Abstract

One of the main open problems in transcendental number theory is Schanuel's Conjecture which was stated in the 1960s.

Schanuel's Conjecture

Let $x_1, \ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then at least $n$ of the $2n$ numbers $x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.

Remark. For almost all tuples (with respect to the Lebesgue measure) the transcendence degree is $2n$.

In other terms, the conclusion is:

If $x_1, \ldots, x_n$ are algebraically independent, then at least $n$ of the $2n$ numbers $x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.

For almost all tuples (with respect to the Lebesgue measure) the transcendence degree is $2n$.

Recently, D. Roy, G.V. Chudnovsky, P. Philippon, Yu. Nesterenko, and more have initiated a $\mathbb{Q}$-linear approach which was developed by Y. V. Nesterenko and more recently D. Roy.
The origin of Schanuel's conjecture is given by S. Lang’s "Introduction to transcendental numbers," attended by my teacher, H. Zagier. The problem of Hilbert’s 14th problem of Hilbert’s 14th problem, solved by E. Bombieri, was attended by M. Negrae, (1237-1908). Let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers with $\alpha_1 \neq 0$, and let $\beta_1, \ldots, \beta_n$ be algebraic numbers with $\beta_1 \neq 0$ and the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ be algebraically independent over $\mathbb{Q}$. Then the numbers are transcendental and there is no nontrivial algebraic relation $\sum_{i=1}^m \lambda_i \beta_i = \sum_{j=1}^n \lambda_j \alpha_j$ for $\lambda_1, \ldots, \lambda_m, \lambda_1, \ldots, \lambda_n \in \mathbb{Q}$ and $\lambda_1 \neq 0$.

Remark: The condition on $\alpha_2$ should be that it is irrational.
Easy consequence of Schanuel’s Conjecture:
numbers $\exp(0), \exp(1), \exp(2), \exp(3), \ldots$ are algebraically independent over $\mathbb{Q}$. According to Schanuel’s Conjecture, the following numbers are algebraically independent over $\mathbb{Q}$:

- $\pi$, $\log(\pi)$, $\log(\log(2))$, $\log(\log(\pi))$, $\ldots$.
Formal analogs

W.V. Brownawell was a student of Schanuel.

P. v. m. ’s Theorem

Version of Schanuel’s Conjecture for power series over \( \mathbb{Q} \) and \( \mathbb{C} \).

Work by W. D. Browanwell.

Ubiquity of Schanuel’s Conjecture

C. Berolini.

Elliptico-Toric Conjecture of Schanuel’s Conjecture over \( \mathbb{C} \).

Case of \( 1 \)-motives:

motives.

Generalization by A. André to the motivic conjecture of a smooth projective variety.

Dimension of the Chow group of a variety.

Conjecture by Grothendieck on the Chow group of a variety.

Conjectures by S. Grothendieck and Y. André.

Ubiquity of Schanuel’s Conjecture

Dipendra Prasad, Preda Mihăilescu


Authors: Preda Mihăilescu

Consequences and some

Title: On Leopoldt’s

Date: Fri. 8 May 2009

arXiv:0905.1274

Preda Mihăilescu

Was a student of Schanuel.

Formal analogs

Gopal Prasad

Dipendra Prasad

### Ubiquity of Schanuel’s Conjecture

- **Analog of A.X. Theorem:**
  - Work by W. D. Browanwell
  - Series over \( \mathbb{Q} \)
- **Elliptico-Toric Conjecture of Schanuel’s:**
  - Case of \( 1 \)-motives
  - Motives
- **Generalization by A. André:**
  - Smooth projective variety
  - Dimension of the Chow group of a variety
- **Conjecture by Grothendieck:**
  - On the Chow group of a variety
- **Conjectures by S. Grothendieck and Y. André:**
The conjecture of xepoldt states that the $p$-adic regulator of a number field does not vanish, but was proved for the abelian case in 1982 by N. M. J. Wiener using complex analysis. If the xepoldt conjecture is false for a Galois field $K$, there is a phantom $\mathbb{Z}_p$-extension of $K\infty$ arising. We show that this is strictly correlated to some infinite Hilbert class fields over $K\infty$. This implies the xepoldt conjecture for arbitrary finite number fields.

Jonas Knigb

Exponential algebraicity in exponential fields

Methods from Logic & Model Theory


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Schanuel’s conjecture for \( n = 1 \)

Schanuel’s conjecture is the Lindemann-Weierstrass theorem. It states that if \( x_1, \ldots, x_n \) are algebraically independent over \( \mathbb{Q} \), then the numbers \( e^{x_1}, \ldots, e^{x_n} \) are algebraically independent over \( \mathbb{Q} \).

**Consequence:**
- The transcendence of numbers like \( e, \pi, \log 2, \sqrt{2} \).

**Not known** for \( n = 2 \)

Schanuel’s conjecture is not yet known for \( n = 2 \).

Let \( x_1, \ldots, x_n \) be algebraic numbers which are algebraically independent over \( \mathbb{Q} \). Then the numbers \( e^{x_1}, \ldots, e^{x_n} \) are algebraically independent over \( \mathbb{Q} \).

If \( x_1 \) is a non-zero algebraic number, then \( e^{x_1} \) is transcendental.

Another equivalent statement is that if \( x_1, \ldots, x_n \) are algebraically independent over \( \mathbb{Q} \), then at least 2 of the numbers \( e^{x_1}, \ldots, e^{x_n} \) are algebraically independent.

A few consequences:

- For \( n = 2 \), Schanuel’s conjecture is not yet known.

**Lindemann-Weierstrass Theorem**

Let \( \beta_1, \ldots, \beta_n \) be algebraic numbers which are linearly independent over \( \mathbb{Q} \). Then the numbers \( e^{\beta_1}, \ldots, e^{\beta_n} \) are algebraically independent over \( \mathbb{Q} \).

**Known**

- \( e, e^\pi \).

**Not Known**

- \( e^\pi \).

**Consequence:**

Transcendence of numbers like \( e, \pi, \log 2, e^\sqrt{2} \).

If \( x_1, x_2 \) are algebraically independent over \( \mathbb{Q} \), then \( \log x_1, \log x_2 \) are algebraically independent.

Another equivalent statement is that if \( x_1, x_2 \) are algebraically independent, then \( e^{x_1}, e^{x_2} \) are algebraically independent.

Transcendental numbers:
- If \( x, y \) is a non-zero algebraic number, then \( e^x, e^y \) is transcendental.
- If \( x \neq 0 \) is a non-zero complex number, then one or the other of the numbers \( e^x, e^{-1/x} \) is transcendental.

For \( n = 1 \), Schanuel's conjecture is the Hermite-Lindemann conjecture.
Hilbert's seventh problem

ArO Gel'fond and ThV Schneider

Solution of Hilbert's seventh problem for the transcendence of \( \alpha^\beta \) and for \( (\log \alpha_1)^{\beta_1} / (\log \alpha_2)^{\beta_2} \), where \( \alpha \) and \( \beta \) are algebraic numbers, \( \alpha \neq 0 \), \( \alpha \neq 1 \), \( \beta \) is irrational, and \( \deg \alpha = d \geq 3 \).

Conjecture: If \( \alpha \) is an algebraic number, \( \alpha \neq 0 \), \( \alpha \neq 1 \), and if \( \beta \) is an irrational algebraic number of degree \( d \geq 2 \), then the \( d-1 \) numbers \( \alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}} \) are algebraically independent.

Special case of Schanuel's conjecture. Take \( x_i = \beta - \log \alpha \) so that \( \{x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}\} \) is algebraically independent.

The conclusion of Schanuel's conjecture is \( \text{tr} \deg Q(\log \alpha, \alpha^\beta, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}) = d \).

Tools

1. Transcendence criterion replaces Liouville's inequality in transcendence proofs.
2. Gel'fond's method of algebraic independence.
3. Transcendence proofs.

Problem of Gel'fond and Schneider

Hilbert's seventh problem

A. O. Gel'fond, A. A. Smirnov, and R. Tijdeman.
Sketch of proof

Assume the transcendence degree over \( k = \mathbb{Q}(\alpha, \beta) \) of the field \( L = k(\alpha \beta, \alpha \beta^2, \ldots, \alpha \beta^{d-1}) \) is \( \leq 1 \). By the Theorem of Gel’fond and Schneider (solution to Hilbert's seventh problem), we know that the transcendence degree is 1. As a matter of fact, the proof of algebraic independence will reprove it.

Consider the exponential functions

\[
e^z, e^{\beta z}, \ldots, e^{\beta^{d-1} z}
\]

which are algebraically independent and satisfy differential equations with coefficients in \( \mathbb{Q}(\beta) \subset k \subset L \). These functions take values in \( L \) when the variable \( z \) is in \( \Gamma = (\mathbb{Z} + \mathbb{Z} \beta \cdots + \mathbb{Z} \beta^{d-1}) \log \alpha \).

Following the approach of Gel’fond and Schneider, one constructs a non-zero polynomial \( F(\cdots, z, \cdots) = 0 \) for sufficiently many points in \( \Gamma \), say

\[
(\begin{array}{c} z_1 - p_1 \\ \vdots \\ z_d - p_d \end{array}) \right) \rightarrow (z) \right) = (z)_d
\]

If the exponential polynomial \( F(z, \cdots, z) = \sum_{j=0}^d \left( \begin{array}{c} d \\ j \end{array} \right) a_j z^j \)

vanishes with some multiplicity at many points in \( \Gamma \), say

\[
F(z_1 - p_1, \cdots, z_d - p_d) = 0
\]

then the exponential polynomial

\[
1 \in \mathbb{Q}[z] \subset \mathbb{Q}(\alpha, \beta, \cdots, \alpha \beta^{d-1}, \log \alpha)
\]

contradicts non-zero polynomial \( \in \mathbb{Q}[\alpha, \alpha \beta, \cdots, \alpha \beta^{d-1}, \log \alpha] \).
Extrapolation of Cauchy–Schwarz inequality. Here another argument is required. This is the transcendence criterion.

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Assume for simplicity that there is a transcendental number \( \theta \) such that all the numbers \( \beta j \) for \( 0 \leq j \leq d - 1 \) belong to \( \mathbb{Z}[\theta] \). Then the number \( \gamma \) which is produced is just in \( \mathbb{Z}[\theta] \) and the size of \( \gamma \) measures the degree and the height of this polynomial. For a transcendence proof, one reaches the conclusion by means of Liouville’s inequality. Here another argument is required. This is the transcendence criterion.

A zero estimate shows that these numbers cannot all vanish. We end up with a non-zero number \( \gamma \) with a very small absolute value, for which we can also bound the size.

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Using Cauchy’s inequalities, we deduce that many more cases have a small modulus.

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From Schwarz’s Lemma we get a sharp upper bound for the maximum modulus of the auxiliary function \( J \) on some disc.

\[
\left| \frac{2n}{p} \right|
\]

Simple form: Given a complex number \( \beta \) if there exists a non-zero polynomial in \( \mathbb{Z}[X] \) with \( \beta \) as a root.

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For all \( u \geq 1 \), then \( \beta \) is algebraic and \( |(\beta)^u d| \)

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Transcendence criterion
Gel'fond's transcendence criterion

The first extension of Gel'fond's transcendence criterion is the following.

For any field \( \mathbb{F} \) and any \( \alpha \in \mathbb{F} \setminus \mathbb{Q} \), let \( d_\alpha \) be the degree of \( \alpha \) over \( \mathbb{Q} \). We say that \( \alpha \) is transcendental over \( \mathbb{Q} \) if \( d_\alpha = \infty \). The transcendence degree of \( \mathbb{F} \) over \( \mathbb{Q} \) is defined as the supremum of the degrees of the algebraic numbers in \( \mathbb{F} \over \mathbb{Q} \).

The conclusion is that the transcendence degree of the field \( \mathbb{F} \) is at least \( 2 \). For all \( \alpha \in \mathbb{F} \setminus \mathbb{Q} \), we have

\[ d_\alpha \geq 2. \]

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An equivalent statement

An inductive process has been suggested by Lang. The special transcendence type

\[ \text{Lang's transcendence type}. \]

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Lang's transcendence type

Further assumption is necessary.
Large transcendence degree

O. Chudnovsky

On the path to

Schanuel’s

Conjecture

Algebraic curves close to a point

General theory of colored sequences

Fields of finite transcendence type and colored sequences

Resultants


14. Fields of finite transcendence degree and colored

1. General theory of colored sequences

2. Algebraic curves close to a point.

C. Diaz (1989): If \( a \) is an algebraic number, \( \beta \neq 0 \), then

\[
\left[ \log_2 d \right] \geq \left[ \frac{d + 1}{2} \right].
\]

A. O. Gel'fond, O. V. Chudnovsky, P. Philippon, Yu. V. Nesterenko.

Partial result on the problem of Gel'fond and Schneider

How could we attack Schanuel’s conjecture?

Let \( x_1, \ldots, x_n \) be \( \alpha \)-independency complex numbers.

Following the transcendence methods of Hermite, Gel'fond, let \( x_1, \ldots, x_n \) be \( \alpha \)-independency complex numbers.

The derivation

For various values of \( m \), \( n \) multiplies

\[
\left( \frac{z^p}{p} \right) = (z) d
\]

The derivatives of \( d \) are given by

\[
(z^p) (z) d = \frac{z^p}{p}
\]

are defined over the ring \( \mathbb{C}[X] \) so that for \( d \in \mathbb{R} \),

\[
\frac{1}{\partial X} + \frac{0}{\partial \theta} d = d
\]

Let denote the derivation

\[
D = \frac{\partial}{\partial X} + X \frac{\partial}{\partial \theta}
\]

over the ring \( \mathbb{C}[X] \) of \( \mathbb{C} \left[ X, \theta \right] \) so that for \( P \in \mathbb{C}[X] \), the

derivatives of \( \left( \frac{z^p}{p} \right) = (z) d \)

are defined by

\[
\frac{d}{dz} \left( \frac{z^p}{p} \right) = (z) d
\]

where \( \partial \) is a non-zero polynomial. One considers

\[
(z^p) (z) d = (z) d
\]

function

Schanuel... one may start by introducing an auxiliary

large transcendence degree.
Roy's conjecture is equivalent to Schanuel's.

Let $\{s_{0}, s_{1}, \ldots, s_{m}\}$ be non-zero complex numbers. Let $\sum_{i=0}^{m} s_{i} z_{i}^{j}$ for $j \geq 0$ be non-zero complex numbers. Assume that for any sufficiently large positive integer $\nu$,

\[
(1 + 0)_{\nu}^{\sum_{i=0}^{m} s_{i} z_{i}^{j}} > \nu > \{1 + 0\}_{\nu}^{g_{0} s_{0} g_{1} \ldots}
\]

and

\[
\{1 s_{0} g_{0}\}_{\nu} \max_{i} \{1 s_{1} g_{1}\}_{\nu} \max_{i} \{1 s_{2} g_{2}\}_{\nu} \ldots
\]

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\{1 s_{0} g_{0}\}_{\nu} \max_{i} \{1 s_{1} g_{1}\}_{\nu} \max_{i} \{1 s_{2} g_{2}\}_{\nu} \ldots
\]
Extending the range

Recently zguyen zgoc mi Van succeeded to extend slightly the range of the admissible values of the parameters $s_0, s_1, t_0, t_1, u$. Such an extension is interesting for both implications of the equivalence between Schanuel’s conjecture and Roy’s conjecture.

\[
\text{For any } k, m \in \mathbb{N} \text{ with } k \leq N_s \text{ and } m \leq N_s,
\]

\[
\left| \left( \partial^k / \partial x^k - \partial^m / \partial \alpha^m \right) \right| \leq e^{-N_u}. 
\]

For any sufficiently large positive integer $N$, there exists a non-zero polynomial $Q \in \mathbb{R}[x_0, x_1]$ with partial degree $\leq N_t$ in $x_0$ and partial degree $\leq N_t$ in $x_1$ and height $H(Q) \leq e^N$ such that

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\[
\left| \left( \partial^k / \partial x^k - \partial^m / \partial \alpha^m \right) \right| \leq e^{-N_u}. 
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The following conditions are equivalent:

- The number $\alpha e^{-x}$ is a root of unity.
- For any sufficiently large positive integer $N$, there exists a non-zero polynomial $Q \in \mathbb{R}[x_0, x_1]$ with partial degree $\leq N_t$ in $x_0$ and partial degree $\leq N_t$ in $x_1$ and height $H(Q) \leq e^N$ such that

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\]

Transcendence criterion with multiplicities

Let $s, \alpha \in \mathbb{C}$ and let $Q \in \mathbb{R}[x_0, x_1]$ be a polynomial of degree $\leq N_t$ in $x_0$ and partial degree $\leq N_t$ in $x_1$ and height $H(Q) \leq e^N$ such that

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\left| \left( \partial^k / \partial x^k - \partial^m / \partial \alpha^m \right) \right| \leq e^{-N_u}. 
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\]

Transcendence criterion with multiplicities

get large transcendence degree.

\[
(\lambda e / \theta) - (\lambda e / \theta) \lambda + (\lambda e / \theta) = \theta
\]

Introduce multiplicity involving the derivative

\[
(\lambda e / \theta) \lambda + (\lambda e / \theta)\lambda X + \cdots + (\lambda e / \theta)\lambda X
\]

Introduce several points

\[
\lambda X
\]

Replace a single variable by two variables

In格尔Fonp's transcendence criterion,

Roy's program towards Schanuel's Conjecture

Equivalence between Schanuel's and Roy's Conjecture

Roy's program towards Schanuel's Conjecture

Due to M. Laurent and D. Roy, applications to algebraic for all $n \geq 1$, assume $t_n / d_n + \alpha \to \infty$. Then $\varphi$ is algebraic.

\[
\varphi \geq \{ a > f > 0 : |(\theta)_{(a,b)}| \}
\]

Transcendence criterion with multiplicities

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Such an extension is interesting for both implications of the range of the admissible values of the parameters $s_0, s_1, t_0, t_1, u$.

Recently Nguyễn Ngọc Al Van succeeded to extend slightly the range of the admissible values of the parameters $s_0, s_1, t_0, t_1, u$.
Given a sequence of complex numbers \( (\vartheta_i)_{i \geq 1} \), assume there exists a sequence \( (P_n)_{n \geq 1} \) of non-zero polynomials in \( \mathbb{Z}[X] \) of degree \( \leq d_n \) and height \( \leq e_n \), such that

\[
\max \{ |P_n(j)_{\vartheta_i}| ; 0 \leq i \leq s_n, 0 \leq j < t_n \} \leq e^{-\nu_n} \text{ for all } n \geq 1.
\]

We wish to deduce that the numbers \( \vartheta_i \) are algebraic. Roy’s small value estimates for the additive group

\[
\text{Small value estimates for the multiplicative group}
\]

Let \( \xi_1, \ldots, \xi_m \) be multiplicatively independent complex numbers in a field of transcendence degree 1. Under suitable assumptions on the parameters \( h, \sigma, \tau, \nu \), for infinitely many positive integers \( n \), there exists no non-zero polynomial \( P \in \mathbb{Z}[Y] \) satisfying

\[
\deg(P) \leq n \quad \text{and} \quad H(P) \leq \exp(nh)
\]

and

\[
\max \{ |P(j_{\xi_i})| ; 0 \leq i \leq m, 0 \leq j < t \} > \exp(-n\nu).
\]

Happy Birthday, Jing Yu,

Heureux Anniversaire,

The end