THIRD KIND ELLIPTIC INTEGRALS AND TRANSCENDENCE

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Abstract. This short appendix aims at giving references on papers related with transcendence results concerning elliptic integrals of the third kind. So far, results on transcendence and linear independence are known, but there are very few results on algebraic independence.

In his book on transcendental numbers [Sc1957], Th. Schneider proposes eight open problems, the third of which is : *Try to find transcendence results on elliptic integrals of the third kind.*

In [La1966, Historical Note of Chapter IV], S. Lang explains the connections between elliptic integrals of the second kind, Weierstrass zeta function and extensions of an elliptic curve by $\mathbb{G}_a$. He applies the so–called Schneider–Lang criterion to the Weierstrass elliptic and zeta functions and deduces the transcendence results due to Th. Schneider on elliptic integrals of the first and second kind. At that time, it was not known how to use this method for proving results on elliptic integrals of the third kind.

The solution came from [Se1979], where J-P. Serre introduces the functions f_q (with the notation of [B2019]) related to elliptic integrals of the third kind, which satisfy the hypotheses of the Schneider-Lang criterion and are attached to extensions of an elliptic curve by $\mathbb{G}_m$. This is how the first transcendence results on these integrals were obtained [Wa1979a, Wa1979b]. In [BeLau1981], D. Bertrand and M. Laurent give further applications of the Schneider-Lang criterion involving elliptic integrals of the third kind. Applications are given in [Be1983a, Be1983b, S1986], dealing with the Neron–Tate canonical height on an elliptic curve (including the p–adic height) and the arithmetic nature of Fourier coefficients of Eisenstein series. A first generalization to abelian integrals of the third kind is quoted in [Be1983b]. Transcendence measures are given in [R1980a].

Properties of the smooth Serre compactification of a commutative algebraic group and of the exponential map, together with the links with integrals, are studied in [FWü1984]. See also [KL1985]. In [M2016, Chapter 20 – Elliptic functions] (see in particular Theorem 20.11 and exercises 20.104 and 20.105) more details are given on the functions associated with elliptic integrals of the third kind, the associated algebraic groups, which are extensions of an elliptic curve by $\mathbb{G}_m$, and the consequences of the Schneider-Lang criterion.

The first results of linear independence of periods of elliptic integrals of the third kind are due to M. Laurent [Lau1980, Lau1982] (he announced his results in [Lau1979a, Lau1979b]). The proof uses Baker's method. More general results on linear independence are due to G. Wüstholz [Wü1984] (see also [BaWü2007, § 6.2]), including the following one, which answers a conjecture that M. Laurent stated in [Lau1982] where he proved special cases of it. Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3. Let ζ be the corresponding Weierstrass zeta function, ω a nonzero period of $\wp$ and η the corresponding quasi-period of ζ. Let $u_1, \dots, u_n$ be complex numbers which are not poles of $\wp$, which are $\mathbb{Q}$

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linearly independent modulo $\mathbb{Z}\omega$ and such that $\wp(u_1), \dots, \wp(u_n)$ are algebraic. Define

$$\lambda(u_i, \omega) = \omega\zeta(u_i) - \eta u_i.$$

Then the $n + 3$ numbers

$$1, \omega, \eta, \lambda(u_1), \dots, \lambda(u_n)$$

are linearly independent over $\overline{\mathbb{Q}}$.

The question of the transcendence of the nonvanishing periods of a meromorphic differential form on an elliptic curve defined over the field of algebraic numbers is now solved [BaWü2007, Theorem 6.6]. See also [HWü2018], as well as [T2017, § 1.5] for abelian integrals of the first and second kind. A reference of historical interest to a letter from Leibniz to Huygens in 1691 is quoted in [BaWü2007, § 6.3] and [Wü20012].

The only results on algebraic independence related with elliptic integrals of the third kind so far are those obtained by É. Reyssat [R1980b, R1982] and by R. Tubbs [T1987, T1990]. We are very far from anything close to the conjectures in [B2019].

For a survey (with an extensive bibliography including 254 entries), see [Wa2008].

The references below are listed by chronological order.

References

[Sc1957] Schneider, Theodor. Einführung in die transzendenten Zahlen. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[La1966] Lang, Serge. Introduction to transcendental numbers. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966.

[Se1979] Serre, Jean–Pierre. Quelques propriétés des groupes algébriques commutatifs. Appendice II de [Wa1979a], 191 – 202.

[Wa1979a] Waldschmidt, Michel. Nombres transcendants et groupes algébriques. With appendices by Daniel Bertrand and Jean-Pierre Serre. Astérisque No. 69-70 (1979), 218 pp.
https://smf.emath.fr/publications/nombres-transcendants-et-groupes-algebriques-2e-edition

[Wa1979b] Waldschmidt, Michel. Nombres transcendants et fonctions sigma de Weierstrass. C. R. Math. Rep. Acad. Sci. Canada 1 (1978/79), no. 2, 111–114.

[Lau1979a] Laurent, Michel. Transcendance de périodes d'intégrales elliptiques. C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 15, 699–701.

[Lau1979b] Laurent, Michel. Transcendance de périodes d'intégrales elliptiques. Séminaire Delange-Pisot-Poitou, 20e année: 1978/1979. Théorie des nombres, Fasc. 1, Exp. No. 13, 4 pp.,

[Lau1980] Laurent, Michel. Transcendance de périodes d'intégrales elliptiques. J. Reine Angew. Math. 316 (1980), 122–139.

[R1980a] Reyssat, Éric. Approximation de nombres liés à la fonction sigma de Weierstrass. Ann. Fac. Sci. Toulouse Math. (5) 2 (1980), no. 1, 79–91.

[R1980b] Reyssat, Éric. Fonctions de Weierstrass et indépendance algébrique. C. R. Acad. Sci. Paris Sér. A-B 290 (1980), no. 10, A439–A441.

[BeLau1981] Bertrand, Daniel & Laurent, Michel. Propriétés de transcendance de nombres liés aux fonctions thêta. C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 16, 747–749.

[Lau1982] Laurent, Michel. Transcendance de périodes d'intégrales elliptiques. II. J. Reine Angew. Math. 333 (1982), 144–161.

[R1982] Reyssat, Éric. Propriétés d'indépendance algébrique de nombres liés aux fonctions de Weierstrass. Acta Arith. 41 (1982), no. 3, 291–310.

[Be1983a] Bertrand, Daniel. Problèmes de transcendance liés aux hauteurs sur les courbes elliptiques. Mathematics, pp. 55–63, CTHS: Bull. Sec. Sci., III, Bib. Nat., Paris, 1981.

[Be1983b] Bertrand, Daniel. Endomorphismes de groupes algébriques; applications arithmétiques. Diophantine approximations and transcendental numbers (Luminy, 1982), 1–45, Progr. Math., 31, Birkhäuser Boston, Boston, MA, 1983.

[Wü1984] Wüstholz, Gisbert. Transzendenzeigenschaften von Perioden elliptischer Integrale. J. Reine Angew. Math. 354 (1984), 164–174.

[FWü1984] Faltings, Gert & Wüstholz, Gisbert. Einbettungen kommutativer algebraischer Gruppen und einige ihrer Eigenschaften. J. Reine Angew. Math. 354 (1984), 175–205.

[KL1985] Knop, Friedrich & Lange, Herbert. Some remarks on compactifications of commutative algebraic groups. Comment. Math. Helv. 60 (1985), no. 4, 497–507.

[S1986] Scholl, Antony. Fourier coefficients of Eisenstein series on noncongruence subgroups. Math. Proc. Cambridge Philos. Soc. 99 (1986), no. 1, 11–17.

[T1987] Tubbs, Robert. Algebraic groups and small transcendence degree. I. J. Number Theory 25 (1987), no. 3, 279–307.

[T1990] Tubbs, Robert. Algebraic groups and small transcendence degree. II. J. Number Theory 35 (1990), no. 2, 109–127.

[BaWü2007] Baker, Alan & Wüstholz, Gisbert. Logarithmic forms and Diophantine geometry. New Mathematical Monographs, 9. Cambridge University Press, Cambridge, 2007.

[Wa2008] Waldschmidt, Michel. Elliptic functions and transcendence. Surveys in number theory, 143–188, Dev. Math., 17, Springer, New York, 2008. https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/SurveyTrdceEllipt2006.pdf

[Wü20012] Wüstholz, Gisbert. Leibniz' conjecture, periods & motives. Colloquium De Giorgi 2009, 33–42, Colloquia, 3, Ed. Norm., Pisa, 2012.

[M2016] Masser, David. Auxiliary polynomials in number theory. Cambridge Tracts in Mathematics, 207. Cambridge University Press, Cambridge, 2016.

[T2017] Tretkoff, Paula. Periods and special functions in transcendence. Advanced Textbooks in Mathematics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

[HWü2018] Huber, Annette & Wüstholz, Gisbert. Periods of 1-motives. https://arxiv.org/abs/1805.10104

[B2019] Bertolin, Cristiana. Third kind elliptic integrals and 1-motives.

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