26/10/2025

Mathematical Research Center, Shandong University, Jinan, Chine.
October 26-27, two day workshop: Number theory and its applications.

# The Table Maker Dilemma and Diophantine Approximation of Transcendental Numbers


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# Abstract

In theoretical computer science, the *Table Maker Dilemma* is the problem of always getting correctly rounded results when computing the elementary functions. A theoretical solution of the TMD problem follows from an effective refinement of the Hermite– Lindemann Theorem on the transcendence of the values of the exponential function at algebraic points. With Yuri Nesterenko we proved such an explicit result of Diophantine Approximation.

# Decimal expansion

Any real number has a decimal expansion :

$$\frac{1}{7} = 0.142857\,142857\,142857\,142857\,142857\ \ldots$$

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,20\ldots$$

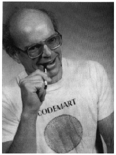
$$\pi = 3.141\,592\,653\,589\,793\,238\,462\,643\,383\,279\ \ldots$$

$$\mathrm{e} = 2.718\,281\,828\,459\,045\,235\,360\,287\,471\,352\ \ldots$$

$$\gamma = 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ \ldots$$

# OEIS

The On-Line Encyclopedia of Integer Sequences




N.J. Sloane

https://oeis.org/A002193
Decimal expansion of square root of $2$.

https://oeis.org/A000796
Decimal expansion of $\pi$ (or digits of $\pi$).

https://oeis.org/A001113
Decimal expansion of $\mathrm{e}$.

https://oeis.org/A001620
Decimal expansion of Euler's constant $\gamma$.

# Decimal expansion

Any decimal expansion (digits $a_i \in \{0, 1, \ldots, 9\}$)

$$0.a_1a_2\ldots a_n\ldots = \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n}{10^n} + \cdots$$

defines a real number.
Not quite a bijective map :

$$\begin{aligned} 0.999\ldots999\ldots &= \frac{9}{10} + \frac{9}{100} + \cdots + \frac{9}{10^n} + \cdots \\ &= \frac{9}{10} \cdot \left(1 + \frac{1}{10} + \frac{1}{100} + \cdots + \frac{1}{10^n} + \cdots\right) \\ &= \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} = 1.000\ldots000\ldots \end{aligned}$$

One definition of the set of real numbers is by means of decimal expansions.

# Binary expansion

Any real number has a binary expansion :

$$\frac{1}{7} = \frac{1}{2^3 - 1} = \frac{1}{2^3} \cdot \frac{1}{1 - \frac{1}{2^3}} = \frac{1}{2^3} \cdot \left( 1 + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots \right)$$

$$=_2 \ 0.001\,001\,001\,001\,001\,001\,001\,001\,001\,001 \ \ldots$$

https://oeis.org/A004539

$$\sqrt{2} =_2 \ 1.011\,010\,100\,000\,100\,111\,100\,110\,011\,001 \ \ldots$$

https://oeis.org/A004601

$$\pi =_2 \ 11.001\,001\,000\,011\,111\,101\,101\,010\,100\,010 \ \ldots$$

https://oeis.org/A004593

$$\mathrm{e} =_2 \ 10.101\,101\,111\,110\,000\,101\,010\,001\,011\,000 \ \ldots$$

https://oeis.org/A104015

$$\gamma =_2 \ 0.100\,100\,111\,100\,010\,001\,100\,111\,111\,000 \ \ldots$$

# Binary expansion

Any binary expansion (digits $a_i \in \{0,1\}$)

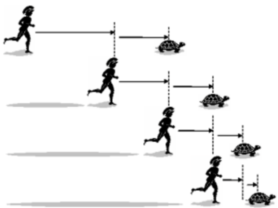
$$0.a_1 a_2 \dots a_n \dots = \frac{a_1}{2} + \frac{a_2}{2} + \dots + \frac{a_n}{2^n} + \dots$$

defines a real number.
Not quite a bijective map :

$$\begin{aligned} 0.11\dots1\dots_2 = {} & \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots \\ & = \frac{1}{2} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots\right) \\ & = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} =_2 1.000\dots000\dots \end{aligned}$$

One definition of the set of real numbers is by means of binary expansions.

# Achilles and the tortoise ( Zeno's paradox)



Two versions :

1. Achilles reduces the distance between him and the tortoise by a coefficient $2$.

2. Achilles reaches the place where the tortoise was when he started, but the tortoise continued to progress.

In both cases he will never catch the tortoise.

# Achilles and the tortoise ( Zeno's paradox)

Achilles and the tortoise are moving at a constant speed and Achilles is moving faster than the tortoise.
When the distance between them is $D$, Achilles needs to reduce the distance to $D/2$, next to $D - D/2 - D/4$, to $D - D/2 - D/4 - D/8$, until the distance is reduced to

$$D - \left(\frac{D}{2} + \frac{D}{4} + \frac{D}{8} + \cdots\right) = D\left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \cdots\right) = 0.$$

If Achilles take a time $t$ to reduce the distance to $D/2$, he will need a time

$$t + \frac{t}{2} + \frac{t}{4} + \frac{t}{8} + \cdots = 2t$$

to catch the tortoise.

# $e^{\pi\sqrt{163}}$

$$
\begin{aligned}
e^{\pi\sqrt{163}} &= 262\,537\,412\,640\,768\,743\,.\,999\,999\,999\,999\,25\,\ldots \\
&\approx 640\,320^3 + 744
\end{aligned}
$$

where

$$640\,320^3 = 262\,537\,412\,640\,768\,000.$$

A special value of the modular function

$$j(\tau) = \frac{1}{q} + 744 + 196\,884q + \cdots \quad \text{where} \quad q = e^{2i\pi\tau}$$

is

$$j\left(\frac{1+\sqrt{-163}}{2}\right) = -640\,320^3.$$

# The modular function

Charles Hermite
1822–1901

Charles Hermite, *Sur la théorie des équations modulaires et la résolution de l'équation du cinquième degré*, Paris : Mallet-Bachelier, 1859.
https://oeis.org/A060295

https://mathshistory.st-andrews.ac.uk

$$e^{\pi\sqrt{163}} = 262\,537\,412\,640\,768\,743{,}999\,999\,999\,999\,25\ldots$$

For

$$\tau = \frac{1 + i\sqrt{163}}{2}, \quad \text{we have} \quad q = e^{2i\pi\tau} = -e^{-\pi\sqrt{163}}.$$

Hence

$$e^{\pi\sqrt{163}} = -\frac{1}{q} = -j(\tau) + 744 - 196\,884 \cdot e^{-\pi\sqrt{163}} + \cdots$$

We have

$$\begin{aligned} -j(\tau) &= 640\,320^3 = 262\,537\,412\,640\,768\,000 \\ \pi\sqrt{163} &= 44.109\,169\,991\ldots, \\ e^{-\pi\sqrt{163}} &= 3.80898093700 \cdot 10^{-18}, \\ 196\,884 \cdot e^{-\pi\sqrt{163}} &= 7.49927402801 8 \cdot 10^{-13} \end{aligned}$$

# Continued fraction for $e^{\pi\sqrt{163}}$

We have

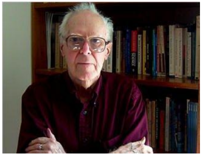
$$e^{\pi\sqrt{163}} = [262537412640768743; 1, 1333462407511, 1, 8, 1, \dots]$$

meaning

$$e^{\pi\sqrt{163}} = 262537412640768743 + \cfrac{1}{1 + \cfrac{1}{1333462407511 + \cfrac{1}{1 + \cfrac{1}{8 + \cfrac{1}{1 + \ddots}}}}}$$

https://oeis.org/A058292

# $e^{\pi\sqrt{163}}$ April Fool's joke by Martin Gardner (1975)



Martin Gardner

1914–2010

Srinivasa Ramanujan

1887–1920

Decimal expansion of $e^{\pi\sqrt{163}}$. https://oeis.org/A060295

This constant is sometimes called " Ramanujan's constant" due to an April Fool's joke by Martin Gardner in which he claimed that Ramanujan conjectured that this constant is an integer, and that a fictitious " John Brillo" of the University of Arizona proved it on May 1974.

In fact, Ramanujan studied similar near-integers of the form $e^{\pi\sqrt{k}}$ (e.g., https://oeis.org/A169624), but not this constant.

# Gel'fond–Schneider Theorem (1934)

$$e^{\pi\sqrt{163}} = (e^{-i\pi})^{i\sqrt{163}} = \alpha^{\beta}, \quad \alpha = e^{-i\pi}, \beta = i\sqrt{163}.$$

Hence $e^{\pi\sqrt{163}}$ is transcendental (not a root of a polynomial with rational coefficients).

Alexandr Osipovich Gel'fond
1906–1968

Theodor Schneider
1911–1988

*If $\alpha$ and $\beta$ are algebraic numbers, $\alpha \neq 0$, $\log\alpha \neq 0$, $\beta \notin \mathbb{Q}$, then $\alpha^{\beta} := e^{\beta\log\alpha}$ is transcendental.*

# Example with binary numbers

www.wolframalpha.com/input?i=15102361/8388608

$$x = \frac{233 \times 64817}{2^{23}} = \frac{15102361}{8388608}$$
$$= [1; 1, 4, 116, 1, 5, 1, 4, 6, 1, 2, 2, 1, 1, 3].$$

Decimal digits :

$$1.8003417253494262695 3125.$$

Binary digits

$$1.11001100111000110011001.$$

wims.univ-cotedazur.fr/wims/wims.cgi
20 or 30 binary digits. Answers

$$1.11001100111000110011000111111111$$

$$1.110011001110001100110010000000$$

# Example with non binary numbers

Let

$$x = \frac{15102361}{8388608} \quad \text{binary digits} \quad 1.11001100111000110011001.$$

$\mathrm{e}^x$ decimal digits $6.051715135574332\ldots$

$\mathrm{e}^x$ binary digits

$$\underbrace{110.00001101001110100110}_{24\text{bits}}\ 0\ \underbrace{111111111111111111111111}_{24\text{bits}}\ 01\ldots$$

Dilemma : $A < \mathrm{e}^x < B$, $\qquad 0 < B - \mathrm{e}^x < 2^{-45}$.

$$A: \ 110.000011010011101001100_2 = \frac{6345683}{2^{20}} = \frac{25382732}{2^{22}}$$

$$B: \ 110.000011010011101001101_2 = \frac{25382733}{2^{22}}.$$

Decimal digits

$A: \ 6.0517148\ldots \qquad B: \ 6.051715135574340\ldots$

# The Table Maker's Dilemma

The Table Maker's Dilemma is the problem of always getting correctly rounded results when computing the elementary functions : $\sin$, $\cos$, $\exp$, $\ln$, $\tan$ and $\arctan$. The values of these functions at rational points cannot be exactly computed in a finite number of steps. The only thing we can do is to compute approximations.

Jean-Michel Muller

Arnaud Tisserand

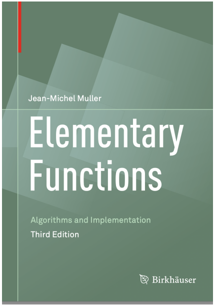
*Towards exact rounding of the elementary functions.*
Alefeld, Götz (ed.) et al., Scientific computing and validated numerics. Math. Res. 90, 59-71 (1996).
https://ieeexplore.ieee.org/document/736435

# Elementary Functions

https://link.springer.com/book/10.1007/978-1-4899-7983-4

Jean-Michel Muller
Elementary Functions — Algorithms and Implementation

Textbook 2016




[360] Y.V. Nesterenko, M. Waldschmidt, On the approximation of the values of exponential function and logarithm by algebraic numbers (in Russian). Mat. Zapiski **2**, 23–42 (1996). Available in English at http://www.math.jussieu.fr/~miw/articles/ps/Nesterenko.ps

# Exact rounding of the elementary functions

https://perso.ens-lyon.fr/jean-michel.muller/Intro-to-TMD.htm

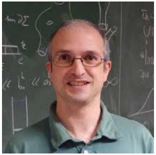
Search for worst cases.

$$\log(110101100.01010000101101000000100111001000101011101110)$$
$$= 110.000011110101001011110011011110101011101100111111100111\,1\,1^{60}\,0101\ldots$$

where $1^{60}$ means "$60$ consecutive ones".

Vincent Lefèvre, Jean-Michel Muller, Arnaud Tisserand

IEEE Transactions on Computers, **47**, No. 11, November 1998, 1235.



Vincent Lefevre

Jean-Michel Muller

Arnaud Tisserand

# Theoretical computer science

A reference to the *Arénaire project in Computer Arithmetic* is http://www.ens-lyon.fr/LIP/Arenaire/

This team works on validated scientific computing : arithmetic. reliability, accuracy, and speed. Their goal is to improve the available arithmetic on computers, processors, dedicated or embedded chips, and they want to achieve more accurate results or getting them more quickly. This has implication in power consumption as well as reliability of numerical software.

Arithmétique des Ordinateurs Projet Arénaire
https://www.ens-lyon.fr/LIP/AriC/

# The AriC project – arithmetic and computing

AriC is a research team of the LIP laboratory, jointly supported by CNRS, ENS de Lyon, Inria and Université Claude Bernard (Lyon 1).

Our overall objective is, through computer arithmetic, to improve computing at large, in terms of performance, efficiency, and reliability. We work on arithmetic algorithms (integer and floating-point arithmetic, complex arithmetic, multiple-precision arithmetic, finite-field arithmetic) and their implementation, approximation methods, Euclidean lattices and cryptology, certified computing and computer algebra.

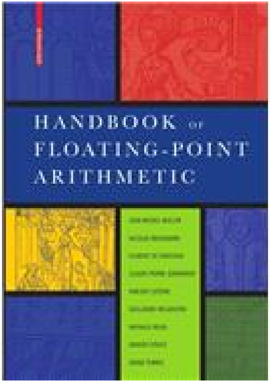
We are interested in computing certified approximations using computer algebra and formal proof systems,

Laboratoire de l'Informatique du Parallélisme ENS de Lyon

Jean-Michel Muller

https://perso.ens-lyon.fr/jean-michel.muller/

# Handbook of floating-point arithmetic




Jean-Michel Muller, Nicolas Brisebarre, Florent de Dinechin, Claude-Pierre Jeannerod, Vincent Lefèvre, Guillaume Melquiond, Nathalie Revol, Damien Stehlé, Serge Torres (auth.)

Birkhäuser 2nd edition 2018

https://link.springer.com/content/pdf/10.1007/978-3-319-76526-6.pdf

# Reducing the problem to the exponential function

The elementary transcendental functions can be expressed by means of the exponential function only.

The main point is that if $a$ is a non zero number from the computer, then $\mathrm{e}^a$ is not a number from the computer : only an approximation of $\mathrm{e}^a$ can be computed.

For almost all $a$, it is easy to find the right approximation to $\mathrm{e}^a$ that we want (nearest integer, closest integer,...).

For some $a$, more computation is required.

One needs an upper bound for the amount of digits needed to compute $\mathrm{e}^a$, valid for all $a$.

# Hermite– Lindemann Theorem

Charles Hermite

1822–1901

Ferdinand von Lindemann

1852–1939

*If* $\alpha$ *and* $\beta$ *are algebraic numbers with* $\beta \neq 0$*, then* $\mathrm{e}^\beta \neq \alpha$*.*

In particular

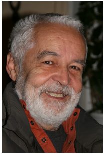
*If* $a$ *and* $b$ *are rational numbers with* $b > 0$*, then* $\mathrm{e}^b \neq a$ *and* $b \neq \log a$ *; hence*

$$\left|\mathrm{e}^b - a\right| > 0.$$

We are interested with explicit lower bounds.

# Kurt Mahler

Kurt Mahler
1903–1988



Maurice Mignotte

Franck Wielonsky

The question of a lower bound for $|e^b - a|$ when $a$ and $b$ are positive integers was considered first by K. Mahler (1953, 1967), then by M. Mignotte (1974), and later by F. Wielonsky (1997).

The sharpest known estimate on Mahler's problem is

$$|e^b - a| > b^{-20b}.$$

# Diophantine approximation

For $a \in \mathbb{Z}$, $a \neq 0$, we have $|a| \geq 1$.
For $a/b \in \mathbb{Q}$ and $c/d \in \mathbb{Q}$ with $a/b \neq c/d$, we have

$$\left|\frac{a}{b} - \frac{c}{d}\right| \geq \frac{1}{bd}.$$

Given $a/b \in \mathbb{Q}$, for any $p/q \in \mathbb{Q}$ with $p/q \neq a/b$, we have

$$\left|\frac{a}{b} - \frac{p}{q}\right| \geq \frac{\kappa}{q}$$

where $\kappa = 1/b$ does not depend on $p/q$.
On the opposite, if $x \in \mathbb{R} \setminus \mathbb{Q}$, there are infinitely many $p/q \in \mathbb{Q}$ such that

$$\left|\frac{a}{b} - \frac{p}{q}\right| < \frac{1}{q^2}.$$

# Height – Liouville's estimate

For $p/q \in \mathbb{Q}$ in lowest terms, define $H(p/q) = \max\{|p|,\, q\}$.

Let $\alpha$ be an algebraic number, root of a polynomial of degree $d$.

*There exists an explicit number $c(\alpha)$ depending only on $\alpha$ such that, for any rational number $p/q$ with $p/q \neq \alpha$ and $q \geq 2$,*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{q^d}.$$

Joseph Liouville

1809–1882

# The main approximation estimate

Yuri Nesterenko

In a joint work with Yu.V. Nesterenko in 1996, we considered an extension of this question when $a$ and $b$ are rational numbers. Then for $a$ and $b$ in $\mathbb{Q}$ with $b \neq 0$, the estimate is

$$|\mathrm{e}^b - a| \geq \exp\{-1.3 \cdot 10^5 (\log A)(\log B)\}$$

where $A = \max\{H(a),\ A_0\}$, $B = \max\{H(b),\ 2\}$. The numerical value of the absolute constant $A_0$ has been explicitly computed.

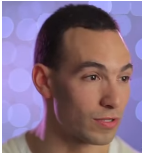
# $|e^b - a|$

With Yuri Nesterenko
*On the approximation of the values of exponential function and logarithm by algebraic numbers*.
http://arxiv.org/abs/math/0002047

(In Russian)
Mat. Zapiski, vol. **2** *Diophantine approximations,*
Proceedings of papers dedicated to the memory of Prof. N.I.Feldman, Centre for applied research under Mech.-Math. Faculty of MSU, Moscow (1996), 23–42.

# $|e^b - a|$

A refinement of our estimate has been obtained by S. Khemira in 2005 and has been sharpened in a joint work of S. Khemira and P. Voutier in 2011.



Samy Khemira

Paul Voutier

Samy Khémira & Paul Voutier
Approximation diophantienne et approximants de Hermite–Padé de type I de fonctions exponentielles.
Ann. Sci. Math. Québec. **35**, No. 1, 85–116 (2011).

# Samy Khémira & Paul Voutier

Zbl 1277.11078

*The aim of this paper is to give (new) explicit lower bounds for* $|\alpha - \mathrm{e}^{\beta}|$ *where* $\alpha$ *and* $\beta$ *are algebraic numbers,* $\beta \neq 0$*. The main result is very detailed, we present one of its corollaries :*

*if* $\alpha$ *and* $\beta$[1] *belong to some imaginary quadratic field and if* $|\beta|$ *is large enough then* $|\alpha - \mathrm{e}^{\beta}| > |\beta|^{-277|\beta|}$*. The proof combines the use of Hermite–Padé approximants and Laurent's interpolation determinants.*

*Maurice Mignotte.*

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1. $\alpha$ and $\beta$ are supposed to be integers

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