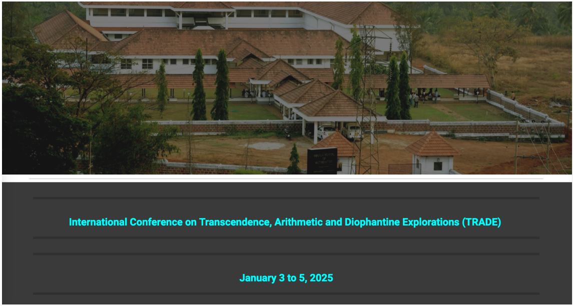

January 3, 2025

SORBONNE UNIVERSITÉ

On Schanuel's Conjecture, elliptic and quasi-elliptic functions

Michel Waldschmidt

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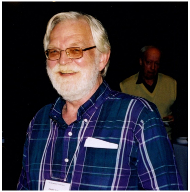
Abstract

Schanuel's Conjecture is a well known and far reaching statement involving the usual exponential function.
We introduce the concept of *Strong Schanuel Property*, where the transcendence degree is $2n$ in place of n. Almost all tuples of complex numbers satisfy this property. We show how to construct an uncountable set of such tuples.
Next we add to the exponential function other functions related with elliptic curves, namely Weierstrass σ, ζ, $\wp$ functions, as well as meromorphic functions introduced by Serre in connexion with elliptic integrals of the third kind. We show that such functions are algebraically independent, and we deduce strong algebraic independence results for almost all tuples.
In a forthcoming joint paper with Cristiana Bertolin we propose some conjectures having a geometric origin for the values of these functions.

Schanuel's Conjecture



Let $x_1, \dots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then at least n of the $2n$ numbers $x_1, \dots, x_n, \ \mathrm{e}^{x_1}, \dots, \mathrm{e}^{x_n}$ are algebraically independent.

In other terms, the conclusion is

$$\operatorname{tr}\deg_{\mathbb{Q}} \mathbb{Q}\big(x_1, \dots, x_n, \ \mathrm{e}^{x_1}, \dots, \mathrm{e}^{x_n}\big) \geqslant n.$$

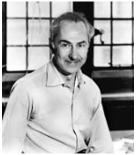
Origin of Schanuel's Conjecture

Course given by Serge Lang (1927–2005) at Columbia in the 60's

S. Lang – *Introduction to transcendental numbers*, Addison-Wesley 1966.

also attended by M. Nagata (1927–2008)
(14*th* Problem of Hilbert).

Nagata's Conjecture solved by E. Bombieri.



S. Lang
1927 – 2005

M. Nagata
1927 – 2008

E. Bombieri

Algebraic independence results

• Lindemann–Weierstrass : if $x_1, \ldots, x_n$ are algebraic and $\mathbb{Q}$–linearly independent, then $e^{x_1}, \ldots, e^{x_n}$ are algebraically independent.

• A.O. Gel'fond : if β is a cubic irrationality and α a nonzero algebraic number with $\log \alpha \neq 0$, then α^β and α^{β^2} are algebraically independent.

• Yu V. Nesterenko : the two numbers π and e^π are algebraically independent.

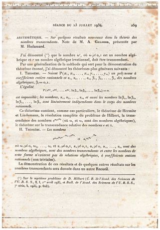
F. Lindemann
1852 – 1939

K. Weierstrass
1815 – 1897

A.O. Gelfond
1906 – 1968

Yu. V. Nesterenko

A.O. Gel'fond CRAS 1934



SÉANCE DU 23 JUILLET 1934. 259

ARITHMÉTIQUE. — *Sur quelques résultats nouveaux dans la théorie des nombres transcendants.* Note de M. A. Gelfond, présentée par M. Hadamard.

J'ai démontré (¹) que le nombre ω^r, où $\omega \neq 0,1$ est un nombre algébrique et r un nombre algébrique irrationnel, doit être transcendant.

Par une généralisation de la méthode qui sert pour la démonstration du théorème énoncé, j'ai démontré les théorèmes plus généraux suivants :

I. Théorème. — *Soient* $P(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$ *un polynome à coefficients entiers rationnels et* $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ *des nombres algébriques,* $\beta_i \neq 0,1$.

L'égalité

$$P(e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}, \ln\beta_1, \ln\beta_2, \ldots, \ln\beta_m) = 0$$

est impossible; les nombres, $\alpha_1, \alpha_2, \ldots, \alpha_n$, *et aussi les nombres* $\ln\beta_1, \ln\beta_2, \ln\beta_3, \ldots, \ln\beta_m$ *sont linéairement indépendants dans le corps des nombres rationnels.*

Ce théorème contient, comme cas particuliers, le théorème de Hermite et Lindemann, la résolution complète du problème de Hilbert, la transcendance des nombres $e^{\omega_1\omega_2}$ (où ω_1 et ω_2 sont des nombres algébriques), le théorème sur la transcendance relative des nombres e et π.

II. Théorème. — *Les nombres*

$$e^{\omega_1^{\omega_2^{\cdots^{\omega_n}}}} \quad \text{et} \quad \alpha_1^{\alpha_2^{\cdots^{\alpha_m}}},$$

où $\omega_1 \neq 0, \omega_2, \ldots, \omega_n$ *et* $\alpha_1 \neq 0,1$, $\alpha_2 \neq 0,1$, $\alpha_i \neq 0$, $\alpha_3, \ldots, \alpha_m$ *sont des nombres algébriques, sont des nombres transcendants et entre les nombres de cette forme n'existent pas de relations algébriques, à coefficients entiers rationnels* (non triviales).

La démonstration de ces résultats et de quelques autres résultats sur les nombres transcendants sera donnée dans un autre Recueil.

(¹) *Sur le septième problème de D. Hilbert* (*C. R. de l'Acad. des Sciences de l'U. R. S. S.*, 2, I, 1er avril 1934, et *Bull. de l'Acad. des Sciences de l'U. R. S. S.*, 7e série, 4, 1934, p. 623).

Statement by Gel'fond (1934)

Let $\beta_1, \ldots, \beta_n$ be $\mathbb{Q}$-linearly independent algebraic numbers and let $\log \alpha_1, \ldots, \log \alpha_m$ be $\mathbb{Q}$-linearly independent logarithms of algebraic numbers. Then the numbers

$$\mathrm{e}^{\beta_1}, \ldots, \mathrm{e}^{\beta_n}, \; \log \alpha_1, \ldots, \log \alpha_m$$

are algebraically independent over $\mathbb{Q}$.

Further statement by Gel'fond

Let $\beta_1, \ldots, \beta_n$ be algebraic numbers with $\beta_1 \neq 0$ and let $\alpha_1, \ldots, \alpha_m$ be algebraic numbers with $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_i \neq 0$. Then the numbers

$$\mathrm{e}^{\beta_1 \mathrm{e}^{\beta_2 \mathrm{e}^{\cdot^{\cdot^{\cdot^{\beta_{n-1}\mathrm{e}^{\beta_n}}}}}}} \quad \text{and} \quad {\alpha_1}^{{\alpha_2}^{\cdot^{\cdot^{\cdot^{\alpha_m}}}}}$$

are transcendental, and there is no nontrivial algebraic relation between such numbers.

`Remark` : The condition on the α_i should be that they are irrational.

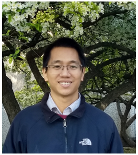
Easy consequence of Schanuel's Conjecture

According to Schanuel's Conjecture, the following numbers are algebraically independent :

$$e+\pi,\ e\pi,\ \pi^e,\ \mathrm{e}^{\mathrm{e}}, \mathrm{e}^{\mathrm{e}^2}, \dots,\ \mathrm{e}^{\mathrm{e}^{\mathrm{e}}}, \dots,\ \pi^{\pi}, \pi^{\pi^2}, \dots\ \pi^{\pi^{\pi}} \dots$$

$$\log\pi,\ \log(\log 2),\ \pi\log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3} \dots$$

Proof : Use Schanuel's Conjecture several times.



Tung T. Nguyen

While working on a cryptography project, we came upon the following problem. Let x *be the real solution of* $x^{x^x} = 100$. *Is* x *a transcendental number ?*

Conditional answer assuming Schanuel's Conjecture : if $x > 0$, $\notin \mathbb{Z}$, is such that x^{x^x} is an integer, then x is transcendental.

Fixed points of the exponential function

Open problem

Federico Pellarin

Let $z_1, \ldots, z_n$ be distinct fixed points of the exponential function :

$$e^{z_i} = z_i \quad i = 1, \ldots, n.$$

Is it true that $z_1, \ldots, z_n$ are $\mathbb{Q}$–linearly independent ?

Schanuel's Conjecture implies that $z_1, \ldots, z_n$ *are algebraically independent.*

Further consequences of Schanuel's Conjecture

Transcendental values of class group L–functions, Gamma values, log Gamma values, digamma function, generalized Euler-Briggs constants, derivatives of L–series and generalized Stieltjes constants ...

Papers by Purusottam Rath, Ram Murty, Sanoli Gun, Kumar Murty, Ekata Saha, Siddhi Pathak, ...

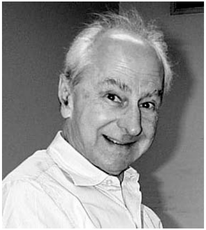
P. Rath, R. Murty, S. Gun

K. Murty

E. Saha

S. Pathak

Lang's exercise



Define $E_0 = \mathbb{Q}$. Inductively, for $n \geq 1$, define E_n as the algebraic closure of the field generated over E_{n-1} by the numbers $\exp(x) = \mathrm{e}^x$, where x ranges over E_{n-1}. Let E be the union of E_n, $n \geq 0$. Then Schanuel's Conjecture implies that the number π does not belong to E.

More precisely : Schanuel's Conjecture implies that the numbers $\pi, \log \pi, \log\log \pi, \log\log\log \pi, \ldots$ are algebraically independent over E.

A variant of Lang's exercise

Define $L_0 = \mathbb{Q}$. Inductively, for $n \geq 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y, where y ranges over the set of complex numbers such that $\mathrm{e}^y \in L_{n-1}$. Let L be the union of L_n, $n \geq 0$. Then Schanuel's Conjecture implies that the number e does not belong to L.

More precisely : Schanuel's Conjecture implies that the numbers $\mathrm{e}, \mathrm{e}^{\mathrm{e}}, \mathrm{e}^{\mathrm{e}^{\mathrm{e}}}, \mathrm{e}^{\mathrm{e}^{\mathrm{e}^{\mathrm{e}}}} \ldots$ are algebraically independent over L.

Arizona Winter School AWS2008, Tucson

Theorem [Jonathan Bober, Chuangxun Cheng, Brian Dietel, Mathilde Herblot, Jingjing Huang, Holly Krieger, Diego Marques, Jonathan Mason, Martin Mereb and Robert Wilson.] *Schanuel's Conjecture implies that the fields E and L are linearly disjoint over $\overline{\mathbb{Q}}$.*

Definition. Given a field extension F/K and two subextensions $F_1, F_2 \subseteq F$, we say F_1, F_2 are linearly disjoint over K when the following holds : *any set $\{x_1, \dots, x_n\} \subseteq F_1$ of K– linearly independent elements is linearly independent over F_2.*

Reference : arXiv.0804.3550 [math.NT] 2008.
J. Number Theory **129** (2009), no. 6, 1464–1467.

C. Bertolin, P. Philippon, B. Saha, E. Saha

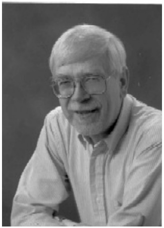
Patrice Philippon, Biswajyoti Saha, Ekata Saha,
An abelian analogue of Schanuel's Conjecture and applications. Ramanujan J., **52**, 2, 381-392 (2020).

Cristiana Bertolin, Patrice Philippon, Biswajyoti Saha, Ekata Saha,
Semi–abelian analogues of Schanuel Conjecture and applications. J. Algebra **596** (2022), pp. 250–288.

Formal analogs

W.D. Brownawell
(was a student of Schanuel)



J. Ax's Theorem (1968) :
Version of Schanuel's Conjecture for power series over $\mathbb{C}$
(and R. Coleman for power series over $\overline{\mathbb{Q}}$)
Work by W.D. Brownawell and K. Kubota on the elliptic analog of Ax's Theorem.

Dale Brownawell and Stephen Schanuel

Strong Schanuel's property

Definition. An n–tuple $(x_1, \dots, x_n)$ of complex numbers satisfies the *strong Schanuel property*, if the $2n$ numbers $x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}$ are algebraically independent over $\mathbb{Q}$.

Theorem. *Almost all n–tuples of complex numbers (for Lebesgue measure of $\mathbb{C}^n$) satisfy the strong Schanuel property.*

The proof uses only the fact that the exponential function $e^z = \exp(z)$ is *transcendental* (over $\mathbb{C}(z)$) : *for $P \in \mathbb{C}(z)[T] \setminus \{0\}$ the function $P(z, e^z)$ is not 0.*

Strong Schanuel property

Let $\psi : \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$ be a decreasing function such that

$$\psi(q) < \mathrm{e}^{-q^4}.$$

Let $x_1, \dots, x_n$ be real numbers. Assume that there exists a sequence $(q_k)_{k \geqslant 0}$ of positive integers such that

$$0 < k^{n-1} \|q_k x_n\| \leqslant \cdots \leqslant k^{i-1} \|q_k x_i\| \leqslant \cdots \leqslant \|q_k x_1\| \leqslant \psi(q_k)$$

for all $k \geqslant 0$. Then the n–tuple $(x_1, \dots, x_n)$ satisfies the strong Schanuel property.

Uncountably many explicit examples

Define a sequence $(q_k)_{k\geqslant 0}$ of positive integers by $q_0 = 1$ and $q_{k+1} = 3^{q_k^4}$ for $k \geqslant 0$. For $\ell \geqslant 1$ and $1 \leqslant i \leqslant n$, let $\epsilon_\ell^{(i)} \in \{-1, +1\}$. For $i = 1, \ldots, n$ and $k \geqslant 1$, set

$$x_i = \sum_{\ell \geqslant 1} \epsilon_\ell^{(i)} \frac{(4(\ell-1))^{n-i}}{q_\ell}$$

and

$$p_k^{(i)} = q_k \sum_{\ell=1}^{k} \epsilon_\ell^{(i)} \frac{(4(\ell-1))^{n-i}}{q_\ell}.$$

Then the n–tuple $(x_1, \ldots, x_n)$ satisfies the strong Schanuel property.

Algebraic independence of values of algebraically independent functions

Theorem. *Let* K *be a finitely generated extension of* $\mathbb{Q}$. *Let* $f_1, \ldots, f_t$ *be meromorphic functions in* $\mathbb{C}$ *which are algebraically independent over* K. *Then for almost all tuples* $(z_1, \ldots, z_n)$ *of complex numbers, the* nt *numbers*

$$f_j(z_i) \qquad (i = 1, \ldots, n, \quad j = 1, \ldots, t)$$

are algebraically independent over K.

Proof

• If F is a nonzero meromorphic function in $\mathbb{C}^n$, then the set $Z(F)$ of zeroes of F in $\mathbb{C}^n$ has Lebesgue measure zero.
• A countable union of sets of Lebesgue measure zero has Lebesgue measure zero.
• The set of polynomials in nt variables with coefficients in K is countable.
For P a nonzero polynomial in nt variables with coefficients in K, define a nonzero meromorphic function F in $\mathbb{C}^n$ by

$$F(z_1, \ldots, z_n) = P\Big((f_j(z_i))_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t}} \Big)$$

and let $Z(F) \subset \mathbb{C}^n$ be the set of zeroes of F. The set of tuples $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that the nt numbers

$$f_j(z_i) \qquad (i = 1, \ldots, n, \quad j = 1, \ldots, t)$$

are algebraically dependent over K is the union of all $Z(F)$ with $P \in K[(X_{ij})_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t}}] \setminus \{0\}$. Hence the result.

Weierstrass functions and Serre functions

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ and $\mathcal{E}$ the elliptic curve $\mathbb{C}/\Omega$. Consider the following Weierstrass functions :

- The canonical product of Weierstrass associated with Ω :

$$\sigma(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

- Weierstrass zeta function $\zeta = \sigma'/\sigma$.
- Weierstrass elliptic function $\wp = -\zeta'$.

Further, for $q \in \mathbb{C} \setminus \Omega$, we introduce Serre function

$$f_q(z) = \frac{\sigma(z+q)}{\sigma(z)\sigma(q)} \mathrm{e}^{-\zeta(q)z}.$$

Weierstrass and Serre functions

K. Weierstrass

1815 – 1897

J-P. Serre

• The periods of the Weierstrass elliptic function $\wp$ are elliptic integrals of the first kind.

• The Weierstrass zeta function ζ has quasi periods $\eta = \zeta(z+\omega) - \zeta(z)$ which are given by elliptic integrals of the second kind.

• Serre functions f_q were introduced in order to study elliptic integrals of the third kind.

Periodicity and quasi-periodicity

For $\omega \in \Omega$,

- $\wp(z+\omega) = \wp(z)$,

- $\zeta(z+\omega) = \zeta(z) + \eta(\omega)$,

- $$\sigma(z+\omega) = \epsilon(\omega)\sigma(z)\exp\left(\eta(\omega)\left(z+\frac{\omega}{2}\right)\right)$$

with $\epsilon(\omega) = 1$ if $\omega/2 \in \Omega$, $\epsilon(\omega) = -1$ if $\omega/2 \notin \Omega$,

- $$f_q(z+\omega) = f_q(z)\exp\left(\omega\zeta(q) - q\eta(\omega)\right).$$

Algebraic independence of elliptic functions

Theorem. *Let* $t_1, \ldots, t_r$ *be complex numbers linearly independent over* $\mathbb{Q}$*. Let* $q_1, \ldots, q_s$ *be complex numbers such that* $\omega_1, \omega_2, q_1, \ldots, q_s$ *are linearly independent over* $\mathbb{Q}$*. Then the* $s + r + 4$ *functions*

$$z, \wp(z), \zeta(z), \sigma(z), \mathrm{e}^{t_1 z}, \ldots, \mathrm{e}^{t_r z}, f_{q_1}(z), \ldots, f_{q_s}(z)$$

are algebraically independent.

Strong elliptic Schanuel property

Theorem. *Let* K *be a finitely generated extension of* $\mathbb{Q}$. *Let* $t_1, \ldots, t_r$ *be complex numbers linearly independent over* $\mathbb{Q}$.
Let $q_1, \ldots, q_s$ *be complex numbers such that* $\omega_1, \omega_2, q_1, \ldots, q_s$ *are linearly independent over* $\mathbb{Q}$.
Then for almost all n*–tuples* $(z_1, \ldots, z_n)$ *of complex numbers, the* $n(s + r + 4)$ *numbers*

$$z_i, \wp(z_i), \zeta(z_i), \sigma(z_i), \mathrm{e}^{t_1 z_i}, \ldots, \mathrm{e}^{t_r z_i}, f_{q_1}(z_i), \ldots, f_{q_s}(z_i)$$

$(i = 1, \ldots, n)$ *are algebraically independent over* $K(g_2, g_3)$.

A conjecture for all tuples ?

$$z_i, \mathrm{e}^{z_i}, \wp(z_i), \zeta(z_i), \sigma(z_i), \mathrm{e}^{t_1 z_i}, \dots, \mathrm{e}^{t_r z_i}, f_{q_1}(z_i), \dots, f_{q_s}(z_i)$$

• We have n free parameters : $z_1, \dots, z_n$. We may select for them algebraic numbers, like in the Lindemann–Weierstrass Theorem. Hence the conclusion of the conjecture one should predict that the transcendence degree is $\geqslant n(s+r+3)$ only.

• Need to replace K by $\mathbb{Q}(g_2, g_3)$.

• Need to assume that the z_i are linearly independent over the field of endomorphisms of the elliptic curve $\mathcal{E}$.

• Need to involve $\wp(q_i)$.

• Better to use a geometric point of view.

Semi-elliptic Conjecture of Schanuel

The next statement *has a geometric origin*.
This is a joint work in progress with Cristiana Bertolin.

Conjecture. *Let* Ω *be a lattice in* $\mathbb{C}$ *such that the elliptic curve* $(\mathcal{E}) = \mathbb{C}/\Omega$ *has no complex multiplication.*

- $q, p_1, \dots, p_n$ *be complex numbers in* $\mathbb{C} \smallsetminus \Omega$ *linearly independent over* $\mathbb{Q}$,
- $q \in \mathbb{C} \setminus \Omega \otimes_{\mathbb{Z}} \mathbb{Q}$,
- $t_1, \dots, t_s$ *be complex numbers* $\mathbb{Q}$*–linearly independent.*

Then at least $3n+s-1$ *of the* $4n+2s+3$ *numbers*

$$g_2, g_3, p_i, \wp(p_i), \zeta(p_i), \wp(q), f_q(p_i), t_k, \mathrm{e}^{t_k} \qquad (1 \leqslant i \leqslant n, \;\; 1 \leqslant k \leqslant s)$$

are algebraically independent over $\mathbb{Q}$.

Algebraic groups

Consider the algebraic group G which is an extension of the elliptic curve $\mathcal{E}$ by the multiplicative group $\mathbb{G}_m$ parametrized by the point $Q = \exp_{\mathcal{E}^*}(q)$ of the dual elliptic curve $\mathcal{E}^*$, that we identify with $\mathcal{E}$.

The semi-elliptic exponential function of G (composed with a projective embedding) is

$$\begin{aligned}\exp_G : \mathrm{Lie} G_{\mathbb{C}} &\longrightarrow G_{\mathbb{C}}(\mathbb{C}) \subset \mathbb{P}^4(\mathbb{C})\\ (z,t) &\longmapsto \sigma(z)^3 \Big[\wp(z) : \wp'(z) : 1 : \mathrm{e}^t f_q(z) :\\ &\qquad \mathrm{e}^t f_q(z) \Big(\wp(z) + \frac{\wp'(z) - \wp'(q)}{\wp(z) - \wp(q)} \Big) \Big].\end{aligned}$$

Reference : David Masser, Chapter 20, Exercise 20.104
Auxiliary polynomials in number theory.
Cambridge Tracts in Mathematics **207**. Cambridge University Press. xviii, 348 p. (2016).

Conjectures by A. Grothendieck and Y. André



Generalized Conjecture on Periods by Grothendieck : Dimension of the Mumford–Tate group of a smooth projective variety.

Generalization by Y. André to motives.

Case of 1–motives : Elliptico-Toric Conjecture of C. Bertolin.

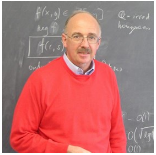
Consequence of Grothendieck's Conjecture

Payman Eskandari
and Kumar Murty

The numbers π, $\log 2$, $\zeta(3)$
are algebraically independent.

Kumar Murty's lecture at the International Conference on Number Theory and Related Topics ICNTRT-2024.
The Institute of Mathematical Sciences (IMSc), Mathematics and Indian Institute of Technology IIT Madras on December 19, 2024.

Appendix : Rational values of the Weierstrass zeta function.



David Masser

Senthil Kumar

A consequence of the result by Senthil Kumar is that *if g_2 and g_3 are algebraic, one at least of $\zeta(3)$, $\zeta(5)$ is transcendental.*

This may be found amusing by those who study another zeta function.

Theorem [D. Masser] *There exist invariants g_2 and g_3 such that $\zeta(3)$, $\zeta(5)$ are rational.*

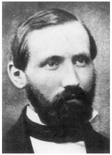
Senthil Kumar. On the values of Weierstrass zeta and sigma functions. Acta Arithmetica **208**.3 (2023), 285 – 294.

Weierstrass vs Riemann

‘

K. Weierstrass

1815 – 1897



B. Riemann

1826 – 1866

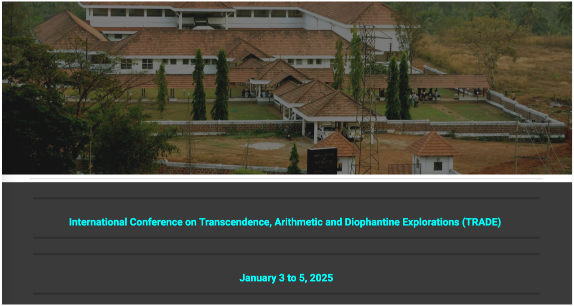
We denote by ζ_R the Riemann zeta function.

Proposition *The four functions* $z, \wp(z), \zeta(z), \zeta_R(z)$ *are algebraically independent.*

Corollary. *For almost all* n*–tuples* $(z_1, \ldots, z_n)$ *of complex numbers, the* $4n$ *numbers*

$$z_1, \ldots, z_n, \wp(z_1), \ldots, \wp(z_n), \zeta(z_1), \ldots, \zeta(z_n), \zeta_R(z_1), \ldots, \zeta_R(z_n)$$

are algebraically independent over $\mathbb{Q}(g_2, g_3)$.



January 3, 2025

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