Abstract

According to Nature News, 10 September 2012, quoting Dorian Goldfeld, the abc Conjecture is "the most important unsolved problem in Diophantine analysis". It is a kind of grand unified theory of Diophantine curves: "The remarkable thing about the abc Conjecture is that it provides a way of reformulating an infinite number of Diophantine problems," says Goldfeld, "and, if it is true, of solving them." Proposed independently in the mid-80s by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6), the abc Conjecture describes a kind of balance or tension between addition and multiplication, formalizing the observation that when two numbers $a$ and $b$ are divisible by large powers of small primes, $a + b$ tends to be divisible by small powers of large primes. The abc Conjecture implies – in a few lines – the proofs of many difficult theorems and outstanding conjectures in Diophantine equations—including Fermat’s Last Theorem.

Abstract (continued)

This talk will be at an elementary level, giving a collection of consequences of the abc Conjecture. It will not include an introduction to the Inter-universal Teichmüller Theory of Shinichi Mochizuki.

As simple as abc

http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html
American Broadcasting Company

American Broadcasting Company


Annapurna Base Camp, October 22, 2014

Mt. Annapurna (8091m) is the 10th highest mountain in the
world and the journey to its base camp is one of the most
popular treks on earth.
annapurna-base-camp-trek.htm

The radical of a positive integer

According to the fundamental theorem of arithmetic, any
integer $n \geq 2$ can be written as a product of prime numbers:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}.$$  

The **radical** (also called **kernel** $\text{Rad}(n)$ of $n$ is the product of
the distinct primes dividing $n$:

$$\text{Rad}(n) = p_1 p_2 \cdots p_t.$$  

$$\text{Rad}(n) \leq n.$$

**Examples**:

$$\text{Rad}(60 \ 500) = \text{Rad}(2^2 \cdot 5^3 \cdot 11^2) = 2 \cdot 5 \cdot 11 = 110,$$

$$\text{Rad}(82 \ 852 \ 996 \ 681 \ 926) = 2 \cdot 3 \cdot 23 \cdot 109 = 15 \ 042.$$  

abc–triples

An abc–triple is a triple of three positive integers $a$, $b$, $c$ which
are coprime, $a < b$ and that $a + b = c$.

**Examples**:

$$1 + 2 = 3, \quad 1 + 8 = 9,$$

$$1 + 80 = 81, \quad 4 + 121 = 125,$$

$$2 + 3^{10} \cdot 109 = 23^5, \quad 11^2 + 3^2 5^6 7^3 = 2^{21} \cdot 23.$$
Following F. Beukers, an 
abc–hit is an abc–triple such
that $\text{Rad}(abc) < c$.

http://www.staff.science.uu.nl/~beuke106/ABCpresentation.pdf

Example: $(1, 8, 9)$ is an abc–hit since $1 + 8 = 9,$
gcd$(1, 8, 9) = 1$ and

$$\text{Rad}(1 \cdot 8 \cdot 9) = \text{Rad}(2^3 \cdot 3^2) = 2 \cdot 3 = 6 < 9.$$  

But $(2, 16, 18)$
is not an abc–hit since these three numbers are not coprime.

Further abc–hits

- $(2, 3^{10} \cdot 109, 23^5) = (2, 6436341, 6436343)$
is an abc–hit since $2 + 3^{10} \cdot 109 = 23^5$ and
$\text{Rad}(2 \cdot 3^{10} \cdot 109 \cdot 23^5) = 15042 < 23^5 = 6436343$.

- $(11^2, 3^2 \cdot 5^6 \cdot 7^3, 2^{21} \cdot 23) = (121, 48234275, 48234496)$
is an abc–hit since $11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 23$ and
$\text{Rad}(2^{21} \cdot 3^2 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 23) = 53130 < 2^{21} \cdot 23 = 48234496$.

- $(1, 5 \cdot 127 \cdot (2 \cdot 3 \cdot 7)^3, 19^6) = (1, 47045880, 47045881)$
is an abc–hit since $1 + 5 \cdot 127 \cdot (2 \cdot 3 \cdot 7)^3 = 19^6$ and
$\text{Rad}(5 \cdot 127 \cdot (2 \cdot 3 \cdot 7)^3 \cdot 19^6) = 5 \cdot 127 \cdot 2 \cdot 3 \cdot 7 \cdot 19 = 506730$.

Some abc–hits

$(1, 80, 81)$ is an abc–hit since $1 + 80 = 81,$ gcd$(1, 80, 81) = 1$
and

$$\text{Rad}(1 \cdot 80 \cdot 81) = \text{Rad}(2^4 \cdot 5 \cdot 3^4) = 2 \cdot 5 \cdot 3 = 30 < 81.$$  

$(4, 121, 125)$ is an abc–hit since $4 + 121 = 125,$
gcd$(4, 121, 125) = 1$ and

$$\text{Rad}(4 \cdot 121 \cdot 125) = \text{Rad}(2^2 \cdot 5^3 \cdot 11^2) = 2 \cdot 5 \cdot 11 = 110 < 125.$$  

Further abc–triples and abc–hits

Among $15 \cdot 10^6$ abc–triples with $c < 10^4,$ we have 120
abc–hits.

Among $380 \cdot 10^6$ abc–triples with $c < 5 \cdot 10^3,$ we have 276
abc–hits.
More $abc$–hits

$$(1, 3^{16} - 1, 3^{16}) = (1, 43046720, 43046721)$$

is an $abc$–hit.

Proof.

$$3^{16} - 1 = (3^8 - 1)(3^8 + 1) = (3^4 - 1)(3^4 + 1)(3^8 + 1) = (3^2 - 1)(3^2 + 1)(3^4 + 1)(3^8 + 1) = (3 - 1)(3 + 1)(3^2 + 1)(3^4 + 1)(3^8 + 1)$$

is divisible by $2^6$.

Hence

$$\operatorname{Rad}((3^{16} - 1) \cdot 3^{16}) \leq \frac{3^{16} - 1}{2^6} \cdot 2 \cdot 3 < 3^{16}.$$ 

Infinitely many $abc$–hits

This argument shows that there exist infinitely many $abc$–triples such that

$$c > \frac{1}{6 \log 3} R \log R$$

with $R = \operatorname{Rad}(abc)$.

Question: Are there $abc$–triples for which $c > \operatorname{Rad}(abc)^2$?

We do not know the answer.

Infinitely many $abc$–hits

**Proposition.** There are infinitely many $abc$–hits.

Take $k \geq 1$, $a = 1$, $c = 3^{2^k}$, $b = c - 1$.

**Lemma.** $2^{k+2}$ divides $3^{2^k} - 1$.

Proof: Induction on $k$ using

$$3^{2^k} - 1 = (3^{2^{k-1}} - 1)(3^{2^{k-1}} + 1).$$

Consequence:

$$\operatorname{Rad}((3^{2^k} - 1) \cdot 3^{2^k}) \leq \frac{3^{2^k} - 1}{2^{k+1}} \cdot 3 < 3^{2^k}.$$ 

Hence

$$(1, 3^{2^k} - 1, 3^{2^k})$$

is an $abc$–hit.

Examples

When $a$, $b$ and $c$ are three positive relatively prime integers satisfying $a + b = c$, define

$$\lambda(a, b, c) = \frac{\log c}{\log \operatorname{Rad}(abc)}.$$ 

Here are the two largest known values for $\lambda(abc)$

<table>
<thead>
<tr>
<th>$a + b$</th>
<th>$c$</th>
<th>$\lambda(a, b, c)$</th>
<th>authors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 + 3^{10} \cdot 109$</td>
<td>$23^5$</td>
<td>1.629912...</td>
<td>É. Reyssat</td>
</tr>
<tr>
<td>$11^2 + 3^25^67^3$</td>
<td>$2^{21} \cdot 23$</td>
<td>1.625990...</td>
<td>B.M. de Weger</td>
</tr>
</tbody>
</table>

At the date of September 11, 2008, 217 $abc$ triples with $\lambda(a, b, c) \geq 1.4$ were known. http://www.math.unicaen.fr/~nitaj/tableabc.pdf

Since August 1, 2015, 238 are known. http://www.math.leidenuniv.nl/~desmit/abc/index.php?sort=1
Eric Reyssat: \( 2 + 3^{10} \cdot 109 = 23^5 \)

Example of Reyssat \( 2 + 3^{10} \cdot 109 = 23^5 \)

\[
a + b = c
\]

\[
a = 2, \quad b = 3^{10} \cdot 109, \quad c = 23^5 = 6436343,
\]

Rad(\( abc \)) = Rad(\( 2 \cdot 3^{10} \cdot 109 \cdot 23^5 \)) = \( 2 \cdot 3 \cdot 109 \cdot 23 = 15042, \)

\[
\lambda(a, b, c) = \frac{\log c}{\log \text{Rad}(abc)} = \frac{5 \log 23}{\log 15042} \approx 1.62991.
\]

**Continued fraction**

\[2 + 109 \cdot 3^{10} = 23^5\]

Continued fraction of \( 109^{1/5} : [2; 1, 1, 4, 77733, \ldots] \),
approximation: \([2; 1, 1, 4] = 23/9\)

\[
109^{1/5} = 2.55555539 \ldots
\]

\[
23/9 = 2.55555555 \ldots
\]

Benne de Weger: \( 11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 23 \)

Rad(\( 2^{21} \cdot 3^2 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 23 \)) = \( 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 53130, \)

\[2^{21} \cdot 23 = 48234496 = (53130)^{1.625990} \ldots\]

N. A. Carella. *Note on the ABC Conjecture*

Explicit abc Conjecture

According to S. Laishram and T. N. Shorey, an explicit version, due to A. Baker, of the abc Conjecture, yields

\[ c < \text{Rad}(abc)^{7/4} \]

for any abc-triple \((a, b, c)\).

Lower bound for the radical of abc

The abc Conjecture is a lower bound for the radical of the product abc:

abc Conjecture. For any \( \varepsilon > 0 \), there exist \( \kappa(\varepsilon) \) such that, if \( a, b \) and \( c \) are relatively prime positive integers which satisfy \( a + b = c \), then

\[ \text{Rad}(abc) > \kappa(\varepsilon)c^{1-\varepsilon}. \]

The abc Conjecture

Recall that for a positive integer \( n \), the radical of \( n \) is

\[ \text{Rad}(n) = \prod_{p|n} p. \]

abc Conjecture. Let \( \varepsilon > 0 \). Then the set of abc triples for which

\[ c > \text{Rad}(abc)^{1+\varepsilon} \]

is finite.

Equivalent statement: For each \( \varepsilon > 0 \) there exists \( \kappa(\varepsilon) \) such that, if \( a, b \) and \( c \) in \( \mathbb{Z}_{>0} \) are relatively prime and satisfy \( a + b = c \), then

\[ c < \kappa(\varepsilon)\text{Rad}(abc)^{1+\varepsilon}. \]

The abc Conjecture of Œsterlé and Masser

The abc Conjecture resulted from a discussion between J. Œsterlé and D. W. Masser in the mid 1980’s.
C.L. Stewart and Yu Kunrui


\[
\log c \leq \kappa R^{1/3}(\log R)^3.
\]

with \( R = \text{Rad}(abc) \):

\[
c \leq e^{\kappa R^{1/3}(\log R)^3}.
\]

Further examples

When \( a, b \) and \( c \) are three positive relatively prime integers satisfying \( a + b = c \), define

\[
g(a, b, c) = \frac{\log abc}{\log \text{Rad}(abc)}.
\]

Here are the two largest known values for \( g(abc) \), found by A. Nitaj.

\[
\begin{array}{ccc}
a + b & = & c \\
13 \cdot 19^6 + 2^{30} \cdot 5 & = & 3^{13} \cdot 11^2 \cdot 31 \\
2^5 \cdot 11^2 \cdot 19^9 + 5^{15} \cdot 37^2 \cdot 47 & = & 3^7 \cdot 7^{11} \cdot 743
\end{array}
\]

On March 19, 2003, 47 abc triples were known with \( 0 < a < b < c, a + b = c \) and \( \gcd(a, b) = 1 \) satisfying \( g(a, b, c) > 4 \).
www.abcathome.com

ABC@home is an educational and non-profit distributed computing project finding abc-triples related to the ABC conjecture.

The ABC conjecture is currently one of the greatest open problems in mathematics. If it is proven to be true, a lot of other open problems can be answered directly from it.

The ABC conjecture is one of the greatest open mathematical questions, one of the holy grails of mathematics. It will teach us something about our very own numbers.
Fermat’s last Theorem for \( n \geq 6 \) as a consequence of the abc Conjecture

Assume \( x^n + y^n = z^n \) with \( \gcd(x, y, z) = 1 \) and \( x < y \). Then \((x^n, y^n, z^n)\) is an abc-triple with

\[
\text{Rad}(x^ny^nz^n) \leq xyz < z^3.
\]

If the explicit abc Conjecture \( c < \text{Rad}(abc)^2 \) is true, then one deduces

\[
z^n < z^6,
\]

hence \( n \leq 5 \) (and therefore \( n \leq 2 \)).

Perfect powers

1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, 196, 216, 225, 243, 256, 289, 324, 343, 361, 400, 441, 484, 512, 529, 576, 625, 676, 729, 784, ...

Nearly equal perfect powers

- Difference 1 : (8, 9)
- Difference 2 : (25, 27), ...
- Difference 3 : (1, 4), (125, 128), ...
- Difference 4 : (4, 8), (32, 36), (121, 125), ...
- Difference 5 : (4, 9), (27, 32), ...

Square, cubes...

- A perfect power is an integer of the form \( a^b \) where \( a \geq 1 \) and \( b > 1 \) are positive integers.
- Squares:
  1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, ...
- Cubes:
  1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, ...
- Fifth powers:
  1, 32, 243, 1024, 3125, 7776, 16807, 32768,...

Perfect powers

1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, 196, 216, 225, 243, 256, 289, 324, 343, 361, 400, 441, 484, 512, 529, 576, 625, 676, 729, 784, ...

Neil J. A. Sloane’s encyclopaedia

http://oeis.org/A001597
Two conjectures

Subbaya Sivasankaranarayana Pillai
(1901-1950)

Eugène Charles Catalan (1814 – 1894)

- Catalan’s Conjecture: In the sequence of perfect powers, 8, 9 is the only example of consecutive integers.

- Pillai’s Conjecture: In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

Pillai’s Conjecture:

- Pillai’s Conjecture: In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

- Alternatively: Let \( k \) be a positive integer. The equation
  \[ x^p - y^q = k, \]
  where the unknowns \( x, y, p \) and \( q \) take integer values, all \( \geq 2 \), has only finitely many solutions \( (x, y, p, q) \).

Results


Catalan was right: the equation \( x^p - y^q = 1 \) where the unknowns \( x, y, p \) and \( q \) take integer values, all \( \geq 2 \), has only one solution \( (x, y, p, q) = (3, 2, 2, 3) \).

Previous work on Catalan’s Conjecture

J.W.S. Cassels, Rob Tijdeman

\[ x^p < y^q < \exp \exp \exp \exp(730) \]

Michel Langevin
Previous work on Catalan’s Conjecture

Maurice Mignotte

Yuri Bilu

Pillai’s conjecture and the abc Conjecture

There is no value of $k \geq 2$ for which one knows that Pillai’s equation $x^p - y^q = k$ has only finitely many solutions.

Pillai’s conjecture as a consequence of the abc Conjecture: if $x^p \neq y^q$, then

$$|x^p - y^q| \geq c(\epsilon) \max\{x^p, y^q\}^{\kappa-\epsilon}$$

with

$$\kappa = 1 - \frac{1}{p} - \frac{1}{q}.$$}

Not a consequence of the abc Conjecture

$p = 3$, $q = 2$

Hall’s Conjecture (1971):

if $x^3 \neq y^2$, then

$$|x^3 - y^2| \geq c \max\{x^3, y^2\}^{1/6}.$$
Generalized Fermat’s equation $x^p + y^q = z^r$

Consider the equation $x^p + y^q = z^r$ in positive integers $(x, y, z, p, q, r)$ such that $x, y, z$ relatively prime and $p, q, r$ are $\geq 2$.

If

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1,$$

then $(p, q, r)$ is a permutation of one of

$$(2, 2, k), \quad (2, 3, 3), \quad (2, 3, 4), \quad (2, 3, 5),$$

$$(2, 4, 4), \quad (2, 3, 6), \quad (3, 3, 3)$$

and in each case the set of solutions $(x, y, z)$ is known (for some of these values there are infinitely many solutions).

On the condition that $x, y, z$ are relatively prime

$$1 + 2^3 = 3^2 \implies 3^6 + 18^3 = 3^8.$$  

Starting with $a^p + b^q = c$, multiply by $c^{pq}$ and get

$$(ac^p)^q + (bc^q)^p = c^{pq+1}.$$  

http://mathoverflow.net/

From Henri Darmon, communicated by Claude Levesque.

Frits Beukers and Don Zagier

For

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

10 solutions $(x, y, z, p, q, r)$ (up to obvious symmetries) to the equation

$$x^p + y^q = z^r$$

are known.

Generalized Fermat’s equation

For

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

the equation

$$x^p + y^q = z^r$$

has the following 10 solutions with $x, y, z$ relatively prime:

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2,$$

$$3^5 + 11^4 = 122^2, \quad 33^8 + 1549034^2 = 15613^3,$$

$$1414^2 + 2213459^2 = 657, \quad 9262^3 + 15312283^2 = 1137^3,$$

$$17^7 + 76271^3 = 21063928^2, \quad 43^8 + 96222^3 = 30042907^2.$$  

http://mathoverflow.net/
Conjecture of Beal, Granville and Tijdeman–Zagier

The equation $x^p + y^q = z^r$ has no solution in positive integers $(x, y, z, p, q, r)$ with each of $p$, $q$, and $r$ at least 3 and with $x$, $y$, $z$ relatively prime.

http://mathoverflow.net/

Andrew Beal

Find a solution with all exponents at least 3, or prove that there is no such solution.


Beal’s Prize


Beal’s Prize : 1,000,000$ US

An AMS-appointed committee will award this prize for either a proof of, or a counterexample to, the Beal Conjecture published in a refereed and respected mathematics publication. The prize money – currently US$1,000,000 – is being held in trust by the AMS until it is awarded. Income from the prize fund is used to support the annual Erdős Memorial Lecture and other activities of the Society.

One of Andrew Beal’s goals is to inspire young people to think about the equation, think about winning the offered prize, and in the process become more interested in the field of mathematics.

http://www.ams.org/profession/prizes-awards/ams-supported/beal-prize
Henri Darmon, Andrew Granville

“Fermat-Catalan” Conjecture (H. Darmon and A. Granville),
consequence of the abc Conjecture: the set of solutions
\((x, y, z, p, q, r)\) to \(x^p + y^q = z^r\) with
\((1/p) + (1/q) + (1/r) < 1\) is finite.

Hint: \(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1\) implies \(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{41}{42}\).

1995 (H. Darmon and A. Granville): for fixed \((p, q, r)\), only
finitely many \((x, y, z)\).

Infinitely many primes are not Wieferich assuming abc

J.H. Silverman: if the abc Conjecture is true, given a
positive integer \(a > 1\), there exist infinitely many primes \(p\)
such that \(p^2\) does not divide \(a^{p-1} - 1\).

Fermat’s Little Theorem

For \(a > 1\), any prime \(p\) not dividing \(a\) divides \(a^{p-1} - 1\).

Hence if \(p\) is an odd prime, then \(p\) divides \(2^{p-1} - 1\).

Wieferich primes (1909): \(p^2\) divides \(2^{p-1} - 1\)

The only known Wieferich primes below \(4 \cdot 10^{12}\) are 1093 and 3511.

Consecutive integers with the same radical

Notice that
\(75 = 3 \cdot 5^2\) and \(1215 = 3^5 \cdot 5\)

hence
\(\text{Rad}(75) = \text{Rad}(1215) = 3 \cdot 5 = 15\).

But also
\(76 = 2^2 \cdot 19\) and \(1216 = 2^6 \cdot 19\)

have the same radical
\(\text{Rad}(76) = \text{Rad}(1216) = 2 \cdot 19 = 38\).
Consecutive integers with the same radical

For $k \geq 1$, the two numbers

$$x = 2^k - 2 = 2(2^{k-1} - 1)$$

and

$$y = (2^k - 1)^2 - 1 = 2^{k+1}(2^{k-1} - 1)$$

have the same radical, and also

$$x + 1 = 2^k - 1$$ \textit{and} $$y + 1 = (2^k - 1)^2$$

have the same radical.

Erdős – Woods Conjecture

There exists an absolute constant $k$ such that, if $x$ and $y$ are positive integers satisfying

$$\text{Rad}(x + i) = \text{Rad}(y + i)$$

for $i = 0, 1, \ldots, k - 1$, then $x = y$.

Erdős – Woods as a consequence of $abc$

M. Langevin: The $abc$ Conjecture implies that there exists an absolute constant $k$ such that, if $x$ and $y$ are positive integers satisfying

$$\text{Rad}(x + i) = \text{Rad}(y + i)$$

for $i = 0, 1, \ldots, k - 1$, then $x = y$. 

Are there further examples of $x \neq y$ with

$$\text{Rad}(x) = \text{Rad}(y) \quad \text{and} \quad \text{Rad}(x + 1) = \text{Rad}(y + 1)?$$

Is it possible to find two distinct integers $x, y$ such that

$$\text{Rad}(x) = \text{Rad}(y),$$

$$\text{Rad}(x + 1) = \text{Rad}(y + 1)$$

and

$$\text{Rad}(x + 2) = \text{Rad}(y + 2)?$$
Erdős Conjecture on $2^n - 1$

In 1965, P. Erdős conjectured that the greatest prime factor $P(2^n - 1)$ satisfies

$$\frac{P(2^n - 1)}{n} \to \infty \quad \text{when} \quad n \to \infty.$$  

In 2002, R. Murty and S. Wong proved that this is a consequence of the abc Conjecture. In 2012, C.L. Stewart proved Erdős Conjecture (in a wider context of Lucas and Lehmer sequences):

$$P(2^n - 1) > n \exp\left(\frac{\log n}{104 \log \log n}\right).$$

Is abc Conjecture optimal?

Let $\delta > 0$. In 1986, C.L. Stewart and R. Tijdeman proved that there are infinitely many abc–triples for which

$$c > R \exp\left((4 - \delta) \frac{(\log R)^{1/2}}{\log \log R}\right).$$

Better than $c > R \log R$.

Conjectures by Machiel van Frankenhuijsen, Olivier Robert, Cam Stewart and Gérald Tenenbaum

Let $\varepsilon > 0$. There exists $\kappa(\varepsilon) > 0$ such that for any abc triple with $R = \text{Rad}(abc) > 8$,

$$c < \kappa(\varepsilon) R \exp\left((4\sqrt{3} + \varepsilon) \left(\frac{\log R}{\log \log R}\right)^{1/2}\right).$$

Further, there exist infinitely many abc–triples for which

$$c > R \exp\left((4\sqrt{3} - \varepsilon) \left(\frac{\log R}{\log \log R}\right)^{1/2}\right).$$

Machiel van Frankenhuijsen, Olivier Robert, Cam Stewart and Gérald Tenenbaum
Heuristic assumption

Whenever \(a\) and \(b\) are coprime positive integers, \(R(a + b)\) is independent of \(R(a)\) and \(R(b)\).


http://blms.oxfordjournals.org/content/46/6/1156.full.pdf


Waring’s Problem

In 1770, a few months before J.L. Lagrange solved a conjecture of Bachet (1621) and Fermat (1640) by proving that every positive integer is the sum of at most four squares of integers, E. Waring wrote :

“Omnis integer numerus vel est cubus, vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, &c., usque ad novemdecim compositus, & sic deinceps”

“Every integer is a cube or the sum of two, three, ... nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree.”

Waring’s functions \(g(k)\) and \(G(k)\)

- **Waring’s function** \(g\) is defined as follows : For any integer \(k \geq 2\), \(g(k)\) is the least positive integer \(s\) such that any positive integer \(N\) can be written \(x_1^k + \cdots + x_s^k\).

- **Waring’s function** \(G\) is defined as follows : For any integer \(k \geq 2\), \(G(k)\) is the least positive integer \(s\) such that any sufficiently large positive integer \(N\) can be written \(x_1^k + \cdots + x_s^k\).

\(g(k) \geq I(k)\)

For each integer \(k \geq 2\), define \(I(k) = 2^k + \lfloor(3/2)^k\rfloor - 2\). It is easy to show that \(g(k) \geq I(k)\) (J. A. Euler, son of Leonhard Euler). Indeed, write

\[3^k = 2^k q + r\]

with \(0 < r < 2^k\), \(q = \lfloor(3/2)^k\rfloor\)

and consider the integer

\[N = 2^k q - 1 = (q - 1)2^k + (2^k - 1)^k\]

Since \(N < 3^k\), writing \(N\) as a sum of \(k\)-th powers can involve no term \(3^k\), and since \(N < 2^k q\), it involves at most \((q - 1)\) terms \(2^k\), all others being \(1^k\); hence it requires a total number of at least \((q - 1) + (2^k - 1) = I(k)\) terms.
The ideal Waring’s Theorem $g(k) = I(k)$

Conjecture (C.A. Bretschneider, 1853) : $g(k) = I(k)$ for any $k \geq 2$, with

$$I(k) = 2^k + \lfloor (3/2)^k \rfloor - 2.$$  

We know that the remainder $r = 3^k - 2^k q$ satisfies $r < 2^k$. A slight improvement of this upper bound would yield the desired result. L.E. Dickson and S.S. Pillai proved independently in 1936 that $g(k) = I(k)$, provided that $r = 3^k - 2^k q$ satisfies

$$r \leq 2^k - q - 2 \quad \text{with} \quad q = \lfloor (3/2)^k \rfloor.$$  

The condition $r \leq 2^k - q - 2$ is satisfied for $4 \leq k \leq 471\,600\,000$.

Mahler’s contribution

- The estimate

  $$r \leq 2^k - q - 2$$  

  is valid for all sufficiently large $k$.

  [Image of Kurt Mahler]

Hence the ideal Waring’s Theorem

$$g(k) = 2^k + \lfloor (3/2)^k \rfloor - 2$$  

holds for all sufficiently large $k$.

Waring’s Problem and the $abc$ Conjecture

$n = x_1^4 + \cdots + x_{19}^4 : g(4) = 19$

Any positive integer is the sum of at most 19 biquadrates


François Dress, R. Balasubramanian, Jean-Marc Deshouillers

S. David : the estimate

$$r \leq 2^k - q - 2$$  

for sufficiently large $k$ follows from the $abc$ Conjecture.

S. Laishram : the ideal Waring’s Theorem

$g(k) = 2^k + \lfloor (3/2)^k \rfloor - 2$ follows from the explicit $abc$ Conjecture.
Conjecture of Alan Baker (1996)

Let \((a, b, c)\) be an \(abc\)-triple and let \(\epsilon > 0\). Then

\[ c \leq \kappa (\epsilon^{-\omega} R)^{1+\epsilon} \]

where \(\kappa\) is an absolute constant, \(R = \text{Rad}(abc)\) and \(\omega = \omega(abc)\) is the number of distinct prime factors of \(abc\).

Remark of Andrew Granville: the minimum of the function on the right hand side over \(\epsilon > 0\) occurs essentially with \(\epsilon = \omega / \log R\). This yields a slightly sharper form of the conjecture:

\[ c \leq \kappa R \frac{(\log R)^\omega}{\omega!}. \]


Let \((a, b, c)\) be an \(abc\)-triple. Then

\[ c \leq \frac{6}{5} R \frac{(\log R)^\omega}{\omega!} \]

with \(R = \text{Rad}(abc)\) the radical of \(abc\) and \(\omega = \omega(abc)\) the number of distinct prime factors of \(abc\).

Shanta Laishram and Tarlok Shorey

The Nagell–Ljunggren equation is the equation

\[ y^q = \frac{x^n - 1}{x - 1} \]

in integers \(x > 1, y > 1, n > 2, q > 1\).

This means that in basis \(x\), all the digits of the perfect power \(y^q\) are 1.

If the explicit \(abc\)-conjecture of Baker is true, then the only solutions are

\[ 11^2 = \frac{3^3 - 1}{3 - 1}, \quad 20^2 = \frac{7^4 - 1}{7 - 1}, \quad 7^3 = \frac{18^3 - 1}{18 - 1}. \]

The \(abc\) conjecture for number fields

\[ \text{Méthodes de transcendance et géométrie diophantienne}, \]

A. Surroca, Thèse Université de Paris 6, 2003.
Mordell’s Conjecture (Faltings’s Theorem)

Using an extension of the \( abc \) Conjecture for number fields, N. Elkies deduces Faltings’s Theorem on the finiteness of the set of rational points on an algebraic curve of genus \( \geq 2 \).


Thue–Siegel–Roth Theorem (Bombieri)

Using the \( abc \) Conjecture for number fields, E. Bombieri (1994) deduces a refinement of the Thue–Siegel–Roth Theorem on the rational approximation of algebraic numbers

\[
\left| \alpha - \frac{p}{q} \right| > \frac{1}{q^{2+\varepsilon}}
\]

where he replaces \( \varepsilon \) by

\[
\kappa (\log q)^{-1/2} (\log \log q)^{-1}
\]

where \( \kappa \) depends only on the algebraic number \( \alpha \).
Siegel’s zeroes (A. Granville and H.M. Stark)

The uniform $abc$ Conjecture for number fields implies a lower bound for the class number of an imaginary quadratic number field, and K. Mahler has shown that this implies that the associated $L$–function has no Siegel zero.

Further consequences of the $abc$ Conjecture

- Erdős’s Conjecture on consecutive powerful numbers.
- Dressler’s Conjecture : between two positive integers having the same prime factors, there is always a prime.
- Squarefree and powerfree values of polynomials.
- Lang’s conjectures : lower bounds for heights, number of integral points on elliptic curves.
- Bounds for the order of the Tate–Shafarevich group.
- Vojta’s Conjecture for curves.
- Greenberg’s Conjecture on Iwasawa invariants $\lambda$ and $\mu$ in cyclotomic extensions.
- Exponents of class groups of quadratic fields.
- Fundamental units in quadratic and biquadratic fields.

$abc$ and meromorphic function fields

Nevanlinna value distribution theory.

Recent work of Hu, Pei–Chu and Yang, Chung-Chun.

$abc$ and Vojta’s height Conjecture

Vojta’s Conjecture on algebraic points of bounded degree on a smooth complete variety over a global field of characteristic zero implies the $abc$ Conjecture.
**ABC Theorem for polynomials**

Let $K$ be an algebraically closed field. The *radical* of a monic polynomial

$$P(X) = \prod_{i=1}^{n} (X - \alpha_i)^{a_i} \in K[X]$$

with $\alpha_i$ pairwise distinct is defined as

$$\text{Rad}(P)(X) = \prod_{i=1}^{n} (X - \alpha_i) \in K[X].$$

---

**The radical of a polynomial as a gcd**

The common zeroes of

$$P(X) = \prod_{i=1}^{n} (X - \alpha_i)^{a_i} \in K[X]$$

and $P'$ are the $\alpha_i$ with $a_i \geq 2$. They are zeroes of $P'$ with multiplicity $a_i - 1$. Hence

$$\text{Rad}(P) = \frac{P}{\gcd(P, P')}.$$
Proof of the $ABC$ Theorem for polynomials

Recall $\gcd(A, B, C) = 1$. Since $\gcd(C', C')$ divides $AC' - A'C = AB' - A'B'$, it divides also

$$\frac{AB' - A'B}{\gcd(A, A') \gcd(B'B')}$$

which is a polynomial of degree

$$< \deg(\text{Rad}(A)) + \deg(\text{Rad}(B)) = \deg(\text{Rad}(AB)).$$

Hence

$$\deg(\gcd(C, C')) < \deg(\text{Rad}(AB))$$

and

$$\deg(C) < \deg(\text{Rad}(C)) + \deg(\text{Rad}(AB)) = \deg(\text{Rad}(ABC)).$$

Papers of Shinichi Mochizuki

- General Arithmetic Geometry
- Intrinsic Hodge Theory
- $p$–adic Teichmüller Theory
- Anabelian Geometry, the Geometry of Categories
- The Hodge-Arakelov Theory of Elliptic Curves
- Inter-universal Teichmüller Theory
In August 2012, Shinichi Mochizuki released a series of four preprints containing a serious claim to a proof of the abc Conjecture.

When an error in one of the articles was pointed out by Vesselin Dimitrov and Akshay Venkatesh in October 2012, Mochizuki posted a comment on his website acknowledging the mistake, stating that it would not affect the result, and promising a corrected version in the near future. He proceeded to post a series of corrected papers of which the latest dated November 24, 2014.

https://en.wikipedia.org/wiki/Abc_conjecture

Workshop on IUT Theory of Shinichi Mochizuki, December 7-11 2015
CMI Workshop supported by Clay Math Institute and Symmetries and Correspondences

Organisers: Ivan Fesenko, Minhyong Kim, Kobi Kremnitzer
Finding the speakers and the program of the workshop: Ivan Fesenko
CMI Workshop supported by Clay Math Institute and Symmetries and Correspondences

The work (currently being refereed) of SHINICHI MOCHIZUKI on inter-universal Teichmüller theory (also known as arithmetic deformation theory) and its application to famous conjectures in diophantine geometry became publicly available in August 2012. This theory, developed over 20 years, introduces a vast collection of novel ideas, methods and objects. Aspects of the theory extend arithmetic geometry to a non-scheme-theoretic setting and, more generally, have the potential to open new fundamental areas of mathematics.

The workshop aims to present and analyse key principles, concepts, objects and proofs of the theory of Mochizuki and study its relations with existing theories in different areas, to help to increase the number of experts in the theory of Mochizuki and stimulate its further applications.

Speakers

Shinichi Mochizuki will answer questions during skype sessions of the workshop. He also responds directly to emailed questions.


Participants

Mahidol University International College November 17, 2016

On the abc Conjecture and some of its consequences

by

Michel Waldschmidt

Université P. et M. Curie (Paris VI)

http://www.imj-prg.fr/~michel.waldschmidt/