Abstract

This lecture will be devoted to a survey of transcendental number theory, including some history, the state of the art and some of the main conjectures, the limits of the current methods and the obstacles which are preventing from going further.

Extended abstract

An algebraic number is a complex number which is a root of a polynomial with rational coefficients. For instance $\sqrt{2}$, $i = \sqrt{-1}$, the Golden Ratio $(1 + \sqrt{5})/2$, the roots of unity $e^{2\pi i/a}$, the roots of the polynomial $X^5 - 6X + 3$ are algebraic numbers. A **transcendental number** is a complex number which is not algebraic.
Rational, algebraic irrational, transcendental

**Goal**: decide upon the arithmetic nature of “given” numbers: rational, algebraic irrational, transcendental.

Rational integers: \( \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots \} \).

Rational numbers: \( \mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1\} \).

Algebraic number: root of a polynomial with rational coefficients.

A **transcendental number** is a complex number which is not algebraic.

Algebraic irrational numbers

Examples of algebraic irrational numbers:

- \( \sqrt{2}, i = \sqrt{-1} \), the Golden Ratio \((1 + \sqrt{5})/2, \)
- \( \sqrt{d} \) for \( d \in \mathbb{Z} \) not the square of an integer (hence not the square of a rational number),
- the roots of unity \( e^{2\pi in/a} \), for \( a/b \in \mathbb{Q} \),
- and, of course, any root of an irreducible polynomial with rational coefficients of degree \( > 1 \).

Rational, algebraic irrational, transcendental

**Goal**: decide whether a “given” real number is rational, algebraic irrational or else transcendental.

- **Question**: what means "given"?

- Criteria for irrationality: development in a given basis (e.g.: decimal expansion, binary expansion), continued fraction.

- Analytic formulae, limits, sums, integrals, infinite products, any limiting process.

Rule and compass; squaring the circle

Construct a square with the same area as a given circle by using only a finite number of steps with compass and straightedge.

Any constructible length is an algebraic number, though not every algebraic number is constructible (for example \( \sqrt{2} \) is not constructible).

Pierre Laurent Wantzel (1814 – 1848)
*Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas. Journal de Mathématiques Pures et Appliquées 1* (2), (1837), 366–372.
Quadrature of the circle

Marie Jacob
La quadrature du cercle
Un problème
à la mesure des Lumières
Fayard (2006).

Resolution of equations by radicals

The roots of the polynomial
\(X^5 - 6X + 3\) are algebraic
numbers, and are not
expressible by radicals.

Evariste Galois
(1811 – 1832)

Gottfried Wilhelm Leibniz

Introduction of the concept of
the transcendental in
mathematics by Gottfried
Wilhelm Leibniz in 1684 :
“Nova methodus pro maximis
et minimis itemque
tangentibus, qua nec fractas,
nec irrationales quantitates
moratur, . . .”

Breger, Herbert. Leibniz' Einführung des Transzendenten, 300
Jahre “Nova Methodus” von G. W. Leibniz (1684-1984),

Serfati, Michel. Quadrature du cercle, fractions continues et autres

§1 Irrationality

Given a basis \(b \geq 2\), a real number \(x\) is rational if and only if
its expansion in basis \(b\) is ultimately periodic.

\(b = 2\) : binary expansion.

\(b = 10\) : decimal expansion.

For instance the decimal number

\[
0.123456789012345678901234567890\ldots
\]

is rational :

\[
\frac{1234567890}{999999999} = \frac{137174210}{1111111111}
\]
First decimal digits of $\sqrt{2}$

http://wims.unice.fr/wims/wims.cgi

141421356237309504880168872420969807856976187537694807317667973
79907324784621073885038753432764157273501384623091229702402483
6055805737212644124970999395381413222665925059275579995050115
27820605714701095599716059702745345968201472851741864088919860
95923292304830871432145083976260362799525140798968725339664633
18088296402601525835239505474575028775996172983557522033753185
7011354374603408498847160386899706990048150305402779013645424
782306949296391862158057846311159666871301310156185689872373258
850926468124949771542183420248256860601468247207714358548741556
57096776573202264880470158588016207584794226572260020855844665
2145839889394437092659180311388246468157082501005948587040031
8648034219897278290641045072636881313739855256117322042050912
27700226941127576272804957381089675040183698683684507257993647
29060762909613804756548237289971803268042746292691248590521
81004459824150591120249441347285314781058036033710773091828693
14710171111683916581726889419758716585212822951848847 . . .

Computation of decimals of $\sqrt{2}$

1542 decimals computed by hand by Horace Uhler in 1951

14000 decimals computed in 1967

1000000 decimals in 1971

137 · 10⁹ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

Motivation : computation of $\pi$.

Square root of 2 on the web

The first decimal digits of $\sqrt{2}$ are available on the web

1, 4, 1, 4, 2, 1, 3, 5, 6, 2, 3, 7, 3, 0, 9, 5, 0, 4, 8, 8, 0, 1, 6, 8, 8, 7, 2, 4, 2, 0, 9, 6, 9, 8, 0, 7, 8, 5, 6, 9, 7, 1, 8, . . .

http://oeis.org/A002193

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

http://oeis.org/
Pythagoras of Samos ~ 569 BC – ~ 475 BC

\[ a^2 + b^2 = c^2 = (a + b)^2 - 2ab. \]

http://www-history.mcs.st-and.ac.uk/Mathematicians/Pythagoras.html

Irrationality in Greek antiquity

Platon, La République: *incommensurable lines, irrational diagonals.*

Theodorus of Cyrene (about 370 BC.) irrationality of \( \sqrt{3}, \ldots, \sqrt{17}. \)

Theetetus: if an integer \( n > 0 \) is the square of a rational number, then it is the square of an integer.

Irrationality of \( \sqrt{2} \)

Pythagorean school

Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, ~800-500 BC.

Émile Borel : 1950

The sequence of decimal digits of \( \sqrt{2} \) should behave like a random sequence, each digit should be occurring with the same frequency \( 1/10 \), each sequence of 2 digits occurring with the same frequency \( 1/100 \) . . .
Émile Borel (1871–1956)


Jahrbuch Database http://www.emis.de/MATH/JFM/JFM.html


Zbl 0035.08302

**Conjecture of Émile Borel**

**Conjecture** (É. Borel). Let $x$ be an irrational algebraic real number, $b \geq 3$ a positive integer and $a$ an integer in the range $0 \leq a \leq b - 1$. Then the digit $a$ occurs at least once in the $b$–ary expansion of $x$.

**Corollary.** Each given sequence of digits should occur infinitely often in the $b$–ary expansion of any real irrational algebraic number. (consider powers of $b$).

- An irrational number with a regular expansion in some basis $b$ should be transcendental.

**Complexity of the $b$–ary expansion of an irrational algebraic real number**

Let $b \geq 2$ be an integer.

- É. Borel (1909 and 1950): the $b$–ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (with respect to Lebesgue’s measure).

- **Remark**: no number satisfies all the laws which are shared by all numbers outside a set of measure zero, because the intersection of all these sets of full measure is empty!

$$\bigcap_{x \in \mathbb{R}} \mathbb{R} \setminus \{x\} = \emptyset.$$

- More precise statements by B. Adamczewski and Y. Bugeaud.

**The state of the art**

There is no explicitly known example of a triple $(b, a, x)$, where $b \geq 3$ is an integer, $a$ is a digit in $\{0, \ldots, b - 1\}$ and $x$ is an algebraic irrational number, for which one can claim that the digit $a$ occurs infinitely often in the $b$–ary expansion of $x$.

A stronger conjecture, also due to Borel, is that algebraic irrational real numbers are normal: each sequence of $n$ digits in basis $b$ should occur with the frequency $1/b^n$, for all $b$ and all $n$. 
What is known on the decimal expansion of $\sqrt{2}$?

The sequence of digits (in any basis) of $\sqrt{2}$ is not ultimately periodic.

Among the decimal digits

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

at least two of them occur infinitely often. Almost nothing else is known.

Complexity of the expansion in basis $b$ of a real irrational algebraic number

Theorem (B. Adamczewski, Y. Bugeaud 2005; conjecture of A. Cobham 1968).

If the sequence of digits of a real number $x$ is produced by a finite automaton, then $x$ is either rational or else transcendental.

Finite automata

The Prouhet – Thue – Morse sequence : A010060 OEIS

$(t_n)_{n \geq 0} = (011010011001011001011001101001 \ldots)$

Write the number $n$ in binary.

If the number of ones in this binary expansion is odd then $t_n = 1$, if even then $t_n = 0$.

Fixed point of the morphism $0 \mapsto 01$, $1 \mapsto 10$.

Start with 0 and successively append the Boolean complement of the sequence obtained thus far.

$t_0 = 0$, $t_{2n} = t_n$, $t_{2n+1} = 1 - t_n$

Sequence without cubes XXX.

§2 Irrationality of transcendental numbers

- The number $e$
- The number $\pi$
- Open problems
**Introductio in analysin infinitorum**

Leonhard Euler (1737)  
(1707 – 1783)  
Introductio in analysin infinitorum

Continued fraction of $e$:

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4 + \ddots}}}}$$

e is irrational.

Joseph Fourier

Fourier (1815) : proof by means of the series expansion

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{N!} + r_N$$

with $r_N > 0$ and $N!r_N \to 0$ as $N \to +\infty$.

Course of analysis at the École Polytechnique Paris, 1815.

**Variant of Fourier's proof : $e^{-1}$ is irrational**

C.L. Siegel : Alternating series

For odd $N$,

$$1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{N!} < e^{-1} < 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(N+1)!}$$

$$\frac{a_N}{N!} < e^{-1} < \frac{a_N}{N!} + \frac{1}{(N+1)!}, \quad a_N \in \mathbb{Z}$$

$$a_N < N!e^{-1} < a_N + 1.$$  
Hence $N!e^{-1}$ is not an integer.

**Irrationality of $\pi$**

Āryabhaṭa, born 476 AD : $\pi \approx 3.1416$.

Nīlakaṇṭha Somayājī, born 1444 AD : Why then has an approximate value been mentioned here leaving behind the actual value? Because it (exact value) cannot be expressed.

Irrationality of $\pi$

Johann Heinrich Lambert (1728 – 1777)


\[ \tan(v) \text{ is irrational when } v \neq 0 \text{ is rational.} \]

As a consequence, $\pi$ is irrational, since $\tan(\pi/4) = 1$.

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Lambert and Frederick II, King of Prussia

— Que savez vous, Lambert ?
— Tout, Sire.
— Et de qui le tenez-vous ?
— De moi-même !

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Known and unknown transcendence results

**Known:**

\[ e, \pi, \log 2, e^{\sqrt{2}}, e^\pi, 2^{\sqrt{2}}, \Gamma(1/4). \]

**Not known:**

\[ e + \pi, e\pi, \log \pi, \pi^e, \Gamma(1/5), \zeta(3), \text{ Euler constant} \]

Why is $e^\pi$ known to be transcendental while $\pi^e$ is not known to be irrational?

**Answer:** $e^\pi = (-1)^{\pi^e}$.

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Catalan’s constant

Is Catalan’s constant

\[ \sum_{n \geq 1} \frac{(-1)^n}{(2n+1)^2} = 0.915 965 594 177 219 015 0 \ldots \]

an irrational number?
Catalan’s constant, Dirichlet and Kronecker

Catalan’s constant is the value at \( s = 2 \) of the Dirichlet L–function \( L(s, \chi_{-4}) \) associated with the Kronecker character

\[
\chi_{-4}(n) = \left( \frac{n}{4} \right) = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
1 & \text{if } n \equiv 1 \pmod{4}, \\
-1 & \text{if } n \equiv -1 \pmod{4}.
\end{cases}
\]

Johann Peter Gustav Lejeune Dirichlet 1805 – 1859
Leopold Kronecker 1823 – 1891

Riemann zeta function

The function

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

was studied by Euler (1707–1783) for integer values of \( s \) and by Riemann (1859) for complex values of \( s \).

Euler: for any even integer value of \( s \geq 2 \), the number \( \zeta(s) \) is a rational multiple of \( \pi^s \).

Examples: \( \zeta(2) = \pi^2/6 \), \( \zeta(4) = \pi^4/90 \), \( \zeta(6) = \pi^6/945 \), \( \zeta(8) = \pi^8/9450 \cdot \cdot \cdot \)

Coefficients: Bernoulli numbers.

Julius Wilhelm Richard Dedekind 1831 – 1916
Georg Friedrich Bernhard Riemann 1826 – 1866

Catalan’s constant, Dedekind and Riemann

The Dirichlet L–function \( L(s, \chi_{-4}) \) associated with the Kronecker character \( \chi_{-4} \) is the quotient of the Dedekind zeta function of \( \mathbb{Q}(i) \) and the Riemann zeta function:

\[
\zeta_{\mathbb{Q}(i)}(s) = L(s, \chi_{-4})\zeta(s)
\]

Riemann zeta function

The number

\[
\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202 056 903 159 594 285 399 738 161 151 \ldots
\]

is irrational (Apéry 1978).

Recall that \( \zeta(s)/\pi^s \) is rational for any even value of \( s \geq 2 \).

Open question: Is the number \( \zeta(3)/\pi^3 \) irrational?
Riemann zeta function

Is the number
\[ \zeta(5) = \sum_{n\geq1} \frac{1}{n^5} = 1.03692775514336926331365486457\ldots \]
irrational?

_T. Rivoal_ (2000) : infinitely many \( \zeta(2n+1) \) are irrational.

Euler–Mascheroni constant

_Euler’s Constant_ is

\[ \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) \]
\[ = 0.57721566490153286060512090082\ldots \]

Is it a rational number?

\[ \gamma = \sum_{k=1}^{\infty} \left( \frac{1}{k} - \log \left( 1 + \frac{1}{k} \right) \right) = \int_1^{\infty} \left( \frac{1}{[x]} - \frac{1}{x} \right) \, dx \]
\[ = -\int_0^1 \int_0^1 \frac{(1-x)\,dy\,dx}{(1-xy)\log(xy)}. \]

Infinitely many odd zeta values are irrational

_Tanguy Rivoal_ (2000)

Let \( \epsilon > 0 \). For any sufficiently large odd integer \( a \), the dimension of the \( \mathbb{Q} \)-vector space spanned by the numbers \( 1, \zeta(3), \zeta(5), \ldots, \zeta(a) \) is at least
\[ \frac{1 - \epsilon}{1 + \log 2} \log a. \]

Euler’s constant

Recent work by _J. Sondow_ inspired by the work of _F. Beukers_ on _Apéry’s proof_.

_F. Beukers_  
_Jonathan Sondow_

[http://home.earthlink.net/~jsondow/](http://home.earthlink.net/~jsondow/)
Euler Gamma function

Is the number
\[ \Gamma(1/5) = 4.590 \, 843 \, 711 \, 998 \, 803 \, 053 \, 204 \, 758 \, 275 \, 929 \, 152 \ldots \]
irrational?

\[ \Gamma(z) = e^{-z} z^{-1} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^{-1} e^{z/n} = \int_{0}^{\infty} e^{-t} t^{z} \cdot \frac{dt}{t} \]

Here is the set of rational values \( r \in (0, 1) \) for which the answer is known (and, for these arguments, the Gamma value \( \Gamma(r) \) is a transcendental number):

\[ r \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, 2, \frac{3}{2}, \frac{3}{4}, \frac{5}{6} \right\} \pmod{1} \]

Georg Cantor (1845 - 1918)

The set of algebraic numbers is countable, not the set of real (or complex) numbers.

Henri Léon Lebesgue (1875 – 1941)

Almost all numbers for Lebesgue measure are transcendental numbers.

Cantor (1874 and 1891).
Most numbers are transcendental

Meta conjecture: any number given by some kind of limit, which is not obviously rational (resp. algebraic), is irrational (resp. transcendental).

Sum of values of a rational function

Let $P$ and $Q$ be non-zero polynomials having rational coefficients and $\deg Q \geq 2 + \deg P$. Consider

$$\sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}.$$

Telescoping series

Examples

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1, \quad \sum_{n=0}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4},$$

$$\sum_{n=0}^{\infty} \left( \frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0$$

$$\sum_{n=0}^{\infty} \left( \frac{1}{5n+2} - \frac{3}{5n+7} + \frac{1}{5n-3} \right) = \frac{5}{6}$$
Transcendental values

\[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2, \]
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \]
\[ \sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3} \]
are transcendental.

Leonardo Pisano (Fibonacci)

The Fibonacci sequence \((F_n)_{n \geq 0}\):

\[
0, 1, 1, 2, 3, 5, 8, 13, 21,
34, 55, 89, 144, 233 \ldots
\]
is defined by

\[ F_0 = 0, \quad F_1 = 1, \]
\[ F_n = F_{n-1} + F_{n-2} \quad (n \geq 2). \]

Transcendental values

\[ \sum_{n=0}^{\infty} \frac{1}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)} = \frac{1}{4320} (192 \log 2 - 81 \log 3 - 7\pi \sqrt{3}) \]
\[ \sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}} = 2.0766740474 \ldots \]
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + 1} = \frac{2\pi}{e^\pi - e^{-\pi}} = 0.272029054982 \ldots \]

Encyclopedia of integer sequences (again)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, \ldots

The Fibonacci sequence is available online

The On-Line Encyclopedia of Integer Sequences

Neil J. A. Sloane

http://oeis.org/A000045
Series involving Fibonacci numbers

The number

\[ \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1 \]

is rational, while

\[ \sum_{n=0}^{\infty} \frac{1}{F_{2n}} = \frac{7 - \sqrt{5}}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1}} = \frac{1 - \sqrt{5}}{2} \]

and

\[ \sum_{n=1}^{\infty} \frac{1}{F_{2n-1} + 1} = \frac{\sqrt{5}}{2} \]

are irrational algebraic numbers.

Each of the numbers

\[ \sum_{n=1}^{\infty} \frac{F_n}{F_n}, \quad \sum_{n=1}^{\infty} \frac{1}{F_n + F_{n+2}}, \quad \sum_{n \geq 1} \frac{1}{F_1 F_2 \cdots F_n} \]

is irrational, but it is not known whether they are algebraic or transcendental.

The first challenge here is to formulate a conjectural statement which would give a satisfactory description of the situation.

The Fibonacci zeta function

For \( \Re(s) > 0 \),

\[ \zeta_F(s) = \sum_{n \geq 1} \frac{1}{F_n^s} \]

\( \zeta_F(2), \ zeta_F(4), \ zeta_F(6) \) are algebraically independent.

Iekata Shiokawa, Carsten Elsner and Shun Shimomura (2006)
§3 Transcendental numbers

- Liouville (1844)
- Hermite (1873)
- Lindemann (1882)
- Hilbert’s Problem 7th (1900)
- Gel’fond–Schneider (1934)
- Baker (1968)
- Nesterenko (1995)

Existence of transcendental numbers (1844)

J. Liouville (1809 - 1882)
gave the first examples of transcendental numbers.

For instance

\[
\sum_{n=1}^{\infty} \frac{1}{10^n} = 0.110
dots
\]

is a transcendental number.

Charles Hermite and Ferdinand Lindemann

Hermite (1873) :
Transcendence of \( e \)
\( e = 2.718281828 \ldots \)

Lindemann (1882) :
Transcendence of \( \pi \)
\( \pi = 3.141592653 \ldots \)

Hermite–Lindemann Theorem

For any non-zero complex number \( z \), one at least of the two numbers \( z \) and \( e^z \) is transcendental.

Corollaries : Transcendence of \( \log \alpha \) and of \( e^\beta \) for \( \alpha \) and \( \beta \)
non-zero algebraic complex numbers, provided \( \log \alpha \neq 0 \).
Transcendental functions

A complex function is called transcendental if it is transcendental over the field \( \mathbb{C}(z) \), which means that the functions \( z \) and \( f(z) \) are algebraically independent: if \( P \in \mathbb{C}[X, Y] \) is a non-zero polynomial, then the function \( P(z, f(z)) \) is not 0.

Exercise. An entire function (analytic in \( \mathbb{C} \)) is transcendental if and only if it is not a polynomial.

Example. The transcendental entire function \( e^z \) takes an algebraic value at an algebraic argument \( z \) only for \( z = 0 \).

Weierstrass question

Is it true that a transcendental entire function \( f \) takes usually transcendental values at algebraic arguments?

Examples: for \( f(z) = e^z \), there is a single exceptional point \( \alpha \) algebraic with \( e^\alpha \) also algebraic, namely \( \alpha = 0 \).
For \( f(z) = e^{P(z)} \) where \( P \in \mathbb{Z}[z] \) is a non–constant polynomial, there are finitely many exceptional points \( \alpha \), namely the roots of \( P \).
The exceptional set of \( e^z + e^{1+z} \) is empty (Lindemann–Weierstrass).
The exceptional set of functions like \( 2^z \) or \( e^{inz} \) is \( \mathbb{Q} \), (Gel’fond and Schneider).

Exceptional sets

Answers by Weierstrass (letter to Strauss in 1886), Strauss, Stäckel, Faber, van der Poorten, Gramain...

If \( S \) is a countable subset of \( \mathbb{C} \) and \( T \) is a dense subset of \( \mathbb{C} \), there exist transcendental entire functions \( f \) mapping \( S \) into \( T \), as well as all its derivatives.

Any set of algebraic numbers is the exceptional set of some transcendental entire function.
Also multiplicities can be included.

van der Poorten: there are transcendental entire functions \( f \) such that \( D^k f(\alpha) \in \mathbb{Q}(\alpha) \) for all \( k \geq 0 \) and all algebraic \( \alpha \).

Integer valued entire functions

An integer valued entire function is a function \( f \), which is analytic in \( \mathbb{C} \), and maps \( \mathbb{N} \) into \( \mathbb{Z} \).

Example: \( 2^z \) is an integer valued entire function, not a polynomial.

Question: Are there integer valued entire function growing slower than \( 2^z \) without being a polynomial?

Let \( f \) be a transcendental entire function in \( \mathbb{C} \). For \( R > 0 \) set
\[
|f|_R = \sup_{|z|=R} |f(z)|.
\]
Integer valued entire functions

G. Pólya (1914): if \( f \) is not a polynomial and \( f(n) \in \mathbb{Z} \) for \( n \in \mathbb{Z}_{\geq 0} \), then \( \limsup_{R \to \infty} 2^{-R} |f|_R \geq 1 \).

Further works on this topic by G.H. Hardy, G. Pólya, D. Sato, E.G. Straus, A. Selberg, Ch. Pisot, F. Carlson, F. Gross, . . .

Transcendence of \( e^\pi \)

A.O. Gel’fond (1929).

If \( e^\pi = 23.140692632779269005729086367 \ldots \) is rational, then the function \( e^{\pi z} \) takes values in \( \mathbb{Q}(i) \) when the argument \( z \) is in \( \mathbb{Z}[i] \).

Expand \( e^{\pi z} \) into an interpolation series at the Gaussian integers.

Hilbert’s Problems

August 8, 1900


Twin primes,

Goldbach’s Conjecture,

Riemann Hypothesis

David Hilbert (1862 - 1943)

Transcendence of \( e^\pi \) and \( 2^{\sqrt{2}} \)

A.O. Gel’fond (1929): growth of entire functions mapping the Gaussian integers into themselves.

Newton interpolation series at the points in \( \mathbb{Z}[i] \).

An entire function \( f \) which is not a polynomial and satisfies \( f(a + ib) \in \mathbb{Z}[i] \) for all \( a + ib \in \mathbb{Z}[i] \) satisfies

\[
\limsup_{R \to \infty} \frac{1}{R^2} \log |f|_R \geq \delta.
\]

F. Gramain (1981): \( \delta = \pi/(2e) = 0.5770863648 \ldots \)

This is best possible: D.W. Masser (1980).
Transcendence of $\alpha^\beta$ and $\log \alpha_1 / \log \alpha_2$ : examples

The following numbers are transcendental:

$$2^{\sqrt{2}} = 2.665 144 142 6\ldots$$

$$\frac{\log 2}{\log 3} = 0.630 929 753 5\ldots$$

$$e^\pi = 23.140 692 632 7\ldots \quad (e^\pi = (-1)^{-i})$$

$$e^{\pi\sqrt{163}} = 262 537 412 640 768 743.999 999 999 999 25\ldots$$

$$e^{\tau} = (-1)^{-i}$$

Example: Transcendence of the number

$$e^{\pi\sqrt{163}} = 262 537 412 640 768 743.999 999 999 999 2\ldots$$

Remark. For

$$\tau = \frac{1 + i \sqrt{163}}{2}, \quad q = e^{2i\pi\tau} = -e^{-\pi\sqrt{163}}$$

we have $j(\tau) = -640 320^3$ and

$$\left| j(\tau) - \frac{1}{q} - 744 \right| < 10^{-12}.$$
Algebraic independence : A.O. Gel’fond 1948

The two numbers $2^{\sqrt{2}}$ and $2^{\sqrt{2}}$ are algebraically independent.

More generally, if $\alpha$ is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if $\beta$ is an algebraic number of degree $d \geq 3$, then two at least of the numbers

$\alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$

are algebraically independent.

Alan Baker 1968

Transcendence of numbers like

$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$

or

$e^{\beta_0 \alpha_1^j} \cdots \alpha_1^{j_1}$

for algebraic $\alpha_i$'s and $\beta_j$'s.

Example (Siegel) :

$\int_0^1 \frac{dx}{1 + x^3} = \frac{1}{3} \left( \log 2 + \frac{\pi}{\sqrt{3}} \right) = 0.835648848 \ldots$

is transcendental.

Gregory V. Chudnovsky

G.V. Chudnovsky (1976)

Algebraic independence of the numbers $\pi$ and $\Gamma(1/4)$.

Also : algebraic independence of the numbers $\pi$ and $\Gamma(1/3)$.

Corollaries : Transcendence of $\Gamma(1/4) = 3.6256099082 \ldots$ and $\Gamma(1/3) = 2.6789385347 \ldots$

Yuri V. Nesterenko

Yu.V.Nesterenko (1996)

Algebraic independence of $\Gamma(1/4)$, $\pi$ and $e^{\pi}$.

Also : Algebraic independence of $\Gamma(1/3)$, $\pi$ and $e^{\pi\sqrt{3}}$.

Corollary : The numbers $\pi = 3.1415926535 \ldots$ and $e^{\pi} = 23.1406926327 \ldots$ are algebraically independent.

Transcendence of values of Dirichlet’s $L$–functions :

Sanoli Gun, Ram Murty and Purusottam Rath (2009).
Weierstraß sigma function

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The canonical product attached to $\Omega$ is the Weierstraß sigma function

$$\sigma(z) = \sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{z(\omega + z/2\omega^2)}.$$

The number

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4} \pi^{1/2} e^{\pi/8} \Gamma(1/4)^{-2}$$

is transcendental.

§4 : Conjectures

Borel 1909, 1950
Schanuel 1964
Grothendieck 1960’s
Rohrlich and Lang 1970’s
André 1990’s

Periods : Maxime Kontsevich and Don Zagier


The number $\pi$

A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Basic example of a period:

$$e^{z + 2i\pi} = e^z$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\pi = \int \int_{x^2 + y^2 \leq 1} \frac{dxdy}{x^2} = 2 \int_{-1}^{1} \sqrt{1 - x^2}dx$$

$$= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1-x^2}.$$
Further examples of periods

\[ \sqrt{2} = \int_{2x^2 \leq 1} \, dx \]

and all algebraic numbers.

\[ \log 2 = \int_{1 < x < 2} \frac{dx}{x} \]

and all logarithms of algebraic numbers.

\[ \pi = \int_{x^2 + y^2 \leq 1} \, dxdy, \]

A product of periods is a period (subalgebra of \( \mathbb{C} \)), but \( 1/\pi \) is expected not to be a period.

**Numbers which are not periods**

Problem (Kontsevich–Zagier): To produce an explicit example of a number which is not a period.

Several levels:

1. **analog of Cantor**: the set of periods is countable. Hence there are real and complex numbers which are not periods (“most” of them).

2. **analog of Liouville**

   Find a property which should be satisfied by all periods, and construct a number which does not satisfy that property.

   Masahiko Yoshinaga, *Periods and elementary real numbers*  
   arXiv:0805.0349

   Compares the periods with hierarchy of real numbers induced from computational complexities. In particular, he proves that periods can be effectively approximated by elementary rational Cauchy sequences.

   As an application, he exhibits a computable real number which is not a period.

3. **analog of Hermite**

   Prove that given numbers are not periods

   Candidates: \( 1/\pi, \, e, \) Euler constant.

   M. Kontsevich: exponential periods

   “The last chapter, which is at a more advanced level and also more speculative than the rest of the text, is by the first author only.”
Relations among periods

1. **Additivity**
   (in the integrand and in the domain of integration)
   \[
   \int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx,
   \]
   \[
   \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx.
   \]

2. **Change of variables**
   If \( y = f(x) \) is an invertible change of variables, then
   \[
   \int_{f(a)}^{f(b)} F(y) \, dy = \int_{a}^{b} F(f(x)) f'(x) \, dx.
   \]

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Conjecture of Kontsevich and Zagier

A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following

**Conjecture (Kontsevich–Zagier).** If a period has two integral representations, then one can pass from one formula to another by using only rules 1, 2, 3 in which all functions and domains of integration are algebraic with algebraic coefficients.

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Conjecture of Kontsevich and Zagier (continued)

In other words, we do not expect any miraculous coincidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.

This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

**Advice:** If you wish to prove a number is transcendental, first prove it is a period.
Conjectures by S. Schanuel and A. Grothendieck

- **Schanuel**: if \(x_1, \ldots, x_n\) are \(\mathbb{Q}\)-linearly independent complex numbers, then \(n\) at least of the \(2n\) numbers \(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}\) are algebraically independent.

- **Periods conjecture by Grothendieck**: Dimension of the Mumford–Tate group of a smooth projective variety.

Consequences of Schanuel’s Conjecture

**Ram Murty**  
**Kumar Murty**  
**N. Saradha**

Purusottam Rath, Ram Murty, Sanoli Gun

Ram and Kumar Murty (2009)

**Ram Murty**  
**Kumar Murty**

Transcendental values of class group \(L\)-functions.

Motives

**Y. André**: generalization of Grothendieck’s conjecture to motives.

**Case of 1-motives**: Elliptico-Toric Conjecture of C. Bertolin.
A simple geometric construction on the moduli spaces \( \mathcal{M}_{0,n} \) of curves of genus 0 with \( n \) ordered marked points is described which gives a common framework for many irrationality proofs for zeta values. This construction yields Apéry’s approximations to \( \zeta(2) \) and \( \zeta(3) \), and for larger \( n \), an infinite family of small linear forms in multiple zeta values with an interesting algebraic structure. It also contains a generalisation of the linear forms used by Ball and Rivoal to prove that infinitely many odd zeta values are irrational.

For \( k, s_1, \ldots, s_k \) positive integers with \( s_1 \geq 2 \), we set \( \mathbf{s} = (s_1, \ldots, s_k) \) and

\[
\zeta(\mathbf{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.
\]

The \( \mathbb{Q} \)-vector space \( \mathcal{Z} \) spanned by the numbers \( \zeta(\mathbf{s}) \) is also a \( \mathbb{Q} \)-algebra. For \( n \geq 2 \), denote by \( \mathcal{Z}_n \) the \( \mathbb{Q} \)-subspace of \( \mathcal{Z} \) spanned by the real numbers \( \zeta(\mathbf{s}) \) where \( \mathbf{s} \) has weight \( s_1 + \cdots + s_k = n \).

The numbers \( \zeta(s_1, \ldots, s_k) \), \( s_1 + \cdots + s_k = n \), where each \( s_i \) is 2 or 3, span \( \mathcal{Z}_n \) over \( \mathbb{Q} \).