence, $\pi$ is irrational, since $\tan(\pi/4) = 1$. 
— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez-vous ?
— De moi-même !
Known and unknown transcendence results

Known:
\[ e, \pi, \log 2, e^{\sqrt{2}}, e\pi, 2^{\sqrt{2}}, \Gamma(1/4). \]

Not known:
\[ e + \pi, e\pi, \log \pi, \pi^e, \Gamma(1/5), \zeta(3), \text{Euler constant} \]

Why is \( e\pi \) known to be transcendental while \( \pi^e \) is not known to be irrational?
Answer: \( e\pi = (-1)^{-i}. \)
Catalan’s constant

Is Catalan’s constant
\[\sum_{n \geq 1} \frac{(-1)^n}{(2n + 1)^2} \]
\[= 0.9159655941772190150\ldots\]
an irrational number?
Catalan’s constant, Dirichlet and Kronecker

Catalan’s constant is the value at $s = 2$ of the Dirichlet $L$–function $L(s, \chi_{-4})$ associated with the Kronecker character

$$\chi_{-4}(n) = \left(\frac{n}{4}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod{4}, \\ -1 & \text{if } n \equiv -1 \pmod{4}. \end{cases}$$

Johann Peter Gustav Lejeune Dirichlet
1805 – 1859

Leopold Kronecker
1823 – 1891
Catalan’s constant, Dedekind and Riemann

The Dirichlet $L$–function $L(s, \chi_{-4})$ associated with the Kronecker character $\chi_{-4}$ is the quotient of the Dedekind zeta function of $\mathbb{Q}(i)$ and the Riemann zeta function:

$$\zeta_{\mathbb{Q}(i)}(s) = L(s, \chi_{-4})\zeta(s)$$

Julius Wilhelm Richard Dedekind
1831 – 1916

Georg Friedrich Bernhard Riemann
1826 – 1866
Riemann zeta function

The function
\[ \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \]
was studied by **Euler** (1707–1783) for integer values of \( s \)
and by **Riemann** (1859) for complex values of \( s \).

**Euler** : for any **even** integer value of \( s \geq 2 \), the number \( \zeta(s) \) is
a rational multiple of \( \pi^s \).

**Examples** : \( \zeta(2) = \pi^2 / 6 \), \( \zeta(4) = \pi^4 / 90 \), \( \zeta(6) = \pi^6 / 945 \),
\( \zeta(8) = \pi^8 / 9450 \ldots \)

**Coefficients** : **Bernoulli numbers**.
Riemann zeta function

The number

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1.202\,056\,903\,159\,594\,285\,399\,738\,161\,511 \ldots$$

is irrational (Apéry 1978).

Recall that $\zeta(s)/\pi^s$ is rational for any even value of $s \geq 2$.  

Open question: Is the number $\zeta(3)/\pi^3$ irrational?
Riemann zeta function

Is the number

$$\zeta(5) = \sum_{n \geq 1} \frac{1}{n^5} = 1.036927755143369926331365486457 \ldots$$

irrational?

*T. Rivoal* (2000): infinitely many $\zeta(2n + 1)$ are irrational.
Infinitely many odd zeta values are irrational

Tanguy Rivoal (2000)

Let $\epsilon > 0$. For any sufficiently large odd integer $a$, the dimension of the $\mathbb{Q}$-vector space spanned by the numbers $1, \zeta(3), \zeta(5), \cdots, \zeta(a)$ is at least

$$\frac{1 - \epsilon}{1 + \log 2} \log a.$$
Euler–Mascheroni constant

Euler’s Constant is

\[ \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) \]

\[ = 0.5772156649015328606512090082 \ldots \]

Is it a rational number?

\[ \gamma = \sum_{k=1}^{\infty} \left( \frac{1}{k} - \log \left( 1 + \frac{1}{k} \right) \right) = \int_1^{\infty} \left( \frac{1}{[x]} - \frac{1}{x} \right) \, dx \]

\[ = -\int_0^1 \int_0^1 \frac{(1-x) \, dx \, dy}{(1-xy) \log(xy)} \cdot \]
Euler’s constant

Recent work by J. Sondow inspired by the work of F. Beukers on Apéry’s proof.

F. Beukers

Jonathan Sondow

http://home.earthlink.net/~jsondow/
\[ \gamma = \int_0^\infty \sum_{k=2}^{\infty} \frac{1}{k^2(t+k)} \, dt \]
\[ \gamma = \lim_{s \to 1^+} \sum_{n=1}^{\infty} \left( \frac{1}{n^s} - \frac{1}{s^n} \right) \]
\[ \gamma = \int_1^\infty \frac{1}{2t(t+1)} F \left( 1, \frac{2}{3}, \frac{2}{t+2} \right) \, dt. \]
Euler Gamma function

Is the number

\[ \Gamma\left(\frac{1}{5}\right) = 4.590 \ 843 \ 711 \ 998 \ 803 \ 053 \ 204 \ 758 \ 275 \ 929 \ 152 \ \ldots \]

irrational?

\[ \Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_{0}^{\infty} e^{-t} t^z \cdot \frac{dt}{t} \]

Here is the set of rational values \( r \in (0, 1) \) for which the answer is known (and, for these arguments, the Gamma value \( \Gamma(r) \) is a transcendental number):

\[ r \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \right\} \pmod{1}. \]
Georg Cantor (1845 - 1918)

The set of algebraic numbers is countable, not the set of real (or complex) numbers.

Cantor (1874 and 1891).
Henri Léon Lebesgue (1875 – 1941)

Almost all numbers for Lebesgue measure are transcendental numbers.
Most numbers are transcendental

Meta conjecture: any number given by some kind of limit, which is not obviously rational (resp. algebraic), is irrational (resp. transcendental).
Special values of hypergeometric series

Jürgen Wolfart

Frits Beukers
Sum of values of a rational function


Let $P$ and $Q$ be non-zero polynomials having rational coefficients and $\deg Q \geq 2 + \deg P$. Consider

$$\sum_{n \geq 0 \atop Q(n) \neq 0} \frac{P(n)}{Q(n)}.$$
Telescoping series

Examples

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1, \quad \sum_{n=0}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}, \]

\[ \sum_{n=0}^{\infty} \left( \frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0 \]

\[ \sum_{n=0}^{\infty} \left( \frac{1}{5n+2} - \frac{3}{5n+7} + \frac{1}{5n-3} \right) = \frac{5}{6} \]
Transcendental values

\[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2, \]

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \]

\[ \sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3} \]

are transcendental.
Transcendental values

\[\sum_{n=0}^{\infty} \frac{1}{(6n + 1)(6n + 2)(6n + 3)(6n + 4)(6n + 5)(6n + 6)} = \frac{1}{4320} (192 \log 2 - 81 \log 3 - 7\pi \sqrt{3})\]

\[\sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}} = 2.0766740474 \ldots\]

\[\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + 1} = \frac{2\pi}{e^\pi - e^{-\pi}} = 0.272029054982 \ldots\]
The Fibonacci sequence $(F_n)_{n \geq 0}$:

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233 \ldots$

is defined by

$F_0 = 0, \ F_1 = 1,$

$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$
Encyclopedia of integer sequences (again)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, ...
Series involving Fibonacci numbers

The number
\[ \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1 \]
is rational, while
\[ \sum_{n=0}^{\infty} \frac{1}{F_{2n}} = \frac{7 - \sqrt{5}}{2} , \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1 - \sqrt{5}}{2} \]
and
\[ \sum_{n=1}^{\infty} \frac{1}{F_{2n-1} + 1} = \frac{\sqrt{5}}{2} \]
are irrational algebraic numbers.
Series involving Fibonacci numbers

The numbers

\[ \sum_{n=1}^{\infty} \frac{1}{F_{2^n}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{4^n}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{6^n}}, \]

\[ \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_{2}^n}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{2n}}, \]

\[ \sum_{n=1}^{\infty} \frac{1}{F_{2^n-1} + F_{2^n+1}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2^{n+1}}} \]

are all transcendental
Series involving Fibonacci numbers

Each of the numbers

\[
\sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}}
\]

is irrational, but it is not known whether they are algebraic or transcendental.

The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.
The Fibonacci zeta function

For $\Re(s) > 0$,

$$\zeta_F(s) = \sum_{n \geq 1} \frac{1}{F^n_s}$$

$\zeta_F(2)$, $\zeta_F(4)$, $\zeta_F(6)$ are algebraically independent.

Iekata Shiokawa, Carsten Elsner and Shun Shimomura (2006)
§3 Transcendental numbers

- Liouville (1844)
- Hermite (1873)
- Lindemann (1882)
- Hilbert’s Problem 7th (1900)
- Gel’fond–Schneider (1934)
- Baker (1968)
- Nesterenko (1995)
Existence of transcendental numbers (1844)

J. Liouville (1809 - 1882)

gave the first examples of transcendental numbers. For instance

$$\sum_{n \geq 1} \frac{1}{10^n!} = 0.110\,001\,000\,000\,0\ldots$$

is a transcendental number.
Charles Hermite and Ferdinand Lindemann

**Hermite (1873):**
Transcendence of $e$

$e = 2.7182818284 \ldots$

**Lindemann (1882):**
Transcendence of $\pi$

$\pi = 3.1415926535 \ldots$
Hermite–Lindemann Theorem

For any non-zero complex number \( z \), one at least of the two numbers \( z \) and \( e^z \) is transcendental.

**Corollaries**: Transcendence of \( \log \alpha \) and of \( e^\beta \) for \( \alpha \) and \( \beta \) non-zero algebraic complex numbers, provided \( \log \alpha \neq 0 \).
A complex function is called **transcendental** if it is transcendental over the field $\mathbb{C}(z)$, which means that the functions $z$ and $f(z)$ are algebraically independent: if $P \in \mathbb{C}[X, Y]$ is a non-zero polynomial, then the function $P(z, f(z))$ is not 0.

**Exercise.** An entire function (analytic in $\mathbb{C}$) is transcendental if and only if it is not a polynomial.

**Example.** The transcendental entire function $e^{z}$ takes an algebraic value at an algebraic argument $z$ only for $z = 0$. 
Weierstrass question

Is it true that a transcendental entire function \( f \) takes usually transcendental values at algebraic arguments?

**Examples:** for \( f(z) = e^z \), there is a single exceptional point \( \alpha \) algebraic with \( e^\alpha \) also algebraic, namely \( \alpha = 0 \).

For \( f(z) = e^{P(z)} \) where \( P \in \mathbb{Z}[z] \) is a non–constant polynomial, there are finitely many exceptional points \( \alpha \), namely the roots of \( P \).

The exceptional set of \( e^z + e^{1+z} \) is empty (Lindemann–Weierstrass).

The exceptional set of functions like \( 2^z \) or \( e^{i\pi z} \) is \( \mathbb{Q} \), (Gel’fond and Schneider).
Exceptional sets

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain...

If $S$ is a countable subset of $\mathbb{C}$ and $T$ is a dense subset of $\mathbb{C}$, there exist transcendental entire functions $f$ mapping $S$ into $T$, as well as all its derivatives.

Any set of algebraic numbers is the exceptional set of some transcendental entire function. Also multiplicities can be included.

van der Poorten: there are transcendental entire functions $f$ such that $D^k f(\alpha) \in \mathbb{Q}(\alpha)$ for all $k \geq 0$ and all algebraic $\alpha$. 
Integer valued entire functions

An integer valued entire function is a function $f$, which is analytic in $\mathbb{C}$, and maps $\mathbb{N}$ into $\mathbb{Z}$.

Example: $2^z$ is an integer valued entire function, not a polynomial.

Question: Are there integer valued entire function growing slower than $2^z$ without being a polynomial?

Let $f$ be a transcendental entire function in $\mathbb{C}$. For $R > 0$ set

$$|f|_R = \sup_{|z| = R} |f(z)|.$$
Integer valued entire functions

G. Pólya (1914) :
if $f$ is not a polynomial and $f(n) \in \mathbb{Z}$ for $n \in \mathbb{Z}_{\geq 0}$, then
$$\limsup_{R \to \infty} 2^{-R} |f|_R \geq 1.$$ 

Further works on this topic by G.H. Hardy, G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross,\ldots
Integer valued entire function on $\mathbb{Z}[i]$

A.O. Gel’fond (1929) : growth of entire functions mapping the Gaussian integers into themselves. Newton interpolation series at the points in $\mathbb{Z}[i]$. 

An entire function $f$ which is not a polynomial and satisfies $f(a + ib) \in \mathbb{Z}[i]$ for all $a + ib \in \mathbb{Z}[i]$ satisfies 

$$\limsup_{R \to \infty} \frac{1}{R^2} \log |f|_R \geq \delta.$$ 

F. Gramain (1981) : $\delta = \pi/(2e) = 0.577 863 674 8 \ldots$. This is best possible : D.W. Masser (1980).
Transcendence of $e^\pi$

A.O. Gel’fond (1929).

If

$$e^\pi = 23.140692632779269005729086367 \ldots$$

is rational, then the function $e^{\pi z}$ takes values in $\mathbb{Q}(i)$ when the argument $z$ is in $\mathbb{Z}[i]$.

Expand $e^{\pi z}$ into an interpolation series at the Gaussian integers.
Hilbert’s Problems

August 8, 1900

David Hilbert (1862 - 1943)


Twin primes,
Goldbach’s Conjecture,
Riemann Hypothesis

Transcendence of $e^\pi$ and $2\sqrt{2}$
A.O. Gel’fond and Th. Schneider

Solution of Hilbert’s seventh problem (1934): Transcendence of $\alpha^\beta$ and of $\frac{\log \alpha_1}{\log \alpha_2}$ for algebraic $\alpha$, $\beta$, $\alpha_1$ and $\alpha_2$. 
Transcendence of $\alpha^\beta$ and $\log \alpha_1/\log \alpha_2$: examples

The following numbers are transcendental:

$$2^{\sqrt{2}} = 2.665\ 144\ 142\ 6\ldots$$

$$\frac{\log 2}{\log 3} = 0.630\ 929\ 753\ 5\ldots$$

$$e^\pi = 23.140\ 692\ 632\ 7\ldots$$

$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.999\ 999\ 999\ 999\ 999\ 25\ldots$$
\[ e^{\pi} = (-1)^{-i} \]

**Example**: Transcendence of the number

\[ e^{\pi \sqrt{163}} = 262 \ 537 \ 412 \ 640 \ 768 \ 743.999 \ 999 \ 999 \ 999 \ 999 \ 2. \ldots. \]

**Remark.** For

\[ \tau = \frac{1 + i \sqrt{163}}{2}, \quad q = e^{2i\pi\tau} = -e^{-\pi \sqrt{163}} \]

we have \( j(\tau) = -640 \ 320^3 \) and

\[ \left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}. \]
Beta values: Th. Schneider 1948

Euler Gamma and Beta functions

\[ B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} \, dx. \]

\[ \Gamma(z) = \int_0^\infty e^{-t} \, t^z \cdot \frac{dt}{t} \]

\[ B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \]
The two numbers $2^{3/2}$ and $2^{3/4}$ are algebraically independent.

More generally, if $\alpha$ is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if $\beta$ is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$$\alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$$

are algebraically independent.
Transcendence of numbers like

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

or

$$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_1^{\beta_1}$$

for algebraic $\alpha_i$’s and $\beta_j$’s.

Example (Siegel) :

$$\int_0^1 \frac{dx}{1 + x^3} = \frac{1}{3} \left( \log 2 + \frac{\pi}{\sqrt{3}} \right) = 0.835648848 \ldots$$

is transcendental.
G.V. Chudnovsky (1976)

Algebraic independence of the numbers $\pi$ and $\Gamma(1/4)$.

Also: algebraic independence of the numbers $\pi$ and $\Gamma(1/3)$.

Corollaries: Transcendence of $\Gamma(1/4) = 3.6256099082\ldots$ and $\Gamma(1/3) = 2.6789385347\ldots$. 
Yuri V. Nesterenko

Yu.V.Nesterenko (1996)
Algebraic independence of
$\Gamma(1/4)$, $\pi$ and $e^\pi$.
Also : Algebraic independence of
$\Gamma(1/3)$, $\pi$ and $e^{\pi \sqrt{3}}$.

Corollary : The numbers $\pi = 3.1415926535\ldots$ and $e^\pi = 23.1406926327\ldots$ are algebraically independent.

Transcendence of values of Dirichlet’s $L$–functions :
Sanoli Gun, Ram Murty and Purusottam Rath (2009).
Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The canonical product attached to $\Omega$ is the *Weierstraß sigma function*

$$\sigma(z) = \sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{(z/\omega)^2 + (z^2/2\omega^2)}.$$ 

The number

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4} \pi^{1/2} e^{\pi/8} \Gamma(1/4)^{-2}$$

is transcendental.
§4 : Conjectures

Borel 1909, 1950

Schanuel 1964

Grothendieck 1960’s

Rohrlich and Lang 1970’s

André 1990’s

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.
The number $\pi$

Basic example of a *period*:

$$e^{z+2i\pi} = e^z$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\pi = \int \int_{x^2+y^2 \leq 1} dx \, dy = 2 \int_{-1}^{1} \sqrt{1-x^2} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1-x^2}.$$
Further examples of periods

\[ \sqrt{2} = \int_{2x^2 \leq 1} dx \]

and all algebraic numbers.

\[ \log 2 = \int_{1 < x < 2} \frac{dx}{x} \]

and all logarithms of algebraic numbers.

\[ \pi = \int_{x^2 + y^2 \leq 1} dxdy, \]

A product of periods is a period (subalgebra of \( \mathbb{C} \)), but \( 1/\pi \) is expected not to be a period.
Relations among periods

1. Additivity
   (in the integrand and in the domain of integration)

   \[
   \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx,
   \]

   \[
   \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
   \]

2. Change of variables:
   if \( y = f(x) \) is an invertible change of variables, then

   \[
   \int_{f(a)}^{f(b)} F(y) \, dy = \int_a^b F(f(x)) f'(x) \, dx.
   \]
Relations among periods (continued)

3. **Newton–Leibniz–Stokes Formula**

\[ \int_a^b f'(x) \, dx = f(b) - f(a). \]
Conjecture of Kontsevich and Zagier

A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following

**Conjecture (Kontsevich–Zagier).** *If a period has two integral representations, then one can pass from one formula to another by using only rules 1, 2, 3 in which all functions and domains of integration are algebraic with algebraic coefficients.*
In other words, we do not expect any miraculous coincidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.

This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

Advice: if you wish to prove a number is transcendental, first prove it is a period.
Conjectures by S. Schanuel and A. Grothendieck

• **Schanuel**: if \( x_1, \ldots, x_n \) are \( \mathbb{Q} \)-linearly independent complex numbers, then \( n \) at least of the \( 2n \) numbers \( x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n} \) are algebraically independent.

• **Periods conjecture by Grothendieck**: Dimension of the Mumford–Tate group of a smooth projective variety.
Consequences of Schanuel’s Conjecture

Ram Murty

Kumar Murty

N. Saradha

Purusottam Rath, Ram Murty, Sanoli Gun
Transcendental values of class group $L$–functions.
Motives

Y. André: generalization of Grothendieck’s conjecture to motives.

Case of 1-motives: Elliptico-Toric Conjecture of C. Bertolin.
A simple geometric construction on the moduli spaces $\mathcal{M}_{0,n}$ of curves of genus 0 with $n$ ordered marked points is described which gives a common framework for many irrationality proofs for zeta values. This construction yields Apéry’s approximations to $\zeta(2)$ and $\zeta(3)$, and for larger $n$, an infinite family of small linear forms in multiple zeta values with an interesting algebraic structure. It also contains a generalisation of the linear forms used by Ball and Rivoal to prove that infinitely many odd zeta values are irrational.
For $k, s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$, we set $s = (s_1, \ldots, s_k)$ and

$$\zeta(s) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$ 

The $\mathbb{Q}$–vector space $\mathfrak{Z}$ spanned by the numbers $\zeta(s)$ is also a $\mathbb{Q}$–algebra. For $n \geq 2$, denote by $\mathfrak{Z}_n$ the $\mathbb{Q}$-subspace of $\mathfrak{Z}$ spanned by the real numbers $\zeta(s)$ where $s$ has weight $s_1 + \cdots + s_k = n$.

The numbers $\zeta(s_1, \ldots, s_k)$, $s_1 + \cdots + s_k = n$, where each $s_j$ is 2 or 3, span $\mathfrak{Z}_n$ over $\mathbb{Q}$. 
Transcendental Number Theory: recent results and open problems.

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