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Workshop on Number Theory and Related Topics.
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Families of Diophantine equations.

by

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Abstract. This is a report on the recent work by Claude Levesque and the author on families of Diophantine equations. This joint work started in 2010 in Rio, and this is still work in progress.

The lecture in Lahore on March 11, 2013 was mainly devoted to a survey of results on Diophantine equations, with the last part dealing with some recent results. Here we describe the content of the recent joint papers listed in the bibliography.

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Fermat–Pell–Mahler

In [1], which is a joint paper involving also Yann Bugeaud, we obtain an upper bound for the number of solutions of the simultaneous Fermat–Pell–Mahler equations

$$\begin{cases} a_1X^2+b_1XZ+c_1Z^2 &= \pm p_1^{m_1}\cdots p_s^{m_s},\\ a_2Y^2+b_2YZ+c_2Z^2 &= \pm p_1^{n_1}\cdots p_s^{n_s}.\end{cases}$$

Consequences of Schmidt's Subspace Theorem

The results of the papers [2, 3, 4] are based on Schmidt's Subspace Theorem, and are very general ones. We obtain families of Thue–Mahler equations having only finitely many solutions and we give upper bounds for the number of solutions, but the method is not effective: we are not able to give upper bounds for the solutions themselves, hence we cannot solve the equations.

A basic introduction to the subject is given in [2]. The main new results are proved in [3]. Consequences on Diophantine approximation are given in [4].

Here is one of the results in [3]. *Let $S=\{p_1,\dots,p_t\}$ be a finite set of prime numbers, $f\in \boldsymbol{Z}[X]$ an irreducible polynomial of degree $d\ge 3$, α a root of f, K the number field $\boldsymbol{Q}(\alpha)$, $\sigma_1,\dots,\sigma_d$ the embeddings of K into $\boldsymbol{C}$. For each S–unit $\varepsilon\in O_S^\times$, define $F_\varepsilon(X,Y)\in\boldsymbol{Z}[X,Y]$ by*

$$F_\varepsilon(X,Y)=a_0\bigl(X-\sigma_1(\alpha\varepsilon)Y\bigr)\bigl(X-\sigma_2(\alpha\varepsilon)Y\bigr)\cdots\bigl(X-\sigma_d(\alpha\varepsilon)Y\bigr).$$

Let $m\in\boldsymbol{Z}\setminus\{0\}$. Then the set of $(x,y,\varepsilon,z_1,\dots,z_t)$ in $\boldsymbol{Z}^2\times O_S^\times\times\boldsymbol{N}^t$ satisfying

$$F_\varepsilon(x,y)=mp_1^{z_1}\cdots p_t^{z_t},$$

with $xy \neq 0$, $\gcd(xy, p_1 \cdots p_t) = 1$ *and* $[\boldsymbol{Q}(\alpha\varepsilon) : \boldsymbol{Q}] \geq 3$, *is finite.*

EFFECTIVE RESULTS

The more recent papers [5, 6, 7] provide effective results for families of Thue equations. The goal is to prove the following conjecture.

CONJECTURE. *Let* α *be an algebraic number of degree* $d \geq 3$ *over* $\boldsymbol{Q}$. *We denote by* K *the algebraic number field* $\boldsymbol{Q}(\alpha)$, *by* $f \in \boldsymbol{Z}[X]$ *the irreducible polynomial of* α *over* $\boldsymbol{Z}$, *by* $\boldsymbol{Z}_K^\times$ *the group of units of* K *and by* r *the rank of the abelian group* $\boldsymbol{Z}_K^\times$. *For any unit* $\varepsilon \in \boldsymbol{Z}_K^\times$ *such that the degree* $\delta = [\boldsymbol{Q}(\alpha\varepsilon) : \boldsymbol{Q}]$ *is* ≥ 3, *we denote by* $f_\varepsilon(X) \in \boldsymbol{Z}[X]$ *the irreducible polynomial of* $\alpha\varepsilon$ *over* $\boldsymbol{Z}$ *(uniquely defined upon requiring that the leading coefficient be* > 0*) and by* F_ε *the irreducible binary form defined by* $F_\varepsilon(X, Y) = Y^\delta f_\varepsilon(X/Y) \in \boldsymbol{Z}[X, Y]$. *Then there exists an effectively computable constant* $\kappa > 0$, *depending only upon* α, *such that , for any* $m \geq 2$, *each solution* $(x, y, \varepsilon) \in \boldsymbol{Z}^2 \times \boldsymbol{Z}_K^\times$ *of the inequation* $|F_\varepsilon(x, y)| \leq m$ *with* $xy \neq 0$ *and* $[\boldsymbol{Q}(\alpha\varepsilon) : \boldsymbol{Q}] \geq 3$ *verifies*

$$\max\{|x|,\ |y|,\ e^{\mathrm{h}(\alpha\varepsilon)}\} \leq m^\kappa.$$

In [5], we prove the conjecture when the field K is a non totally real cubic field. In [6], we prove the conjecture in the more general case where the field K has at most one real embedding. In [7], we prove the conjecture when one requests the unknown ε to belong to a subset of the group of units of K, and we show that this subset contains a positive proportion of all units as soon as the degree of K is at least 4.

The proofs of the effective results rely on lower bounds for linear forms in logarithms.

References

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