16/09/2013

Solving effectively some families of Thue Diophantine equations

Claude Levesque and *Michel Waldschmidt*

Abstract

Let α be an algebraic number of degree $d \geq 3$ and let K be the algebraic number field $\mathbf{Q}(\alpha)$. When ε is a unit of K such that $\mathbf{Q}(\alpha\varepsilon) = K$, we consider the irreducible polynomial $f_\varepsilon(X) \in \mathbf{Z}[X]$ such that $f_\varepsilon(\alpha\varepsilon) = 0$. Let $F_\varepsilon(X, Y)$ be the irrreducible binary form of degree d associated to $f_\varepsilon(X)$ under the condition $F_\varepsilon(X, 1) = f_\varepsilon(X)$. For each positive integer m, we want to exhibit an effective upper bound for the solutions (x, y, ε) of the diophantine inequation $|F_\varepsilon(x, y)| \leq m$. We achieve this goal by restricting ourselves to a subset of units ε which we prove to be sufficiently large as soon as the degree of K is ≥ 4.

AMS Classification: Primary 11D61 Secondary 11D41, 11D59

1 The conjecture and the main result

Let α be an algebraic number of degree $d \geq 3$ over $\mathbf{Q}$. We denote by K the algebraic number field $\mathbf{Q}(\alpha)$, by $f \in \mathbf{Z}[X]$ the irreducible polynomial of α over $\mathbf{Z}$, by $\mathbf{Z}_K^\times$ the group of units of K and by r the rank of the abelian group $\mathbf{Z}_K^\times$. For any unit $\varepsilon \in \mathbf{Z}_K^\times$ such that the degree $\delta = [\mathbf{Q}(\alpha\varepsilon) : \mathbf{Q}]$ be ≥ 3, we denote by $f_\varepsilon(X) \in \mathbf{Z}[X]$ the irreductible polynomial of $\alpha\varepsilon$ over $\mathbf{Z}$ (uniquely defined upon requiring that the leading coefficient be > 0) and by F_ε the irreductible binary form defined by $F_\varepsilon(X, Y) = Y^\delta f_\varepsilon(X/Y) \in \mathbf{Z}[X, Y]$.

The purpose of this paper is to investigate the following conjecture.

Conjecture 1. *There exists an effectively computable constant $\kappa_1 > 0$, depending only upon α, such that, for any $m \geq 2$, each solution $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathbf{Z}_K^\times$ of the inequation $|F_\varepsilon(x, y)| \leq m$ with $xy \neq 0$ and $[\mathbf{Q}(\alpha\varepsilon) : \mathbf{Q}] \geq 3$ verifies*

$$\max\{|x|,\ |y|,\ e^{\mathrm{h}(\alpha\varepsilon)}\} \leq m^{\kappa_1}.$$

We noted h the absolute logarithmic height (see (1) below).

To prove this conjecture, it suffices to restrict ourselves to units ε of K such that $\mathbf{Q}(\alpha\varepsilon) = K$: as a matter of fact, the field K has but a finite number of subfields. An equivalent formulation of the conjecture 1 is then the following one: *if $xy \neq 0$ and $\mathbf{Q}(\alpha\varepsilon) = K$, then*

$$|\mathrm{N}_{K/\mathbf{Q}}(x - \alpha\varepsilon y)| \geq \kappa_2 \max\{|x|,\ |y|,\ e^{\mathrm{h}(\alpha\varepsilon)}\}^{\kappa_3}$$

with effectively computable positive constants κ_2 and κ_3, depending only upon α.

The finiteness of the set of solutions $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathbf{Z}_K^\times$ of the inequation $|F_\varepsilon(x, y)| \leq m$ with $xy \neq 0$ and $[\mathbf{Q}(\alpha\varepsilon) : \mathbf{Q}] \geq 3$ follows from Corollary 3.6 of [1] (which deals with Thue–Mahler equations, while in this paper we restrict ourselves to Thue equations). The proof in [1] rests on Schmidt's subspace theorem; it allows to exhibit explicitly an upper bound for the number of solutions as a function of m, d and the height of α, but it does not allow to give an upper bound for the solutions. The particular case of the conjecture 1, in which the form F is of degree 3 and the rank of the unit group of the cubic field $\mathbf{Q}(\alpha)$ is 1, was taken care of in [2]. In [3], we considered a slightly more general case, namely when the number of real embeddings of K into $\mathbf{C}$ is 0 or 1, while restricting to units ε such that $\mathbf{Q}(\alpha\varepsilon) = K$. In this paper, we prove that the conjecture is true at least for a subset $\tilde{\mathcal{E}}_\nu^{(\alpha)}$ of units, the definition of which is given in the following.

Denote by $\Phi = \{\sigma_1, \ldots, \sigma_d\}$ the set of embeddings of K into $\mathbf{C}$ and by $\lceil\overline{\gamma}\rceil$ the *house* of an algebraic number γ, defined to be the maximum of the moduli of the Galois conjuguates of γ in $\mathbf{C}$. In symbols, for $\gamma \in K$,

$$\lceil\overline{\gamma}\rceil = \max_{1 \leq i \leq d} |\sigma_i(\gamma)|.$$

The *absolute logarithmic height* is noted h and involves the *Mahler measure* M:

$$\mathrm{h}(\alpha) = \frac{1}{d} \log \mathrm{M}(\alpha) \quad \text{with} \quad \mathrm{M}(\alpha) = a_0 \prod_{1 \leq i \leq d} \max\{1, |\sigma_i(\alpha)|\}, \tag{1}$$

a_0 being the leading coefficient of the irreducible polynomial of α over $\mathbf{Z}$.

The set

$$\mathcal{E}^{(\alpha)} = \{\varepsilon \in \mathbf{Z}_K^\times \;\mid\; \mathbf{Q}(\alpha\varepsilon) = K\}$$

depends only upon α; (we have supposed $\mathbf{Q}(\alpha) = K$). When ν is a real number in the interval $]0, 1[$, we denote by $\mathcal{E}_\nu^{(\alpha)}$ the set of units $\varepsilon \in \mathcal{E}^{(\alpha)}$ for which there exist two distinct elements φ_1 and φ_2 of Φ such that

$$|\varphi_1(\alpha\varepsilon)| = \lceil\overline{\alpha\varepsilon}\rceil \quad \text{and} \quad |\varphi_2(\alpha\varepsilon)| \geq \lceil\overline{\alpha\varepsilon}\rceil^\nu.$$

We also denote by $\tilde{\mathcal{E}}_\nu^{(\alpha)}$ the set of units $\varepsilon \in \mathcal{E}_\nu^{(\alpha)}$ such that $\varepsilon^{-1} \in \mathcal{E}_\nu^{(1/\alpha)}$.

Let us state our main result.

Theorem 1. *Let $\nu \in]0, 1[$. There exist two effectively computable positive constants κ_4, κ_5, depending only upon α and ν, which have the following properties:*

(a) *For any $m \geq 2$, each solution $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathcal{E}_\nu^{(\alpha)}$ of the inequation $|F_\varepsilon(x, y)| \leq m$ with $0 < |x| \leq |y|$ satisfies*

$$\max\{|y|,\ e^{\mathrm{h}(\alpha\varepsilon)}\} \leq m^{\kappa_4}.$$

(b) *For any* $m \geq 2$, *each solution* $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \tilde{\mathcal{E}}_\nu^{(\alpha)}$ *of the inequation* $|F_\varepsilon(x,y)| \leq m$ *with* $xy \neq 0$ *satisfies*

$$\max\{|x|,\ |y|,\ e^{\mathrm{h}(\alpha\varepsilon)}\} \leq m^{\kappa_5}.$$

Proposition 1, stated below and proved in §13, means that $\tilde{\mathcal{E}}_\nu^{(\alpha)}$ for $d \geq 4$ has a positive density in the set $\mathcal{E}^{(\alpha)}$. Since the case of a non-totally real cubic field has been taken care of in [2], it is only in the case of a totally real cubic field that our main result provides no effective bound for an infinite family of Thue equations.

When N is a real positive number and $\mathcal{F}$ is a subset of $\mathbf{Z}_K^\times$, we define

$$\mathcal{F}(N) = \{\varepsilon \in \mathcal{F} \ \mid\ \overline{|\alpha\varepsilon|} \leq N\} \quad \text{and} \quad |\mathcal{F}(N)| = \mathrm{Card}\mathcal{F}(N),$$

so

$$\mathcal{F}(N) = \mathbf{Z}_K^\times(N) \cap \mathcal{F}.$$

Proposition 1. (a) *The limit*

$$\lim_{N\to\infty} \frac{|\mathbf{Z}_K^\times(N)|}{(\log N)^r}$$

exists and is positive.

(b) *One has*

$$\liminf_{N\to\infty} \frac{|\mathcal{E}^{(\alpha)}(N)|}{|\mathbf{Z}_K^\times(N)|} > 0.$$

(c) *For* $0 < \nu < 1/2$, *one has*

$$\liminf_{N\to\infty} \frac{|\mathcal{E}_\nu^{(\alpha)}(N)|}{(\log N)^r} > 0.$$

(d) *For* $0 < \nu < 1$ *and* $d \geq 4$, *one has*

$$\liminf_{N\to\infty} \frac{|\tilde{\mathcal{E}}_\nu^{(\alpha)}(N)|}{(\log N)^r} > 0.$$

Let us write the irreductible polynomial f of α over $\mathbf{Z}$ as

$$f(X) = a_0X^d + a_1X^{d-1} + \cdots + a_{d-1}X + a_d \in \mathbf{Z}[X],$$

whereupon

$$f(X) = a_0 \prod_{i=1}^{d} \bigl(X - \sigma_i(\alpha)\bigr)$$

and its associated irreducible binary form F is

$$F(X,Y) = Y^d f(X/Y) = a_0X^d + a_1X^{d-1}Y + \cdots + a_{d-1}XY^{d-1} + a_dY^d.$$

For $\varepsilon \in \mathbf{Z}_K^\times$ verifying $\mathbf{Q}(\alpha\varepsilon) = K$, we have

$$F_\varepsilon(X,Y) = a_0 \prod_{i=1}^{d} \bigl(X - \sigma_i(\alpha\varepsilon)Y\bigr) \in \mathbf{Z}[X,Y].$$

Given $(x,y,\varepsilon) \in \mathbf{Z}^2 \times \mathbf{Z}_K^\times$, we define

$$\beta = x - \alpha\varepsilon y.$$

Therefore

$$F_\varepsilon(x,y) = a_0 \sigma_1(\beta) \cdots \sigma_d(\beta). \tag{2}$$

Dirichlet's unit theorem provides the existence of units $\epsilon_1, \ldots, \epsilon_r$ in K, the classes modulo $K_{\mathrm{tors}}^\times$ of which form a basis of the free abelian group $\mathbf{Z}_K^\times / K_{\mathrm{tors}}^\times$. Effective versions (see for instance [4]) provide bounds for the heights of these units as a function of $\mathrm{h}(\alpha)$ and d.

Steps of the proof. In §2 we quote useful lemmas, the most powerful being a proposition of [5] involving transcendence methods and giving lower bounds for the distance between 1 and a product of powers of algebraic numbers. Each time we will use that proposition, we will write that we are using a diophantine argument. After introducing some parameters A and B in §3, we eliminate x and y between the equations $\varphi(\beta) = x - \varphi(\alpha\varepsilon)y$, $\varphi \in \Phi$. In §5 we introduce four privileged embeddings, denoted by σ_a, σ_b, τ_a, τ_b, and four useful sets of embeddings $\Sigma_a(\nu)$, $\Sigma_b(\nu)$, $T_a(\nu)$, $T_b(\nu)$, depending on a parameter ν. Applying some results from [3], we show in §6 that we may suppose A and B sufficiently large, namely $\geq \kappa \log m$, via a diophantine argument. In §7 and in §8, we prove that A is bounded from above by κB and that B is bounded from above by $\kappa' A$. In §9 we prove that τ_b is unique. In §10 we give an upper bound for $|\tau_b(\alpha\varepsilon)|$. In §11 we deduce that σ_a is unique. In §12 we complete the proof of Theorem 1. In §13 we give the proof of Proposition 1.

2 Tools

This chapter contains the auxiliary lemmas we shall need. The details of the proofs are in [3]. We start with an equivalence of norms (Lemma 1). Then we state Lemma 2, which appeared as Lemma 2 of [2] and also as Lemma 6 of [3]. Next we quote Proposition 2 (which is Corollary 9 of [3]) involving a lower bound of a linear form in logarithms of algebraic numbers.

2.1 Equivalence of norms

Let K be an algebraic number field of degree d over $\mathbf{Q}$. Let us recall that $\epsilon_1, \ldots, \epsilon_r$ denote the elements of a basis of the unit group of K modulo $K_{\mathrm{tors}}^\times$ and that we are supposing $r \geq 1$.

There exists an effectively computable positive constant κ_6, depending only upon $\epsilon_1, \dots, \epsilon_r$, such that, if $c_1, \dots, c_r$ are rational integers and if we let

$$C = \max\{|c_1|, \dots, |c_r|\}, \quad \gamma = \epsilon_1^{c_1} \cdots \epsilon_r^{c_r},$$

then

$$e^{-\kappa_6 C} \le |\varphi(\gamma)| \le e^{\kappa_6 C} \tag{3}$$

for each embedding φ of K into $\mathbf{C}$.

The following lemma (see Lemma 5 of [3]) shows that the two inequalities of (3) are optimal.

Lemma 1. *There exists an effectively computable positive constant κ_7, which depends only upon $\epsilon_1, \dots, \epsilon_r$, with the following property. If $c_1, \dots, c_r$ are rational integers and if we let*

$$C = \max\{|c_1|, \dots, |c_r|\}, \quad \gamma = \epsilon_1^{c_1} \cdots \epsilon_r^{c_r},$$

then there exist two embeddings σ and τ of K into $\mathbf{C}$ such that

$$|\sigma(\gamma)| \ge e^{\kappa_7 C} \quad \textit{and} \quad |\tau(\gamma)| \le e^{-\kappa_7 C}.$$

Remark. *Under the hypotheses of Lemma* 1*, if γ_0 is a nonzero element of K and if we let $\gamma_1 = \gamma_0 \gamma$, one deduces*

$$e^{-\kappa_6 C - d\mathrm{h}(\gamma_0)} \le \min_{\varphi \in \Phi} |\varphi(\gamma_1)| \le e^{-\kappa_7 C + d\mathrm{h}(\gamma_0)}$$

and

$$e^{\kappa_7 C - d\mathrm{h}(\gamma_0)} \le \max_{\varphi \in \Phi} |\varphi(\gamma_1)| \le e^{\kappa_6 C + d\mathrm{h}(\gamma_0)}.$$

2.2 On the norm

The following lemma is a consequence of Lemma A.15 of [4] (see also Lemma 2 of [2] and Lemma 6 of [3]).

Lemma 2. *Let K be a field of algebraic numbers of degree d over $\mathbf{Q}$ with regulator R. There exists an effectively computable positive constant κ_8, depending only on d and R, such that, if γ is an element of $\mathbf{Z}_K$, the norm of which has an absolute value $\le m$ with $m \ge 2$, then there exists a unit $\varepsilon \in \mathbf{Z}_K^\times$ such that*

$$\max_{1 \le j \le d} |\sigma_j(\varepsilon\gamma)| \le m^{\kappa_8}. \tag{4}$$

2.3 Diophantine tool

We will use the particular case of Theorem 9.1 of [5] (stated in Corollary 9 of [3]). Such estimates (known as *lower bounds for linear forms in logarithms of algebraic numbers*) first occurred in the work of A.O. Gel'fond, then in the work of A. Baker - a historical survey is given in [3].

Proposition 2. *Let s and D two positive integers. There exists an effectively computable positive constant κ_9, depending only upon s and D, with the following property. Let $\gamma_1, \ldots, \gamma_s$ be nonzero algebraic numbers generating a number field of degree $\leq D$. Let $c_1, \ldots, c_s$ be rational integers and let $H_1, \ldots, H_s$ be real numbers ≥ 1 satisfying $H_j \leq H_s$ for $1 \leq j \leq s$ and*

$$H_i \geq \mathrm{h}(\gamma_i) \quad (1 \leq i \leq s).$$

Let C be a real number subject to

$$C \geq 2, \quad C \geq \max_{1\leq j\leq s} \left\{ \frac{H_j}{H_s} |c_j| \right\}.$$

Suppose also $\gamma_1^{c_1} \cdots \gamma_s^{c_s} \neq 1$. Then

$$|\gamma_1^{c_1} \cdots \gamma_s^{c_s} - 1| > \exp\{-\kappa_9 H_1 \cdots H_s \log C\}.$$

3 Introduction of the parameters $\tilde{A}$, A, $\tilde{B}$, B

From now on, we fix a solution $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathbf{Z}_K^\times$ of the Thue inequation $|F_\varepsilon(x, y)| \leq m$ with $xy \neq 0$ and $\mathbf{Q}(\alpha\varepsilon) = K$. Up to §11 inclusively, we suppose

$$1 \leq |x| \leq |y|.$$

Let

$$\tilde{A} = \max\{1, \mathrm{h}(\alpha\varepsilon)\}.$$

Write

$$\varepsilon = \zeta \epsilon_1^{a_1} \cdots \epsilon_r^{a_r}$$

with $\zeta \in K_{\mathrm{tors}}^\times$ and $a_i \in \mathbf{Z}$ for $1 \leq i \leq r$ and define

$$A = \max\{1, |a_1|, \ldots, |a_r|\}.$$

Thanks to (3) and to Lemma 1, we have

$$\kappa_{10} A \leq \tilde{A} \leq \kappa_{11} A.$$

Next define

$$\tilde{B} = \max\{1, \mathrm{h}(\beta)\}.$$

Since $|F_\varepsilon(x, y)| \leq m$, it follows from (4) and (2) that there exists $\rho \in \mathbf{Z}_K$ verifying

$$\mathrm{h}(\rho) \leq \kappa_{12} \log m \tag{5}$$

with $\kappa_{12} > 0$ such that $\eta = \beta/\rho$ is a unit of $\mathbf{Z}_K$ of the form

$$\eta = \epsilon_1^{b_1} \cdots \epsilon_r^{b_r}$$

with rational integers $b_1, \ldots, b_r$; define

$$B = \max\{1,\ |b_1|, |b_2| \ldots,\ |b_r|\}.$$

Because of the relation $\beta = \rho\eta$, we deduce from (3),

$$\tilde{B} \leq \kappa_{13}(B + \log m)$$

and from Lemma 1,

$$B \leq \kappa_{14}(\tilde{B} + \log m).$$

Since $xy \neq 0$ and $\mathbf{Q}(\alpha\varepsilon) = K$, we deduce that for φ and σ in Φ, we have

$$\varphi = \sigma \iff \varphi(\alpha\varepsilon) = \sigma(\alpha\varepsilon) \iff \varphi(\beta) = \sigma(\beta) \iff \sigma(\alpha\varepsilon)\varphi(\beta) = \sigma(\beta)\varphi(\alpha\varepsilon).$$

Here is an example of application of Proposition 2. The following lemma will be used in the proof of Lemma 9.

Lemma 3. *There exists an effectively computable positive constant κ_{15} with the following property. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ be elements of Φ with $\varphi_1(\alpha\varepsilon)\varphi_2(\beta) \neq \varphi_3(\alpha\varepsilon)\varphi_4(\beta)$. Then*

$$\left| \frac{\varphi_1(\alpha\varepsilon)\varphi_2(\beta)}{\varphi_3(\alpha\varepsilon)\varphi_4(\beta)} - 1 \right| \geq \exp\left\{ -\kappa_{15}(\log m)\log\left(2 + \frac{A+B}{\log m}\right)\right\}.$$

Proof. Write

$$\frac{\varphi_1(\alpha\varepsilon)\varphi_2(\beta)}{\varphi_3(\alpha\varepsilon)\varphi_4(\beta)}$$

as $\gamma_1^{c_1} \cdots \gamma_s^{c_s}$ with $s = 2r+1$, and

$$\gamma_j = \frac{\varphi_1(\epsilon_j)}{\varphi_3(\epsilon_j)}, \quad c_j = a_j, \quad \gamma_{r+j} = \frac{\varphi_2(\epsilon_j)}{\varphi_4(\epsilon_j)}, \quad c_{r+j} = b_j \quad (j = 1, \dots, r),$$

$$\gamma_s = \frac{\varphi_1(\alpha\zeta)\varphi_2(\rho)}{\varphi_3(\alpha\zeta)\varphi_4(\rho)}, \quad c_s = 1.$$

We have $\mathrm{h}(\gamma_s) \leq \kappa_{16}\log m$, thanks to the upper bound (5) for the height of ρ. Write

$$H_1 = \cdots = H_{2r} = \kappa_{17}, \quad H_s = \kappa_{17}\log m, \quad C = 2 + \frac{A+B}{\log m}\cdot$$

The hypothesis

$$\max_{1 \leq j \leq s} \frac{H_j}{H_s}|c_j| \leq C$$

of Proposition 2 is satisfied. Lemma 3 follows from this proposition.

□

4 Elimination

4.1 Expressions of x and y in terms of $\alpha\varepsilon$ and β

Let φ_1, φ_2 be two distinct elements of Φ, namely two distinct embeddings of K into $\mathbf{C}$. We eliminate x (resp. y) between the two equations

$$\varphi_1(\beta) = x - \varphi_1(\alpha\varepsilon)y \quad \text{and} \quad \varphi_2(\beta) = x - \varphi_2(\alpha\varepsilon)y,$$

to obtain

$$y = \frac{\varphi_1(\beta) - \varphi_2(\beta)}{\varphi_2(\alpha\varepsilon) - \varphi_1(\alpha\varepsilon)}, \quad x = \frac{\varphi_2(\alpha\varepsilon)\varphi_1(\beta) - \varphi_1(\alpha\varepsilon)\varphi_2(\beta)}{\varphi_2(\alpha\varepsilon) - \varphi_1(\alpha\varepsilon)}. \tag{6}$$

4.2 The unit equation

Let $\varphi_1, \varphi_2, \varphi_3$ be embeddings of K into $\mathbf{C}$. Let

$$u_i = \varphi_i(\alpha\varepsilon), \quad v_i = \varphi_i(\beta) \qquad (i = 1, 2, 3).$$

We eliminate x and y between the three equations

$$\begin{cases} \varphi_1(\beta) &= x - \varphi_1(\alpha\varepsilon)y \\ \varphi_2(\beta) &= x - \varphi_2(\alpha\varepsilon)y \\ \varphi_3(\beta) &= x - \varphi_3(\alpha\varepsilon)y \end{cases}$$

by writing that the determinant of this nonhomogeneous system of three equations in two unknowns, which is equal to

$$\begin{vmatrix} 1 & \varphi_1(\alpha\varepsilon) & \varphi_1(\beta) \\ 1 & \varphi_2(\alpha\varepsilon) & \varphi_2(\beta) \\ 1 & \varphi_3(\alpha\varepsilon) & \varphi_3(\beta) \end{vmatrix} = \begin{vmatrix} 1 & u_1 & v_1 \\ 1 & u_2 & v_2 \\ 1 & u_3 & v_3 \end{vmatrix},$$

is 0, and this leads to

$$u_1v_2 - u_1v_3 + u_2v_3 - u_2v_1 + u_3v_1 - u_3v_2 = 0. \tag{7}$$

5 Four sets of privileged embeddings

We denote by σ_a (resp. σ_b) an embedding of K into $\mathbf{C}$ such that $|\sigma_a(\alpha\varepsilon)|$ (resp. $|\sigma_b(\beta)|$) be maximal among the elements $|\varphi(\alpha\varepsilon)|$ (resp. among the elements $|\varphi(\beta)|$) for $\varphi \in \Phi$. Therefore

$$|\sigma_a(\alpha\varepsilon)| = \left\lceil\overline{\alpha\varepsilon}\right\rceil \quad \text{and} \quad |\sigma_b(\beta)| = \left\lceil\overline{\beta}\right\rceil.$$

Next we denote by τ_a (resp. τ_b) an embedding of K into $\mathbf{C}$ such that $|\tau_a(\alpha\varepsilon)|$ (resp. $|\tau_b(\beta)|$) be minimal among the elements $|\varphi(\alpha\varepsilon)|$ (resp. among the elements $|\varphi(\beta)|$) for $\varphi \in \Phi$. Therefore

$$\left|\tau_a\left((\alpha\varepsilon)^{-1}\right)\right| = \frac{1}{\left\lceil\overline{(\alpha\varepsilon)}\right\rceil} \quad \text{and} \quad \left|\tau_b\left(\beta^{-1}\right)\right| = \frac{1}{\left\lceil\overline{\beta}\right\rceil}.$$

Since there are at least three distinct embeddings of K into $\mathbf{C}$, we may suppose $\tau_b \neq \sigma_b$ and $\tau_a \neq \sigma_a$. By definition of σ_a, σ_b, τ_a and τ_b, for any $\varphi \in \Phi$ we have

$$|\tau_a(\alpha\varepsilon)| \leq |\varphi(\alpha\varepsilon)| \leq |\sigma_a(\alpha\varepsilon)| \quad \text{and} \quad |\tau_b(\beta)| \leq |\varphi(\beta)| \leq |\sigma_b(\beta)|.$$

Let ν be a real number in the open interval $]0,1[$. Let us denote by $\Sigma_a(\nu)$, $\Sigma_b(\nu)$, $T_a(\nu)$, $T_b(\nu)$ the sets of embeddings of K into $\mathbf{C}$ defined by the following conditions:

$$\begin{cases} \Sigma_a(\nu) &= \{\varphi \in \Phi \mid |\sigma_a(\alpha\varepsilon)|^\nu \leq |\varphi(\alpha\varepsilon)| \leq |\sigma_a(\alpha\varepsilon)|\}, \\ \Sigma_b(\nu) &= \{\varphi \in \Phi \mid |\sigma_b(\beta)|^\nu \leq |\varphi(\beta)| \leq |\sigma_b(\beta)|\}, \\ T_a(\nu) &= \{\varphi \in \Phi \mid |\tau_a(\alpha\varepsilon)| \leq |\varphi(\alpha\varepsilon)| \leq |\tau_a(\alpha\varepsilon)|^\nu\}, \\ T_b(\nu) &= \{\varphi \in \Phi \mid |\tau_b(\beta)| \leq |\varphi(\beta)| \leq |\tau_b(\beta)|^\nu\}. \end{cases}$$

Of course, we have

$$\sigma_a \in \Sigma_a(\nu), \quad \sigma_b \in \Sigma_b(\nu), \quad \tau_a \in T_a(\nu), \quad \tau_b \in T_b(\nu).$$

We will see in §6 that we have

$$|\sigma_a(\alpha\varepsilon)| > 2, \quad |\sigma_b(\beta)| > 2, \quad |\tau_a(\alpha\varepsilon)| < \frac{1}{2}, \quad |\tau_b(\beta)| < \frac{1}{2},$$

from which we will deduce

$$T_a(\nu) \cap \Sigma_a(\nu) = \emptyset, \quad T_b(\nu) \cap \Sigma_b(\nu) = \emptyset.$$

6 Lower bounds for A and B

Thanks to Lemma 15 in §7.2 of [3] and to Lemma 17 in §7.3 of [3], we may suppose, without loss of generality, that A and B have a lower bound given by $\kappa_{18} \log m$ for a sufficiently large effectively computable positive constant κ_{18}, depending only on α:

$$A \geq \kappa_{18} \log m, \quad B \geq \kappa_{18} \log m. \tag{8}$$

In particular, we deduce that A, B, $|\sigma_a(\alpha\varepsilon)|$ and $|\sigma_b(\beta)|$ are sufficiently large and also that $|\tau_a(\alpha\varepsilon)|$ and $|\tau_b(\beta)|$ are sufficiently small.

By using Lemma 1 with the estimates (3), we deduce that there exist some effectively computable positive constants κ_{19} et κ_{20}, depending only on α, such that

$$\begin{cases} e^{\kappa_{20}A} \leq |\sigma_a(\alpha\varepsilon)| \leq e^{\kappa_{19}A}, \\ e^{\kappa_{20}B} \leq |\sigma_b(\beta)| \leq e^{\kappa_{19}B}, \\ e^{-\kappa_{19}A} \leq |\tau_a(\alpha\varepsilon)| \leq e^{-\kappa_{20}A}, \\ e^{-\kappa_{19}B} \leq |\tau_b(\beta)| \leq e^{-\kappa_{20}B}. \end{cases} \tag{9}$$

Therefore we have

$$
\begin{cases}
e^{\kappa_{20}\nu A} & \le & |\varphi(\alpha\varepsilon)| & \le & e^{\kappa_{19} A} & \text{for} & \varphi \in \Sigma_a(\nu),\\
e^{\kappa_{20}\nu B} & \le & |\varphi(\beta)| & \le & e^{\kappa_{19} B} & \text{for} & \varphi \in \Sigma_b(\nu),\\
e^{-\kappa_{19} A} & \le & |\varphi(\alpha\varepsilon)| & \le & e^{-\kappa_{20}\nu B} & \text{for} & \varphi \in T_a(\nu),\\
e^{-\kappa_{19} B} & \le & |\varphi(\beta)| & \le & e^{-\kappa_{20}\nu B} & \text{for} & \varphi \in T_b(\nu).
\end{cases}
$$

7 Upper bounds for A, $|x|$, $|y|$ in terms of B

From the relation (6) we deduce in an elementary way the following upper bounds. Recall the assumption $1 \le |x| \le |y|$ made in §3.

Lemma 4. *One has*

$$A \le \kappa_{21} B \quad \textit{and} \quad |x| \le |y| \le e^{\kappa_{22} B}.$$

Proof. There is no restriction in supposing that A and B are larger than a constant times $\log m$. From the inequality $|\sigma_a(\alpha\varepsilon)| \ge 2|\tau_a(\alpha\varepsilon)|$, we deduce

$$|\sigma_a(\alpha\varepsilon) - \tau_a(\alpha\varepsilon)| \ge \frac{1}{2}|\sigma_a(\alpha\varepsilon)|.$$

Then we use (6) with $\varphi_2 = \sigma_a$ and $\varphi_1 = \tau_a$:

$$y\big(\sigma_a(\alpha\varepsilon) - \tau_a(\alpha\varepsilon)\big) = \tau_a(\beta) - \sigma_a(\beta).$$

From the upper bound

$$|\sigma_a(\beta) - \tau_a(\beta)| \le 2|\sigma_b(\beta)|,$$

we deduce

$$|y\sigma_a(\alpha\varepsilon)| \le 4|\sigma_b(\beta)|. \tag{10}$$

With the help of (9), one obtains the inequalities

$$e^{\kappa_{20} A} \le |\sigma_a(\alpha\varepsilon)| \le |y\sigma_a(\alpha\varepsilon)| \le 4|\sigma_b(\beta)| \le 4e^{\kappa_{19} B}$$

which imply $A \le \kappa_{21} B$. From (10) and because $|\sigma_a(\alpha\varepsilon)| > 2$, we get the upper bound $\log|y| \le \kappa_{22} B$. We can conclude the proof by using the hypothesis $|x| \le |y|$ (cf. §3). □

8 Upper bound of B in terms of A

We use the unit equation (7) of §4.2 with three different embeddings τ_b, σ_b and φ, where φ is an element of Φ different from τ_b and σ_b.

Lemma 5. *One has*

$$B \le \kappa_{23} A.$$

Proof. Let $\varphi \in \Phi$ with $\varphi \neq \sigma_b$ and $\varphi \neq \tau_b$. We take advantage of the relation (7) with $\varphi_1 = \sigma_b$, $\varphi_2 = \varphi$, $\varphi_3 = \tau_b$, written in the form

$$\varphi(\beta)\big(\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)\big) - \sigma_b(\beta)\big(\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)\big) + \tau_b(\beta)\big(\varphi(\alpha\varepsilon) - \sigma_b(\alpha\varepsilon)\big) = 0$$

and we divide by $\sigma_b(\beta)\big(\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)\big)$ (which is different from 0):

$$\frac{\varphi(\beta)}{\sigma_b(\beta)} \cdot \frac{\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)} - 1 = -\frac{\tau_b(\beta)}{\sigma_b(\beta)} \cdot \frac{\varphi(\alpha\varepsilon) - \sigma_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)} \cdot \tag{11}$$

The right side of (11) is different from 0. Let us show that an upper bound of its modulus is given by

$$e^{\kappa_{24} A} e^{-\kappa_{25} B}.$$

As a matter of fact, on the one hand, from (9) we have

$$|\tau_b(\beta)| \leq e^{-\kappa_{20} B}, \quad \text{and} \quad |\sigma_b(\beta)| \geq e^{\kappa_{20} B};$$

on the other hand, the height of the number

$$\delta = \frac{\varphi(\alpha\varepsilon) - \sigma_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}$$

is bounded from above by $e^{\kappa_{26} A}$. From this upper bound for the height we derive the upper bound for the modulus $|\delta|$, namely $|\delta| \leq e^{\kappa_{27} A}$, hence

$$\left| \frac{\tau_b(\beta)}{\sigma_b(\beta)} \cdot \delta \right| \leq \frac{e^{\kappa_{27} A}}{e^{2\kappa_{20} B}} \cdot$$

Let us write the term

$$\frac{\varphi(\beta)}{\sigma_b(\beta)} \cdot \frac{\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}$$

appearing on the left side of (11) in the form $\gamma_1^{c_1} \cdots \gamma_s^{c_s}$ with $s = r + 1$ and

$$\gamma_j = \frac{\varphi(\epsilon_j)}{\sigma_b(\epsilon_j)}, \quad c_j = b_j \quad (j = 1, \ldots, r),$$

$$\gamma_s = \frac{\sigma_b(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)}{\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)} \cdot \frac{\varphi(\rho)}{\sigma_b(\rho)}, \quad c_s = 1.$$

Thanks to (5) and (8), we have

$$\mathrm{h}(\gamma_s) \leq \kappa_{28} A + 2\mathrm{h}(\varrho) \leq \kappa_{29} A.$$

Define

$$H_1 = \cdots = H_r = \kappa_{30}, \quad H_s = \kappa_{28} A, \quad C = 2 + \frac{B}{\kappa_{31} A} \cdot$$

We check that the hypothesis

$$\max_{1 \leq j \leq s} \frac{H_j}{H_s} |c_j| \leq C$$

of Proposition 2 is satisfied. We deduce from this proposition that a lower bound for the modulus of the left member of (11) is given by $\exp\{-\kappa_{32} H_s \log C\}$. Consequently,

$$\kappa_{25} B \leq \kappa_{24} A + \kappa_{32} H_s \log C.$$

Hence $C \leq \kappa_{33} \log C$, which allows to conclude that $C \leq \kappa_{34}$, and this secures the inequality $B \leq \kappa_{23} A$ we wanted to prove. □

9 Unicity of τ_b

We want to prove that no other embedding plays the same role as τ_b. This will be achieved by proving the next lemma, which exhibits a contradiction to (8).

Lemma 6. *Suppose $T_b(\nu) \neq \{\tau_b\}$. Then $B \leq \kappa_{35} \log m$.*

Proof. Let $\varphi \in T_b(\nu)$. Suppose $\varphi \neq \tau_b$. Let us use (6) with $\varphi_1 = \varphi$, $\varphi_2 = \tau_b$, in the form

$$\frac{\varphi(\alpha\varepsilon)}{\tau_b(\alpha\varepsilon)} - 1 = \frac{\tau_b(\beta) - \varphi(\beta)}{y\tau_b(\alpha\varepsilon)} \cdot$$

From the inequality

$$|x - \tau_b(\alpha\varepsilon)y| = |\tau_b(\beta)| < \frac{1}{2}$$

obtained from (9), we deduce

$$|\tau_b(\alpha\varepsilon)y| \geq |x| - \frac{1}{2} \geq \frac{1}{2} \cdot$$

Since $|\tau_b(\beta)| \leq |\varphi(\beta)|$, we also have

$$|\varphi(\alpha\varepsilon) - \tau_b(\alpha\varepsilon)| = \frac{1}{y}|\varphi(\beta) - \tau_b(\beta)| \leq \frac{2|\varphi(\beta)|}{|y|} \cdot$$

Consequently,

$$\left| \frac{\varphi(\alpha\varepsilon)}{\tau_b(\alpha\varepsilon)} - 1 \right| \leq \frac{2|\varphi(\beta)|}{|\tau_b(\alpha\varepsilon)y|} \leq 4|\varphi(\beta)| \leq 4e^{-\kappa_{20}\nu B}.$$

The left side is not 0 since $\varphi \neq \tau_b$. Let us write

$$\frac{\varphi(\alpha\varepsilon)}{\tau_b(\alpha\varepsilon)} = \gamma_1^{c_1} \cdots \gamma_s^{c_s}$$

with $s = r + 1$, and

$$\gamma_i = \frac{\varphi(\epsilon_i)}{\tau_b(\epsilon_i)}, \quad c_i = a_i, \quad (i = 1, \ldots, r), \quad \gamma_s = \frac{\varphi(\alpha\zeta)}{\tau_b(\alpha\zeta)}, \quad c_s = 1.$$

From Proposition 2 with

$$H_1 = \cdots = H_s = \kappa_{36}, \quad C = A,$$

we deduce $B \leq \kappa_{37} \log A$. Then we use the upper bound $A \leq \kappa_{21} B$ of Lemma 4 to get $B \leq \kappa_{38} \log B$ and $A \leq \kappa_{39} \log A$. We use (9) to conclude the proof of Lemma 6. □

Therefore Lemma 6 now allows us to suppose that for any $\varphi \in \Phi$ different from τ_b, we have $|\varphi(\beta)| > |\tau_b(\beta)|^\nu$. In particular, the embedding τ_b is then real. This is the end of the proof in the totally imaginary case, cf. [3].

From now on, we suppose $T_b(\nu) = \{\tau_b\}$.

10 Upper bound for $|\tau_b(\alpha\varepsilon)|$

An upper bound for $|\tau_b(\alpha\varepsilon)|$ is exhibited.

Lemma 7. *One has* $|\tau_b(\alpha\varepsilon)| \leq 2$.

Proof. We have

$$|x - \tau_b(\alpha\varepsilon)y| = |\tau_b(\beta)| < \frac{1}{2} < |x|,$$

wherupon we deduce

$$|\tau_b(\alpha\varepsilon)y| \leq 2|x| \leq 2|y|,$$

since $|x| \leq |y|$.

□

11 Unicity of σ_a

Since $|\tau_b(\beta)|$ is very small, x is close to $\tau_b(\alpha\varepsilon)y$. Now, for any $\varphi \in \Phi$, we have

$$\varphi(\beta) = x - \varphi(\alpha\varepsilon)y.$$

Consequently, if $|\varphi(\alpha\varepsilon)|$ is smaller than $|\tau_b(\alpha\varepsilon)|$, then $\varphi(\beta)$ is close to x, while if $|\varphi(\alpha\varepsilon)|$ is larger than $|\tau_b(\alpha\varepsilon)|$, then $\varphi(\beta)$ is close to $-\varphi(\alpha\varepsilon)y$. Let us justifty these claims.

Lemma 8. *Let* $\varphi \in \Phi$.
(a) *Let* λ *be a real number in the interval* $]0, 1[$. *If* $|\varphi(\alpha\varepsilon)| \leq \lambda|\tau_b(\alpha\varepsilon)|$, *then*

$$|\varphi(\beta) - x| \leq \lambda|x| + \lambda e^{-\kappa_{20}B}.$$

(b) *Let* μ *be a real number* > 1. *If* $|\varphi(\alpha\varepsilon)| \geq \mu|\tau_b(\alpha\varepsilon)|$, *then*

$$|\varphi(\beta) + \varphi(\alpha\varepsilon)y| \leq \frac{1}{\mu}|\varphi(\alpha\varepsilon)y| + e^{-\kappa_{20}B}.$$

Proof. We have $|\tau_b(\beta)| \leq e^{-\kappa_{20}B}$, namely

$$|x - \tau_b(\alpha\varepsilon)y| \leq e^{-\kappa_{20}B}.$$

We also have

$$\varphi(\beta) = x - \varphi(\alpha\varepsilon)y.$$

Because of the hypothesis (a) we get

$$|\varphi(\beta) - x| = |\varphi(\alpha\varepsilon)y| \leq \lambda|\tau_b(\alpha\varepsilon)y| \leq \lambda|x| + \lambda e^{-\kappa_{20}B}.$$

Because of the hypothesis (b), we have

$$|\varphi(\beta) + \varphi(\alpha\varepsilon)y| = |x| \leq |\tau_b(\alpha\varepsilon)y| + e^{-\ \kappa_{20}B} \leq \frac{1}{\mu}|\varphi(\alpha\varepsilon)y| + e^{-\ \kappa_{20}B}.$$

□

Lemma 9. *Let* $\varphi \in \Phi$ *with* $\varphi \neq \sigma_a$. *Then*

$$|\varphi(\beta)| \leq 2|x| \exp\left\{\kappa_{40}(\log m)\log\left(2 + \frac{A+B}{\log m}\right)\right\}$$

and

$$|\varphi(\alpha\varepsilon)| \leq \max\left\{\frac{3}{2}|\tau_b(\alpha\varepsilon)|\ ,\ 8\frac{|x|}{|y|}\exp\left\{\kappa_{40}(\log m)\log\left(2 + \frac{A+B}{\log m}\right)\right\}\right\}.$$

Proof. From the relation (6) with $\varphi_1 = \sigma_a$ et $\varphi_2 = \varphi$ we deduce

$$x = \frac{\varphi(\alpha\varepsilon)\sigma_a(\beta) - \sigma_a(\alpha\varepsilon)\varphi(\beta)}{\varphi(\alpha\varepsilon) - \sigma_a(\alpha\varepsilon)},$$

hence

$$\frac{\varphi(\alpha\varepsilon)\sigma_a(\beta)}{\sigma_a(\alpha\varepsilon)\varphi(\beta)} - 1 = \frac{\varphi(\alpha\varepsilon) - \sigma_a(\alpha\varepsilon)}{\sigma_a(\alpha\varepsilon)\varphi(\beta)} \cdot x.$$

The member of the right side is nonzero, and its modulus is bounded from above by $2|x|/|\varphi(\beta)|$ since $|\varphi(\alpha\varepsilon)| \leq |\sigma_a(\alpha\varepsilon)|$. The upper bound of $|\varphi(\beta)|$ follows from Lemma 3 with $\varphi_1 = \varphi_4 = \varphi$ et $\varphi_2 = \varphi_3 = \sigma_a$.

To establish the upper bound given in Lemma 9 for $|\varphi(\alpha\varepsilon)|$, we may suppose

$$|\varphi(\alpha\varepsilon)| > \frac{3}{2}|\tau_b(\alpha\varepsilon)|,$$

otherwise the conclusion is trivial. Then we may use Lemma 8 (b) with $\mu = 3/2$ to deduce

$$|\varphi(\alpha\varepsilon)y| - |\varphi(\beta)| \leq |\varphi(\beta) + \varphi(\alpha\varepsilon)y| \leq \frac{2}{3}|\varphi(\alpha\varepsilon)y| + e^{-\ \kappa_{20}B},$$

hence

$$|\varphi(\alpha\varepsilon)y| \leq 3|\varphi(\beta)| + 3e^{-\ \kappa_{20}B} \leq 4\max\{|\varphi(\beta)|, 1\}.$$

We can conclude by using the upper bound of $|\varphi(\beta)|$ which we just established.

□

From Lemma 9 we deduce the following.

Corollary 1. *Assuming* (8), *we have* $\Sigma_a(\nu) = \{\sigma_a\}$.

Proof. Let us remind that $|x| \leq |y|$. Since $|\tau_b(\alpha\varepsilon)| \leq 2$ (Lemma 7), with $\sigma \in \Sigma_a(\nu)$, we have

$$|\sigma(\alpha\varepsilon)| > \frac{3}{2}|\tau_b(\alpha\varepsilon)|.$$

If there were $\sigma \in \Sigma_a(\nu)$ with $\sigma \neq \sigma_a$, by using Lemma 9 with $\varphi = \sigma$, we would deduce $A \leq \kappa_{41} \log m$ and thanks to (8) we could conclude that σ_a is the only element of $\Sigma_a(\nu)$. □

12 Proof of of the main result

Let us concentrate on the

Proof of Theorem 1. For the part (a) of Theorem 1, we take $\varepsilon \in \mathcal{E}_\nu^{(\alpha)}$; by definition of $\mathcal{E}_\nu^{(\alpha)}$, there exists $\varphi \in \Phi$, $\varphi \neq \sigma_a$, with

$$|\varphi(\alpha\varepsilon)| \geq |\sigma_a(\alpha\varepsilon)|^\nu,$$

namely $\varphi \in \Sigma_a(\nu)$. Since $\Sigma_a(\nu)$ contains more than one element, Corollary 1 shows that the inequalities (8) are not satisfied. This completes the proof of part (a) of Theorem 1.

To prove the part (b), we will use the reciprocal polynomial of f_ε, defined by

$$Y^d f_\varepsilon(1/Y) = a_d Y^d + \cdots + a_0 = a_d \prod_{i=1}^{d} \big(Y - \sigma_i(\alpha'\varepsilon')\big),$$

with $\alpha' = \alpha^{-1}$ and $\varepsilon' = \varepsilon^{-1}$ and we will write the binary form F_ε as

$$F_\varepsilon(X, Y) = a_d \prod_{i=1}^{d} \big(Y - \sigma_i(\alpha'\varepsilon')X\big).$$

The part (a) of Theorem 1 not only indicates that any solution $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathcal{E}_\nu^{(\alpha)}$ of the inequation $|F_\varepsilon(x, y)| \leq m$ with $0 < |x| \leq |y|$ verifies

$$\max\{|y|,\ e^{\mathrm{h}(\alpha\varepsilon)}\} \leq m^{\kappa_4(\alpha)},$$

but also shows that any solution $(x, y, \varepsilon) \in \mathbf{Z}^2 \times \mathcal{E}_\nu^{(\alpha)}$ of the inequation $|F_\varepsilon(x, y)| \leq m$ with $0 < |y| \leq |x|$ verifies

$$\max\{|x|,\ e^{\mathrm{h}(\alpha'\varepsilon')}\} \leq m^{\kappa_4(\alpha')}.$$

Since $\mathrm{h}(\alpha'\varepsilon') = \mathrm{h}(\alpha\varepsilon)$ and since $\tilde{\mathcal{E}}_\nu^{(\alpha)}$ is the set of $\varepsilon \in \mathcal{E}_\nu^{(\alpha)}$ such that $\varepsilon' \in \mathcal{E}_\nu^{(\alpha')}$, it follows that each solution of the inequation $|F_\varepsilon(x, y)| \leq m$ with $xy \neq 0$ verifies

$$\max\{|x|,\ |y|,\ e^{\mathrm{h}(\alpha\varepsilon)}\} \leq m^{\kappa_5}.$$

□

13 Proof of Proposition 1

Let us index the elements of Φ in such a way that $\sigma_1,\ldots,\sigma_{r_1}$ are the real embeddings and $\sigma_{r_1+1},\ldots,\sigma_d$ are the non-real embeddings, with $\sigma_{r_1+j}=\overline{\sigma}_{r_1+r_2+j}$ $(1\le j\le r_2)$. We have $d=r_1+2r_2$ and $r=r_1+r_2-1$. The logarithmic embedding of K is the group homomorphism $\underline{\lambda}$ of $K^\times$ into $\mathbf{R}^{r+1}$ defined by

$$\underline{\lambda}(\gamma)=\bigl(\delta_1\log|\sigma_1(\gamma)|,\ldots,\delta_{r+1}\log|\sigma_{r+1}(\gamma)|\bigr),$$

where

$$\delta_i=\begin{cases}1 & \text{for } i=1,\ldots,r_1,\\ 2 & \text{for } i=r_1+1,\ldots,r_1+r_2.\end{cases}$$

Its kernel is the finite subgroup $K^\times_{\mathrm{tors}}$ of torsion elements of $K^\times$, which are the roots of unity belonging to K. By Dirichlet's theorem, the image of $\mathbf{Z}_K^\times$ under $\underline{\lambda}$ is a lattice of the hyperplane $\mathcal{H}$ of equation

$$t_1+\cdots+t_{r+1}=0 \tag{12}$$

in $\mathbf{R}^{r+1}$. For $M>0$, define

$$\mathcal{H}(M)=\{(t_1,\ldots,t_{r+1})\in\mathcal{H}\ \mid\ \max\{\delta_1^{-1}t_1,\ldots,\delta_{r+1}^{-1}t_{r+1}\}\le M\}.$$

For all elements $(t_1,\ldots,t_{r+1})$ of $\mathcal{H}(M)$ we have

$$\max_{1\le i\le r+1}t_i\le 2M.$$

Further, the inequality

$$t_1+\cdots+t_{r+1}\le \min_{1\le i\le r+1}t_i+r\max_{1\le i\le r+1}t_i$$

together with the equation of $\mathcal{H}$ implies

$$\max_{1\le i\le r+1}-t_i=-\min_{1\le i\le r+1}t_i\le r\max_{1\le i\le r+1}t_i\le 2rM,$$

hence this set $\mathcal{H}(M)$ is bounded: namely, for $(t_1,\ldots,t_{r+1})\in\mathcal{H}(M)$,

$$\max_{1\le i\le r+1}|t_i|\le 2rM.$$

The r–dimension volume of $\mathcal{H}(M)$ is the product of the volume of $\mathcal{H}(1)$ by M^r while the volume of $\mathcal{H}(1)$ is an effectively computable positive constant, depending only upon r_1 and r_2.

Proof of the part (a). Since $\underline{\lambda}(\mathbf{Z}_K^\times)$ is a lattice of the hyperplane $\mathcal{H}$, the limit

$$\lim_{M\to\infty}\frac{1}{M^r}\left|\underline{\lambda}(\mathbf{Z}_K^\times)\cap\mathcal{H}(M)\right|$$

exists and is a positive number.

The image of $\varepsilon \in \mathbf{Z}_K^\times$ by $\underline{\lambda}$ is

$$\underline{\lambda}(\varepsilon) = (t_1, \dots, t_{r+1}) \quad \text{with} \quad t_i = \delta_i \log|\sigma_i(\varepsilon)| \quad (i = 1, \dots, r+1).$$

If on the one hand $\varepsilon \in \mathbf{Z}_K^\times(N)$, then

$$\delta_i^{-1} t_i \le \log N - \log|\sigma_i(\alpha)| \quad (1 \le i \le r+1);$$

therefore

$$\max_{1 \le i \le r+1} \delta_i^{-1} t_i \le \log N + \log \lceil \overline{\alpha^{-1}} \rceil.$$

Consequently, if we define

$$M_+ = \log N + \log \lceil \overline{\alpha^{-1}} \rceil,$$

we have $\underline{\lambda}(\varepsilon) \in \mathcal{H}(M_+)$. On the other hand, we have

$$\log|\sigma_i(\alpha\varepsilon)| = \delta_i^{-1} t_i + \log|\sigma_i(\alpha)| \le \delta_i^{-1} t_i + \log \lceil \overline{\alpha} \rceil.$$

If we define

$$M_- = \log N - \log \lceil \overline{\alpha} \rceil,$$

then for any $\underline{\lambda}(\varepsilon) \in \underline{\lambda}(\mathbf{Z}_K^\times) \cap \mathcal{H}(M_-)$ we have $\varepsilon \in \mathbf{Z}_K^\times(N)$. Therefore,

$$\underline{\lambda}(\mathbf{Z}_K^\times) \cap \mathcal{H}(M_-) \subset \underline{\lambda}(\mathbf{Z}_K^\times(N)) \subset \underline{\lambda}(\mathbf{Z}_K^\times) \cap \mathcal{H}(M_+).$$

Now we can conclude that the part (a) of Proposition 1 is proved.

Recall that a CM field is a totally imaginary number field which is a quadratic extension of its maximal totally real subfield. Let us prove that for a CM field the number of elements ε of $\mathbf{Z}_K^\times(N)$ such that $\mathbf{Q}(\alpha\varepsilon) \neq K$ is negligible with respect to the number of elements ε of $\mathbf{Z}_K^\times(N)$ such that $\mathbf{Q}(\alpha\varepsilon) = K$. Denote by $\mathcal{F}^{(\alpha)}$ the complement of $\mathcal{E}^{(\alpha)}$ in $\mathbf{Z}_K^\times$:

$$\mathcal{F}^{(\alpha)} = \{\varepsilon \in \mathbf{Z}_K^\times \mid \mathbf{Q}(\alpha\varepsilon) \neq K\}.$$

Lemma 10. *Assume K is not a CM field. Then*

$$\limsup_{N \to \infty} \frac{1}{(\log N)^{r-1}} \left| \underline{\lambda}\big(\mathcal{F}^{(\alpha)}(N)\big) \right| < \infty.$$

Proof. The set of subfields L of K is finite. Since K is not a CM field, the rank ϱ of the unit group of such a subfield L strictly contained in K is smaller than r. Therefore the number of $\varepsilon \in \mathbf{Z}_K^\times$ such that $\mathbf{Q}(\alpha\varepsilon) = L$ and $\lambda(\alpha) \in \mathcal{H}(M)$ is bounded by a constant times M^ϱ. The proof of Lemma 10 is then secured. □

Proof of the part (b) *of Proposition* 1. If K is not a CM field, the stronger estimate
$$\lim_{N\to\infty} \frac{|\mathcal{E}^{(\alpha)}(N)|}{|\mathbf{Z}_K^\times(N)|} = 1.$$
follows from the part (a) and from Lemma 10.

Assume now that K is CM field with maximal totally real subfield L_0. Since $K = \mathbf{Q}(\alpha)$, for any $\varepsilon \in \mathbf{Z}_{L_0}^\times$, we have $\mathbf{Q}(\alpha\varepsilon) \neq L_0$. As we have seen, for each subfield L of K different from K and from L_0, the set of $\varepsilon \in \mathbf{Z}_K^\times$ such that $\mathbf{Q}(\alpha\varepsilon) = L$ and $\lambda(\alpha) \in \mathcal{H}(M)$ is bounded by a constant times M^{r-1}. The other elements $\varepsilon \in \mathbf{Z}_K^\times$ with $\lambda(\alpha) \in \mathcal{H}(M)$ have $\mathbf{Q}(\alpha\varepsilon) = K$. This completes the proof of the part (b) of Proposition 1.

Before completing the proof of Proposition 1, one introduces a change of variables $t_i = \delta_i x_i$: we call H the hyperplane of $\mathbf{R}^{r+1}$ of equation
$$\delta_1 x_1 + \cdots + \delta_{r+1} x_{r+1} = 0$$
and for $M > 0$, we consider
$$H(M) = \bigl\{(x_1, \ldots, x_{r+1}) \in \mathcal{H} \ \mid\ \max\{x_1, \ldots, x_{r+1}\} \leq M\bigr\}.$$

Proof of the part (c).

Let ν be a real number in the interval $]0,1[$. Let us take $M = \log N$. Define some subsets $D_\nu(M)$ and $D'_\nu(M)$ of $H(M)$ the following way:
$$D_\nu(M) = \bigl\{(x_1, \ldots, x_{r+1}) \in H(M) \ \mid \ \text{there exists } i,j \text{ with } i \neq j \text{ and } 1 \leq i,j \leq r_1 \text{ such that } x_i \geq \nu M \text{ and } x_j \geq \nu M\bigr\}$$
and
$$D'_\nu(M) = \bigl\{(x_1, \ldots, x_{r+1}) \in H(M) \ \mid \ \text{there exists } i \text{ with } r_1 < i \leq r+1, \text{ such that } x_i \geq \nu M\}\bigr\}.$$
If $D_\nu(M)$ is not empty, then $r_1 \geq 2$ while if $D'_\nu(M)$ is not empty, then $r_2 \geq 1$. We show that if $r_1 \geq 2$ and $0 < \nu < \delta_{r+1}/2$, then $D_\nu(1)$ has a positive volume while if $r_2 \geq 1$ and $0 < \nu < \delta_r/2$, then $D'_\nu(1)$ has a positive volume. This will show that, for a number field of degree ≥ 3 and for $0 < \nu < 1/2$, at least one of the two sets $D_\nu(1)$ and $D'_\nu(1)$ has a positive volume.

Assume $r_1 \geq 2$, hence $\delta_1 = \delta_2 = 1$, and $0 < \nu < \delta_{r+1}/2$. Let a, b, c be positive real numbers with $\nu \leq a < b < \delta_{r+1}/2$, $c < 1$ and $c < \delta_{r+1} - 2b$. Then $D_\nu(1)$ contains the set of $(x_1, x_2, \ldots, x_{r+1}) \in H$ verifying [1]
$$a \leq x_1, x_2 \leq b, \quad \frac{-c}{\delta_i(r-2)} \leq x_i \leq \frac{c}{\delta_i(r-2)} \quad (3 \leq i \leq r),$$

[1]Notice that one does not divide by 0: if $r = 2$ the last conditions for $3 \leq i \leq r$ disappear.

because these bounds and the equation $\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0$ of H, imply

$$-1 \leq x_{r+1} \leq 1.$$

This shows that $D_\nu(1)$ has positive volume.

Next assume $r_2 \geq 1$, hence $\delta_{r+1} = 2$, and $0 < \nu < \delta_r/2$. Let a, b, c be positive real numbers with $\nu \leq a < b < \delta_r/2$ and $c < \delta_r - 2b$. Then $D'_\nu(1)$ contains the set of $(x_1, x_2, \ldots, x_{r+1}) \in H$ verifying

$$a \leq x_{r+1} \leq b, \quad \frac{-c}{\delta_i(r-1)} \leq x_i \leq \frac{c}{\delta_i(r-1)} \quad (1 \leq i \leq r-1),$$

because these bounds, together with the equation $\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0$ of H, imply

$$-1 \leq x_r \leq 1.$$

This shows that $D'_\nu(1)$ has positive volume.

Once we know that the r–dimension volume of $D_\nu(1)$ (resp. $D'_\nu(1)$) in H is positive, we deduce that the r–dimension volume of $D_\nu(M)$ (resp. $D'_\nu(M)$) is bounded below by an effectively computable positive constant times M^r — as a matter of fact, $D_\nu(M)$ (resp. $D'_\nu(M)$) is equal to the product of M^r by the effectively computable constant $D_\nu(1)$ (resp. $D'_\nu(1)$). Since $\underline{\lambda}(\alpha) + \underline{\lambda}(\mathbf{Z}_K^\times)$ is a translate of the lattice $\underline{\lambda}(\mathbf{Z}_K^\times)$, the cardinality of the set

$$\big(\underline{\lambda}(\alpha) + \underline{\lambda}(\mathbf{Z}_K^\times)\big) \cap \big(D_\nu(M) \cup D'_\nu(M)\big)$$

is bounded below by an effectively computable positive constant times M^r.

Let $\varepsilon \in \mathbf{Z}_K^\times$ be such that $\underline{\lambda}(\alpha\varepsilon) \in D_\nu(M) \cup D'_\nu(M)$. We have

$$\log \max_{1 \leq j \leq d} |\sigma_j(\alpha\varepsilon)| \leq M$$

and there exist two distinct elements φ_1, φ_2 of Φ such that

$$\log |\varphi_i(\alpha\varepsilon)| \geq \nu M \quad (i = 1, 2).$$

Consequently,

$$\lceil \alpha\varepsilon \rceil \leq e^M, \quad |\varphi_1(\alpha\varepsilon)| \geq \lceil \alpha\varepsilon \rceil^\nu, \quad |\varphi_2(\alpha\varepsilon)| \geq \lceil \alpha\varepsilon \rceil^\nu$$

and finally, since $N = e^M$, we conclude $\varepsilon \in \mathcal{E}_\nu^{(\alpha)}(N)$.

Proof of the part (d). Suppose $d \geq 4$. For $M > 0$, define

$$\begin{aligned}
\tilde{D}_\nu(M) &= \big\{(x_1, \ldots, x_{r+1}) \in D_\nu(M) \;\mid\; (-x_1, \ldots, -x_{r+1}) \in D_\nu(M)\big\}, \\
D''_\nu(M) &= \big\{(x_1, \ldots, x_{r+1}) \in D_\nu(M) \;\mid\; (-x_1, \ldots, -x_{r+1}) \in D'_\nu(M)\big\}, \\
\tilde{D}'_\nu(M) &= \big\{(x_1, \ldots, x_{r+1}) \in D'_\nu(M) \;\mid\; (-x_1, \ldots, -x_{r+1}) \in D'_\nu(M)\big\}.
\end{aligned}$$

If $\tilde{D}_\nu(M)$ is not empty, then $r_1 \geq 4$. If $D''_\nu(M)$ is not empty, then $r_1 \geq 2$ and $r_2 \geq 1$. If $\tilde{D}'_\nu(M)$ is not empty, then $r_2 \geq 2$.

Let us show conversely that if $r_1 \geq 4$, then $\tilde{D}_\nu(1)$ has a positive volume, that if $r_1 \geq 2$ and $r_2 \geq 1$, then $\tilde{D}'_\nu(1)$ has a positive volume and that if $r_2 \geq 2$, then $\tilde{D}'_\nu(1)$ has a positive volume.

Let a, b, c be three positive numbers such that

$$\nu < a < b < 1 \quad \text{and} \quad c + 2b < 2a + 1.$$

For instance

$$a = \frac{1+\nu}{2}, \quad b = \frac{3+\nu}{4}, \quad c = \frac{1+\nu}{4}\cdot$$

Assume $r_1 \geq 4$, hence $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$. Then $\tilde{D}_\nu(1)$ contains the set of $(x_1, x_2, \ldots, x_{r+1}) \in H$ verifying

$$a \leq x_1, x_2 \leq b, \quad -b \leq x_3, x_4 \leq -a, \quad \frac{-c}{\delta_i(r-4)} \leq x_i \leq \frac{c}{\delta_i(r-4)} \quad (5 \leq i \leq r),$$

because these bounds, together with the equation $\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0$ of H, imply

$$-1 \leq x_{r+1} \leq 1.$$

This shows that $\tilde{D}_\nu(1)$ has a positive volume.

Assume $r_1 \geq 2$ and $r_2 \geq 1$, hence $\delta_1 = \delta_2 = 1$ and $\delta_{r+1} = 2$. Then $\tilde{D}_\nu(1)$ contains the set of $(x_1, x_2, \ldots, x_{r+1}) \in H$ verifying

$$a \leq x_1, x_2 \leq b, \quad -b \leq x_{r+1} \leq -a, \quad \frac{-c}{\delta_i(r-3)} \leq x_i \leq \frac{c}{\delta_i(r-3)} \quad (3 \leq i \leq r-1),$$

because these bounds and the equation $\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0$ of H, imply

$$-1 \leq x_r \leq 1.$$

Therefore $\tilde{D}'_\nu(1)$ has a positive volume.

Finally, assume $r_2 \geq 2$, hence $\delta_r = \delta_{r+1} = 2$. Then $\tilde{D}'_\nu(1)$ contains the set of $(x_1, x_2, \ldots, x_{r+1}) \in H$ verifying

$$a \leq x_{r+1} \leq b, \quad -b \leq x_r \leq -a, \quad \frac{-c}{\delta_i(r-2)} \leq x_i \leq \frac{c}{\delta_i(r-2)} \quad (2 \leq i \leq r-1),$$

because these bounds, together with the equation $\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0$ of H imply $-1 \leq x_1 \leq 1$. Hence the r–dimension volume of $\tilde{D}'_\nu(1)$ is positive.

Since $d \geq 4$, in all cases the volume of $\tilde{D}_\nu(M) \cup \tilde{D}'_\nu(M) \cup \tilde{D}'_\nu(M)$ is bounded below by an effectively computable positive constant times M^r. The number of elements in the intersection of this set with $\underline{\lambda}(\alpha) + \underline{\lambda}(\mathbf{Z}_K^\times)$ is bounded below by an effectively computable positive constant times M^r.

Let $\varepsilon \in \mathbf{Z}_K^\times$ be such that $\underline{\lambda}(\alpha\varepsilon) \in \tilde{D}_\nu(M) \cup \tilde{D}'_\nu(M) \cup \tilde{D}'_\nu(M)$. We have

$$\log \max_{1 \le j \le d} |\sigma_j(\alpha\varepsilon)| \le M, \quad \log \min_{1 \le j \le d} |\sigma_j(\alpha\varepsilon)| \ge -M$$

and there exist four distinct elements $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ of Φ such that

$$\log |\varphi_i(\alpha\varepsilon)| \ge \nu M \quad (i = 1, 2) \quad \text{and} \quad \log |\varphi_j(\alpha\varepsilon)| \le -\nu M \quad (j = 3, 4).$$

Consequently

$$\lceil \alpha\varepsilon \rceil \le e^M, \quad \lceil \alpha \rceil^\nu \le |\varphi_i(\alpha\varepsilon)| \le \lceil \alpha \rceil \quad (i = 1, 2)$$

and

$$\left\lceil (\alpha\varepsilon)^{-1} \right\rceil \le e^M, \quad \left\lceil (\alpha\varepsilon)^{-1} \right\rceil^{-1} \le |\varphi_j(\alpha\varepsilon)| \le \left\lceil (\alpha\varepsilon)^{-1} \right\rceil^{-\nu} \quad (j = 2, 3),$$

whereupon finally $\varepsilon \in \tilde{\mathcal{E}}_\nu^{(\alpha)}(e^M)$.

The part (d) of Proposition 1 is then proved.

Acknowledgements

The second author thanks the ASSMS (Abdus Salam School of Mathematical Sciences) of Lahore for a fruitful stay in October 2011.

References

[1] C. Levesque and M. Waldschmidt, *Familles d'équations de Thue–Mahler n'ayant que des solutions triviales*, Acta Arith., **155** (2012), 117–138.

[2] C. Levesque and M. Waldschmidt, *Families of cubic Thue equations with effective bounds for the solutions*, J.M. Borwein et al. (eds.), Number Theory and Related Fields: In Memory of Alf van der Poorten, Springer Proceedings in Mathematics & Statistics **43** (2013) 229–243.

[3] C. Levesque and M. Waldschmidt, *Solving simultaneously and effectively Thue Diophantine equations: almost totally imaginary case*, Proceedings of the International Meeting on Number Theory HRI 2011, in honor of R. Balasubramanian, HBA, India (submitted).

[4] T.N. Shorey and R. Tijdeman, *Exponential Diophantine equations*, vol. 87 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1986.

[5] M. Waldschmidt, *Diophantine approximation on linear algebraic groups*, volume **326** of *Grundlehren der Mathematischen Wissenschaften* (Springer-Verlag, Berlin, 2000).

Claude LEVESQUE
Département de mathématiques et de statistique
Université Laval
Québec (Québec)
CANADA G1V 0A6
Claude.Levesque@mat.ulaval.ca

Michel WALDSCHMIDT
Université Pierre et Marie Curie (Paris 6),
Théorie des Nombres Case courrier 247
4 Place Jussieu
F – 75252 PARIS Cedex 05, FRANCE
miw@math.jussieu.fr