Abstract

After the proof by R. Apery of the irrationality of \( \zeta(3) \) in 1976, a number of papers have been devoted to the study of Diophantine properties of values of the Riemann zeta function.

We review more recent results including contributions by Stischler, Zudilin,\( \ldots \)

Michel Waldschmidt

Michel Waldschmidt

Recent Diophantine results on zeta values:

Zeta

\[
\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \geq 2} \left(1 - \frac{1}{p^s}\right)^{-1} = (s)^{\zeta(s)}
\]
Special values of Riemann zeta function

\[ \zeta \left( \frac{1}{2} \right) = \pi \frac{i}{\sqrt{2}} \]

Conjecture. There is no relation at all between the numbers

\( \zeta \left( \frac{1}{2} \right), \zeta \left( \frac{3}{2} \right), \zeta \left( 2 \right) \).

Diophantine question: For \( n \geq 1 \), is the number

\( \zeta \left( \frac{2}{1} \right) \) rational?

Beroulli numbers:

\[ \frac{B_{2k}}{2} = \frac{(2k)!}{2^{2k+1}k!} \pi + 1 \]

Transcendence of even zeta values:

For \( k \geq 1 \), the numbers

\( \zeta \left( \frac{2k}{1} \right) \) are conjectured to be transcendental.
Values of $\zeta$ at odd positive integers

The number $\zeta_{ixl} = \sum_{n \geq 1} \frac{u_n}{n^3} = v$. . .

is irrational.

Infinitely many $\zeta_{ivk\nu}$ are irrational.

Let $\epsilon > 0$. For any sufficiently large odd integer $a$ and the dimension of the $\mathbb{Q}$–vector space spanned by the numbers $u_{ixl}$, $\zeta_{ixl}$, $\zeta_{i5l}$, . . .

is at least $u - \epsilon n \log v$. . .

Infinitely many odd zeta are irrational.

Let $a > 0$. For any sufficiently large odd integer $a$ and the dimension of the $\mathbb{Q}$–vector space spanned by the numbers $u_{ixl}$, $\zeta_{ixl}$, $\zeta_{i5l}$, . . .

is at least $u - \epsilon n \log v$. . .

are $\mathbb{Q}$–linearly independent.

There exists an odd integer $a$ such that

is irrational.

At least one of the four numbers $\zeta_{i5l}$, $\zeta_{i7l}$, $\zeta_{i9l}$, $\zeta_{iul}$ is irrational.

There exists an odd integer $j$ in the range $[5, 69]$ such that the three numbers $u_{ixl}$, $\zeta_{ixj}$, $\zeta_{iul}$

are $\mathbb{Q}$–linearly independent.

Infinitely many $\zeta_{ixl}$ are irrational.

Infinitely many $\zeta_{ixl}$ are irrational.

References to works on zeta values by

Rivoal (2000), Zudilin

Rivoal and Zudilin

Rivoal and Zudilin

Rivoal and Zudilin


Valdivy Zudilin (2000)
Irrationality of zeta values

\[ \text{ST sischler} \]

Ir\`{e}nalit\'{e} de valeurs de z\'{e}ta d'apr\`{e}s Yperyd Rivoal

\[ \text{S\'{e}minaire Bourbaki} \]

R\'{e}sum\'{e} : \text{Christian Krattenthaler et Tanguy Rivoal}

Hyperg\'eom\'etrie et fonction z\'{e}ta de Riemann

\[ \text{Christian Krattenthaler and Tanguy Rivoal} \]

Irrationality measures at the state of the art

\[ \varrho \in \mathbb{R}, \quad |\varrho - \frac{p}{q}| \geq \frac{u}{q^{\mu}} + \epsilon \quad \iff \varrho \text{ is not a Liouville number} \]

\[ \varrho \] is not a Liouville number if

\[ \lim_{q \to \infty} \frac{1}{q} \cdot \frac{1}{|\varrho - \frac{p}{q}|} \leq \infty \]

\[ \mu \in \mathbb{R} \]

\[ \text{Irrationality of zeta values} \]
Irrationality measure for \( \zeta(2) \) and \( \zeta(3) \).

Then \( I \delta^n \) are linearly independent over \( \mathbb{Q} \).

\[ I(\delta^n) = \left| \frac{p}{q} \right|^{\frac{1}{n}} \quad (n \in \mathbb{Z}, q > 0) \]

Let \( \vartheta_1, \ldots, \vartheta_m \) be complex numbers.

Criterion of Vl. V. Nesterenko (qualitative)

1. Example: \( m = 1 \) – irrationality criterion.

Then \( I \delta^n \) are linearly independent over \( \mathbb{Q} \).

Georges Rhin and Carlo Viola


1. \[ \delta^n > 3.19 \]

2. \[ \delta^n > 3.18 \]

3. \[ \delta^n > 3.09 \]

4. \[ \delta^n > 2.96 \]

5. \[ \delta^n > 2.78 \]

6. \[ \delta^n > 2.69 \]

7. \[ \delta^n > 2.53 \]

8. \[ \delta^n > 2.37 \]

9. \[ \delta^n > 2.24 \]

10. \[ \delta^n > 2.1 \]

1. \[ \delta^n > 2.01 \]

2. \[ \delta^n > 1.96 \]

3. \[ \delta^n > 1.93 \]

4. \[ \delta^n > 1.86 \]

5. \[ \delta^n > 1.8 \]

6. \[ \delta^n > 1.75 \]

7. \[ \delta^n > 1.7 \]

8. \[ \delta^n > 1.56 \]

9. \[ \delta^n > 1.51 \]

10. \[ \delta^n > 1.20 \]

1. \[ \delta^n > 1.18 \]

2. \[ \delta^n > 1.16 \]

3. \[ \delta^n > 1.14 \]

4. \[ \delta^n > 1.12 \]

5. \[ \delta^n > 1.08 \]

6. \[ \delta^n > 1.06 \]

7. \[ \delta^n > 1.02 \]

8. \[ \delta^n > 1.00 \]

9. \[ \delta^n > 0.97 \]

10. \[ \delta^n > 0.92 \]
Recent developments

Stéphane Fischler and Wadim Zudilin

A refinement of Nesterenko’s linear independence criterion with applications to zeta values

To appear in Math. Annalen

Preprint: IMP 2009.35

Fischler and Zudilin, 2009

Multizeta values

There exist positive odd integers \(i \leq 9x\) and \(j \leq u5u\) such that the numbers
\[
\zeta^{i} = \sum_{n>\cdots>n_k>1} n_{s_1} \cdots n_{s_k}
\]
are linearly independent over \(\mathbb{Q}\).

Multizeta values

For \(s_1, \ldots, s_k\) positive integers with \(s_k \geq 2\),

There exist positive odd integers \(i \leq 9x\) and \(j \geq 139\) such that the numbers
\[
\zeta^{i} = \sum_{n>\cdots>n_k>1} n_{s_1} \cdots n_{s_k}
\]
are linearly independent over \(\mathbb{Q}\).

Simplified proof of Nesterenko’s Theorem

Raffaele Marcolli, Pierre Bel, Francesco Amoroso

Refinements: Raffaele Marcolli, Pierre Bel

Refinements: Raffaele Marcolli, Pierre Bel
The Multiple Zeta Value Data Mine

The calculator gives numerical values of \( \zeta(n_1, \ldots, n_k) \) with up to 100 decimal places accuracy. The calculator also has a function to look for relations of linear dependence.

To make it easy \((\leq 100)\) looks for a vanishing linear combination of \( a_1, a_2, \ldots, a_k \) with integer coefficients. This makes it easy to discover new identities.

This project was a collaboration of

- D. Broadhurst, J.A.M. Vermaseren
- M. Hoffman
- W. Zudilin
- R. Baik
- F. Beukers

The Multiple Zeta Value Data Mine

\[
\Gamma(i) = \int_0^\infty e^{-t} t^i \, dt = e^{-\gamma} \sum_{n=1}^{\infty} \frac{1}{n^i},
\]

\[
\Gamma(n) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{n^i}\right) \Gamma(i).
\]

\[
\psi(x) = \int_1^x \frac{1}{t} \Gamma(t) \, dt.
\]

\[
\psi(x) = (\log x)^2 + \gamma \log x + \frac{1}{2} \gamma^2 + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n^2} \right) \frac{1}{x^n}.
\]

\[
\psi(x) = \frac{1}{x} \Gamma'(1/2) \Gamma(1/2 - x).
\]

\[
\psi(x) = \frac{1}{x} + \gamma + \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{1}{n} - \frac{1}{n^2} \right) \frac{1}{x^n}.
\]

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\psi(x) = \frac{1}{x} + \gamma + \frac{1}{2} \gamma^2 + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n^2} \right) \frac{1}{x^n}.
\]
Complex multiplication \( t \) of \( \mathbb{Q} (i) \)

\[ \wp'_{2} = y \wp_{3} - y \wp_{2}, \]

\[ g_{2} = y, \quad g_{3} = t, \]

\[ \omega_{1} = \int_{1}^{0} dx \sqrt{x - x^3} = u \frac{v}{y}, \quad u/v = \frac{\Gamma (i) u}{y^2}, \]

\[ \eta_{1} = \frac{\pi \sqrt{x}}{\omega_{1}} = \frac{v}{y^{3/2}}, \quad \eta_{2} = -i \eta_{1}. \]

Transcendence of special values of Weierstraß functions

Theory (J-B. Boas: slopes inequalities).


Lower bounds for linear combinations of elliptic logarithms.

Diophantine approximation

are transcendental.

\[ \frac{\wp}{i} (t/1), \]

and

\[ \frac{\wp}{i} (t/1), \]

The Schneider (1934,)

Functions

Transcendence of special values of Weierstraß.

Complex multiplication \( (i) \) of \( O \).
Sinnou David and Noriko Hirata

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yinear forms in elliptic logarithms

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–belian varieties

For a and b in Q with a d and b and a n b not in Z d the number

\[ b
\frac{\log b}{\Gamma(i/a)} \Gamma(i/b) \Gamma(i/a + b) \]

is transcendental.

The proof involves Abelain integrals of higher genus related with the wacobian of a sermat curve.

Theorem

Two at least of the numbers

\[ g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2 \]

are algebraically independent.

Corollary:

\[ \pi \]

and

\[ \frac{\Gamma(i/a)}{\Gamma(i/b)} \]

are algebraically independent.

Chudnovsky’s algebraic independence theorem

G.V. Chudnovsky (1978)

Corollary:

\[ \pi \]

and

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Chudnovsky's method

\[ \frac{\mu_v}{\xi_v} = (b)H' \quad \mu = (b)\lambda \quad \frac{\mu}{\xi} = (b)\lambda \]

\[ \cdots = \frac{\mu_v}{\xi_v} = 1 \quad \frac{\mu_v}{\xi_v} = b \quad \lambda = 1 \]

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Ramanujan Functions

\[ P_i q = \left(1 - \frac{1}{q^2}ight) \sum_{n=1}^\infty \frac{n}{q^n} \]

\[ Q_i q = \left(1 - \frac{1}{q^2}ight) \sum_{n=1}^\infty \frac{n^3}{q^n} \]

\[ R_i q = \left(1 - \frac{1}{q^2}ight) \sum_{n=1}^\infty \frac{n^5}{q^n} \]

Special values

\[ \tau = i, \quad q = e^{-2\pi i}, \quad \omega_1 = \Gamma(1/4)^2\sqrt{8\pi} = 2.6220575542 \ldots \]

\[ P(q) = 3\pi, \quad Q(q) = 3(\omega_1\pi)^4, \quad R(q) = 0 \]

\[ \tau = \eta, \quad q = e^{-\pi\sqrt{3}}, \quad \omega_1 = \Gamma(1/3)^3/2\sqrt{3\pi} = 2.428650648 \ldots \]

\[ P(q) = 2\sqrt{3}\pi, \quad Q(q) = 0, \quad R(q) = 27(\omega_1\pi)^6 \]

Eisenstein Series

\[ \sum_{n=-\infty}^{\infty} \frac{3H(1-n)}{\eta^3} + 1 = (z)^3 \Phi \]

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Eisenstein Series

\[ \sum_{n=-\infty}^{\infty} \frac{3H(1-n)}{\eta^3} + 1 = (z)^3 \Phi \]
For any \( q \in \mathbb{C} \) with \( t < |q| < u \), there are at least three of the four numbers \( q, P, Q, R \) are algebraically independent.

**Tools:**

The functions \( P, Q, R, R \) are algebraically independent.

The number \( \frac{d}{dq} \) satisfies a system of differential equations:

\[
\frac{d}{dq} - d = \frac{d}{dq} = \frac{d}{dq} = \frac{d}{dq}
\]

Consequences of Nesterenko's Theorem

The three numbers \( \pi, e^\pi, \Gamma(iu/yl) \) are algebraically independent.

The three numbers \( \pi, e^{\pi \sqrt{3}}, \Gamma(iu/yl) \) are algebraically independent.

**Consequences of Nesterenko's Theorem**

Special values of Weierstrass sigma functions

The number \( \sigma_{\mathbb{C}}[i] \) is transcendental.

The number \( \frac{1}{\zeta} \) is transcendental.

**Fibonacci zeta values**

\[ F_0 = t, F_1 = u, F_{n+1} = F_n + F_n \]

**Tools:**

The functions \( \sigma, \mu, \frac{d}{dq} \) are algebraically independent.

For any \( \gamma \in \mathbb{C} \) with \( 0 < \gamma < 1 \),

**Theorem (Nesterenko, 1996)**
Fibonacci zeta values

\[ \zeta_F(s) = \sum_{n \geq 1} \frac{u_n}{n^s} = \zeta_F(ivl, vl), \]

for \( s \geq t \) and \( \nu, \gamma \in \mathbb{Z} \). For any positive integer \( n \),

\[ \frac{(nu)_j n^{(j/2)}(nu)_j}{v} = (n - 1) \Gamma(n), \]

\[ (nu)_j n^{(j/2)}(nu)_j = (1 + v) \Gamma(n). \]

For \( s \geq t \) and \( \nu, \gamma \in \mathbb{Z} \), let \( \nu, \gamma \) be distinct positive integers. The numbers

\[ (a^n \nu)_j \mathcal{O} \supseteq (1 + \nu) \Gamma(n), \]

are algebraically dependent if and only if the three integers \( s, \nu, \gamma \) are odd. For \( s \geq 2, s, \nu \in \mathbb{Z} \), with some \( \mathcal{O} \in \mathbb{Z} \),

\[ (a^n \nu)_j \mathcal{O} \supseteq (1 + s + \nu)^d, \]

for \( s \geq 0 \), \( \nu \in \mathbb{Z} \), and \( \mathcal{O} \in \mathbb{Z} \),

\[ \mathcal{O} \supseteq (a^n \nu)_j \supseteq (1 + s + \nu)^d. \]

Small Gamma Products with Simple Values

The two previous examples are due respectively to

Albert Meyer, Small Gamma products with Simple Values

and to

N. J. A. Sloane, Small Gamma products with Simple Values

The numbers

\[ \zeta_F(ivl, vl), \]

for \( s \geq t \) and \( \nu, \gamma \in \mathbb{Z} \), are algebraically dependent if and only if the three integers \( s, \nu, \gamma \) are odd. For \( s \geq 2, s, \nu \in \mathbb{Z} \), with some \( \mathcal{O} \in \mathbb{Z} \),

\[ (a^n \nu)_j \mathcal{O} \supseteq (1 + s + \nu)^d, \]

for \( s \geq 0 \), \( \nu \in \mathbb{Z} \), and \( \mathcal{O} \in \mathbb{Z} \),

\[ \mathcal{O} \supseteq (a^n \nu)_j \supseteq (1 + s + \nu)^d. \]
Lang's Conjecture

Conjecture

algebraic dependence relation

among the numbers

\[ \pi, \Gamma_i \]

with \( a \in \mathbb{Q} \) lies in the ideal generated by the standard relations:

\[ f_i \]

Universal odd distribution

Consequence of the Rohrlich–Lang Conjecture

An example: The Rohrlich–Lang Conjecture implies that for any \( q > 1 \), the transcendence degree of the field generated by

\[ \pi, \Gamma_i/a/q \]

is at most one.

Variant of the Rohrlich–Lang Conjecture

In consequence is that for any \( q > 1 \) the numbers

\[ \log \Gamma_i/a/q \]

are linearly independent over the field of algebraic numbers. For any \( 1 < b \), the numbers

\[ f_i(\chi), \]

primitive odd character \( \chi \) modulo \( b \), for which

A consequence is that for any \( 1 < b \) there is at most one

primitive odd character \( \chi \) modulo \( b \)

Consequence of the Rohrlich–Lang Conjecture

\[ (\chi^*1)_b \]

Transcendental values of class group \( L \)-functions.

Ram and Kumar Murty (2009)

Transcendental values of class group \( L \)-functions.

Consequence of the Rohrlich–Lang Conjecture

\[ (b/n)_{\chi} + 1, \]

\[ (b/n)_{\chi} \]

numbers

Any \( 1 < b \) the transcendence degree of the field generated by

Consequence of the Rohrlich–Lang Conjecture

\[ (\chi^*1)_b \]

Transcendental values of class group \( L \)-functions.
The number \( \sum_{n=2}^{\infty} \frac{\zeta(n)}{n^{p/q}} \) is transcendental.

It is a consequence of Nesterenko’s theorem.

\[ a(z + u) \sum_{n=0}^{\infty} = (z^{'})^{c} \]

Here the number \( \sum_{n=2}^{\infty} \frac{u^{n}}{n} = \frac{1}{1 - e^{u}} \).

The number is transcendental.

\[ \sum_{n=2}^{\infty} \frac{2.076}{n} \]

is transcendental over \( \mathbb{Q} \) for \( s \neq 1 \).

The transcendence of this number for even integers \( s \geq 2 \) would follow as a consequence of Schanuel’s conjecture.
The conjecture of Chowla and Milnor

For $k$ and $q$ integers $>u$ and $\phi_i/q_l$ numbers $\zeta_i/a/q_l$, $u \leq a \leq q$, $i \neq a$, $q_l = u$
are linearly independent over $Q$.

Hence the strong Chowla-Milnor conjecture holds for $\gamma = 2$
for at least one of the two values $a = b$ and $a$.

Strong Chowla-Milnor conjecture holds for this value of $a$ and only if the
for $u < 1$ odd. The number $\zeta_i$ is irrational if and only if the
are linearly independent over $Q$

$1 = (b'/a) \leq n \leq 1 = (b/a, n)$ for $\gamma = 2$

The strong Chowla-Milnor conjecture for $b = \gamma$ implies the

The digamma function

$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \Gamma'(x)/\Gamma(x) = 1 \gamma - 1 \gamma \sum_{n=1}^{\infty} (z^n/n) = (z)^{\gamma}

Thus $\int_{a}^{z} = (z)\gamma$

For $u$, $v$, $\gamma$ and $\alpha_i$
are linearly independent over $Q$:

For $k$ and $q$ integers $< 1$, the numbers $(b)^p < 1$.

Strong Chowla-Milnor conjecture (2009): For $u > 1$, $n \geq 1$.

The conjecture of Chowla and Milnor

Sanoli Gun, Ram Murty and Purusottam Rath

Linear independence of polylogarithms

(1931 - 1995) John William Milnor
(1975 - 1995) Sarvadaman Chowla
Special values of the digamma function

\[ \psi(iu) = -\gamma, \]

\[ \psi(u \nu) = -v \log ivl - \gamma, \]

\[ \psi(v \kappa - u \nu) = -v \log ivl - \gamma, \]

\[ \psi(u \nu) = -\pi v - x \log ivl - \gamma, \]

\[ \psi(x \nu) = \pi v - x \log ivl - \gamma. \]

Euler-Mascheroni constant

\[ \psi(iu) \]

Hence

\[ \underbrace{\gamma - (b \nu) \log \nu - \frac{\gamma}{\nu}}_{= \psi(b \nu)} = \left( \frac{1}{\nu} \right) \phi \]

\[ \underbrace{\gamma - (b \nu) \log \nu - \frac{\gamma}{\nu}}_{= \psi(b \nu)} = \left( \frac{1}{\nu} \right) \phi \]

\[ \frac{\pi}{1} \sum_{1 \leq n \leq \nu} \frac{1}{n} \log n - \gamma = \left( \frac{C}{1} - \gamma \right) \phi \]

\[ \gamma - (b \nu) \log b - \frac{\gamma}{b} = \left( \frac{C}{b} \right) \phi, \quad \gamma = (1) \phi \]
\[
\left(\frac{p}{z} + 1\right)^{\sum_{k=2}^{\infty} u_k z^k} = (z)_{\infty}^z
\]

Euler's constant \( \gamma \)

\[
\gamma = \lim\limits_{s \to 1^+} \sum_{n=1}^{\infty} \left( u_n^s - u_s^n \right)
\]

\[
\gamma = \int_{\infty}^{1} u v t v t n ul F(u, v, v x, t n) dt.
\]

Euler's constant

Open Problems

- Is the Euler constant \( \gamma \) irrational?
- Is \( \gamma \) transcendental?
- Is the Euler constant rational?

Thakur Gamma function

\[
\prod_{a \in \mathbb{A}} \frac{1}{\mathbb{A} + a} = \frac{1}{\mathbb{A}} \sum_{n=1}^{\infty} \left( \mathbb{A} - n \right)
\]

Carlini zeta values: For \( s \in \mathbb{Z}^+ \)

- \((s+1)(s+2)\mathbb{A} = \infty \)
- \((s+2)\mathbb{A} = \infty \)
- Prime polynomials in \( \mathbb{A} \)
- Monic polynomials in \( \mathbb{A} \)
- \([n]_s \mathbb{A} = \mathbb{A} \)

Leonard Carlitz (1907 - 1999)

\[
\psi \left( \frac{\gamma + 1}{2}, \frac{1}{2}, \frac{1}{z} \right) \mathbb{A} I \sum_{n=1}^{\infty} = \gamma
\]

\[
\psi \left( \frac{\gamma + 1}{2}, \frac{1}{2}, \frac{1}{z} \right) \mathbb{A} I \sum_{n=1}^{\infty} = \gamma
\]

Carlitz zeta values

Jonathan Sondow

http://home.earthlink.net/~jsondow/
Carlitz – Bernoulli numbers

\[ \forall m \in \mathbb{N}, (m)^{\nu} \in \mathbb{Q}. \]

For \( m \) a multiple of \( h - 1 \),

\[ \prod_{n=1}^{\infty} \left( \frac{1 - 11_{m}\cdot 11_{n}}{1 - 11_{n}} - 1 \right) = \pi \]

Define

Carlitz zeta values at even \( \mathcal{A} \)-integers

transcendental over \( \mathfrak{A} \).

For \( m \) a positive integer not a multiple of \( h - 1 \), \( (m)^{\nu} \in \mathbb{Q} \).

For \( m \) a positive integer, \( (m)^{\nu} \) is transcendental over \( \mathfrak{A} \).

Greg Anderson, Dimesh Thakur, Jing Yu

Bourbaki Seminar

aspects de l’indépendance algébrique en caractéristique non nulle
Chieh-Yu Chang, Matthew A. Papanikolas, Jing Yu

Title: Geometric Gamma values and zeta values in positive characteristic

Abstract: We consider the values at proper fractions of the geometric gamma function and the values at positive integers of the Carlitz zeta values. We prove that, when considered together, all of the algebraic relations among these special values arise from the standard functional equations of the gamma function and from the Carlitz zeta relations and the Frobenius $p$th power relations of the zeta function.

Chieh-Yu Chang, Matthew A. Papanikolas, Jing Yu

Title: Algebraic independence of arithmetic gamma values

Abstract: We consider the values at proper fractions of the arithmetic gamma function and the values at positive integers of the Carlitz zeta function and provide complete algebraic independence results for them.

Chieh-Yu Chang, Matthew A. Papanikolas, Jing Yu

Title: Periods of third kind for rank 2 Drinfeld modules and algebraic independence of logarithms

Abstract: In analogy with the periods of abelian integrals of differentials of third kind for an elliptic curve defined over a number field, we introduce a notion of periods of third kind for a rank 2 Drinfeld $\mathbb{F}_q[t]$-module $\rho$. When $\rho$ has complex multiplication over a separable extension, we prove the algebraic independence of these special values that arise from the standard functional equations of the gamma function and from the Carlitz zeta relations. Together with the main result in [CP08], we completely determine all the algebraic relations among the periods of first, second and third kind for rank 2 Drinfeld $\mathbb{F}_q[t]$-modules in odd characteristic.