Valeurs spéciales de polylogarithmes multiples

par

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The Harmonic Algebra, Quasisymmetric Series and stuffle relations between polylogarithms in several variables

On introduit l'algèbre harmonique de M. Hoffman, on étudie sa structure, le lien avec les fonctions quasisymétriques, et on applique ces résultats aux polylogarithmes multiples en plusieurs variables pour en déduire les deuxièmes relations de mélange entre polyzêta.

1. The Harmonic Algebra $\mathfrak{H}_\star$

There is another shuffle-like law on $\mathfrak{H}$, called the *harmonic product* by M. Hoffman [H 1997] and *stuffle* by other authors [B^{3}L 2001], again denoted with as $\star$ (*), which also gives rise to subalgebras

$$\mathfrak{H}^0_\star \subset \mathfrak{H}^1_\star \subset \mathfrak{H}_\star.$$

It is defined as follows. First on X^*, the map $\star : X^* \times X^* \rightarrow \mathfrak{H}$ is defined by induction, starting with

$$x_0^n \star w = w \star x_0^n = w x_0^n$$

for any $w \in X^*$ and any $n \geq 0$ (for $n = 0$ it means $e \star w = w \star e = w$ for all $w \in X^*$), and then

$$(y_s u) \star (y_t v) = y_s\big(u \star (y_t v)\big) + y_t\big((y_s u) \star v\big) + y_{s+t}(u \star v)$$

for u and v in X^*, s and t positive integers.

We shall not use so many parentheses later: in a formula where there are both concatenation products and either shuffle of star products, we agree that concatenation is always performed first, unless parentheses impose another priority:

$$y_s u \star y_t v = y_s(u \star y_t v) + y_t(y_s u \star v) + y_{s+t}(u \star v)$$

(*) There should be no confusion with the rational operation $S \mapsto S^*$ on power series, where the star is written * and is always in the exponent. Beware that we shall write $S^{\star 2}$ for $S \star S$; the square of S^* will never occur here, but if would be written $(S^*)^2$

Again this law is extended to all of $\mathfrak{H}$ by distributivity with respect to addition:

$$\sum_{u\in X^*}(S|u)u \;\star \sum_{v\in X^*}(T|v)v = \sum_{u\in X^*}\sum_{v\in X^*}(S|u)(T|v)u\star v.$$

`Remark.` From the definition (by induction on the length of uv) one deduces

$$(ux_0^m)\star(vx_0^m) = (u\star v)x_0^m$$

for $m\geq 0$, u and v in X^*.

`Example.`

$$y_2^{\star 3} = y_2\star y_2\star y_2 = 6y_2^3+3y_2y_4+3y_4y_2+y_6.$$

Hoffman's Theorem [H 1997] gives the structure of the harmonic algebra $\mathfrak{H}_\star$:

Theorem 1.3. *The harmonic algebras are polynomial algebras on Lyndon words:*

$$\mathfrak{H}_\star = K[\mathcal{L}]_\star,\quad \mathfrak{H}_\star^0 = K\big[\mathcal{L}\setminus\{x_0,x_1\}\big]_\star \quad \textit{et} \quad \mathfrak{H}_\star^1 = K\big[\mathcal{L}\setminus\{x_0,x_1\}\big]_\star.$$

For instance the 10 non-Lyndon words of weight ≤ 3 are polynomials in the 5 Lyndon words:

$$x_0 < x_0x_1 < x_0^2x_1 < x_0x_1^2 < x_1.$$

as follows:

$$\begin{array}{ll}
e=e, & x_0^2 = x_0\star x_0,\\
x_0^3 = x_0\star x_0\star x_0, & x_0x_1x_0 = x_0\star x_0x_1,\\
x_1x_0 = x_0\star x_1, & x_1x_0^2 = x_0\star x_0\star x_1,\\
x_1x_0x_1 = x_0x_1\star x_1 - x_0^2x_1 - x_0x_1^2, & x_1^2 = \frac12 x_1\star x_1 - \frac12 x_0x_1,\\
x_1^2x_0 = \frac12 x_0\star x_1\star x_1 - \frac12 x_0\star x_0x_1, & x_1^3 = \frac16 x_1\star x_1\star x_1 - \frac12 x_0x_1\star x_1 + \frac13 x_0^2x_1.
\end{array}$$

In the same way as Corollary 1.2 follows from Theorem 1.1, we deduce from Theorem 1.3:

Corollary 1.4. *We have*

$$\mathfrak{H}_\star = \mathfrak{H}_\star^1[x_0]_\star = \mathfrak{H}_\star^0[x_0,x_1]_\star \quad \textit{et} \quad \mathfrak{H}_\star^1 = \mathfrak{H}_\star^0[x_1]_\star.$$

`Remark.` Consider the diagram

$$\begin{array}{ccc}
\mathfrak{H}_{ш} & \longrightarrow & K[\mathcal{L}]_{ш}\\
\Big\downarrow f & & \Big\downarrow g\\
\mathfrak{H}_\star & \longrightarrow & K[\mathcal{L}]_\star
\end{array}$$

The horizontal maps are just the identity: $\mathfrak{H}_{ш} = K[\mathcal{L}]_{ш}$ and $\mathfrak{H}_\star = K[\mathcal{L}]_\star$. The vertical map f is also the identity on $\mathfrak{H}$, since the algebras $\mathfrak{H}_{ш}$ and $\mathfrak{H}_\star$ have the same underlying set $\mathfrak{H}$ (only the law differs). But the map g is not a morphism of algebras: it maps each Lyndon word on itself, but consider for instance the image of the word x_0^2:, as a polynomial in $K[\mathcal{L}]_\star$, $x_0^2 = x_0\star x_0 = x_0^{\star 2}$, but, as a polynomial in $K[\mathcal{L}]_{ш}$, $x_0^2 = (1/2)x_0 ш x_0 = (1/2)x_0^{ш 2}$.

2. Quasi-Symmetric Series

The harmonic product is closely connected with the theory of *quasi-symmetric series* as follows (work of Stanley, 1974 [R 1993]).

Denote by $\underline{t} = (t_1, t_2, \ldots)$ a sequence of commutative variables. To $\underline{s} = (s_1, \ldots, s_k)$, where each s_j is an integer ≥ 1, associate the series

$$M_{\underline{s}}(\underline{t}) = \sum_{\substack{n_1\geq 1,\ldots,n_k\geq 1\\ n_1,\ldots,n_k \text{pairwise distinct}}} t_{n_1}^{s_1}\cdots t_{n_k}^{s_k}.$$

The space of power series spanned by these $M_{\underline{s}}$ is denoted by Sym and its elements are called *symmetric series.* A basis of Sym is given by the series $M_{\underline{s}}$ with $s_1 \geq s_2 \geq \cdots \geq s_k$ and $k \geq 0$.

A *quasi-symmetric series* is an element of the algebra QSym spanned by the series

$$QM_{\underline{s}}(\underline{t}) = \sum_{n_1>\cdots>n_k\geq 1} t_{n_1}^{s_1}\cdots t_{n_k}^{s_k},$$

where $\underline{s}$ ranges over the set of tuples $(s_1, \ldots, s_k)$ with $k \geq 0$ and $s_j \geq 1$ for $1 \leq j \leq k$. Notice that, for $\underline{s} = (s_1, \ldots, s_k)$ of length k,

$$M_{\underline{s}} = \sum_{\tau\in\mathfrak{S}_k} QM_{\underline{s}^\tau},$$

where $\mathfrak{S}_k$ is the symmetric group on k elements and $\underline{s}^\tau = (s_{\tau(1)}, \ldots, s_{\tau(k)})$. Hence any symmetric series is also quasi-symmetric. Therefore Sym is a subalgebra of QSym.

Proposition 2.1. *The K-linear map $\phi : \mathfrak{H}^1 \to$ QSym defined by $y_{\underline{s}} \mapsto QM_{\underline{s}}$ is an isomorphism of K-algebras from $\mathfrak{H}^1$ to* QSym*.*

In other terms, if we write

$$y_{\underline{s}} \star y_{\underline{s}'} = \sum_{\underline{s}''} y_{\underline{s}''}, \tag{2.2}$$

then

$$QM_{\underline{s}}(\underline{t})\, QM_{\underline{s}'}(\underline{t}) = \sum_{\underline{s}''} QM_{\underline{s}''}(\underline{t}),$$

which means

$$\sum_{n_1>\cdots>n_k\geq 1} t_{n_1}^{s_1}\cdots t_{n_k}^{s_k} \sum_{n'_1>\cdots>n'_k\geq 1} t_{n'_1}^{s'_1}\cdots t_{n'_k}^{s'_k} = \sum_{\underline{s}''}\sum_{n''_1>\cdots>n''_k\geq 1} t_{n''_1}^{s''_1}\cdots t_{n''_k}^{s''_k}.$$

The star (stuffle) law gives an explicit way of writing the product of two quasi-symmetric series as a sum of quasi-symmetric series: from the definition of $\star$ it follows that in (2.2), $\underline{s}''$ runs over the tuples $(s''_1, \ldots, s''_{k''})$ obtained from $\underline{s} = (s_1, \ldots, s_k)$ and $\underline{s}' = (s'_1, \ldots, s'_{k'})$ by inserting, in all possible ways, some 0 in the string $(s_1, \ldots, s_k)$ as well as in the string $(s'_1, \ldots, s'_{k'})$ (including in front and at the end), so that the new strings have the same length k'', with $\max\{k, k'\} \le k'' \le k+k'$, and by adding the two sequences term by term. Here is an example:

$$\begin{array}{cccccccc} \underline{s} & s_1 & s_2 & 0 & s_3 & s_4 & \cdots & 0 \\ \underline{s}' & 0 & s'_1 & s'_2 & 0 & s'_3 & \cdots & s'_{k'} \\ \underline{s}'' & s_1 & s_2+s'_1 & s'_2 & s_3 & s_4+s'_3 & \cdots & s'_{k'}. \end{array}$$

Let QSym^0 be the subspace of QSym spanned by the $QM_{\underline{s}}(\underline{t})$ for which $s_1 \ge 2$. The restriction of ϕ to $\mathfrak{H}^0$ gives an isomorphism of K-algebra from $\mathfrak{H}^0$ to QSym^0. The specialization $t_n \to 1/n$ for $n \ge 1$ restricted QSym^0 maps $QM_{\underline{s}}$ onto $\zeta(\underline{s})$. Hence we have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{H} & & \\ \bigcup & & \\ \mathfrak{H}^1 & \overset{\phi}{\underset{\sim}{\longrightarrow}} & \mathrm{QSym} \\ \bigcup & & \bigcup \\ \mathfrak{H}^0 & \overset{\sim}{\longrightarrow} & \mathrm{QSym}^0 \\ \hat{\zeta}\Big\downarrow & \swarrow & \\ \mathbb{R} & & \end{array} \qquad\qquad \begin{array}{ccc} y_{\underline{s}} & \longmapsto & QM_{\underline{s}}(\underline{t}) \\ \downarrow & \swarrow & \\ \zeta(\underline{s}) & & \end{array}$$

Lemma 2.3. *The following syntaxic identity holds:*

$$y_2^* \star (-y_2)^* = (-y_4)^*.$$

Proof. From the definition of ϕ in Proposition 2.1 we have

$$\phi(y_2^*) = \sum_{k=0}^{\infty} \sum_{n_1 > \cdots > n_k \ge 1} t_{n_1}^2 \cdots t_{n_k}^2,$$

$$\phi\big((-y_2)^*\big) = \sum_{k=0}^{\infty} (-1)^k \sum_{n_1 > \cdots > n_k \ge 1} t_{n_1}^2 \cdots t_{n_k}^2$$

and

$$\phi\big((-y_4)^*\big) = (-1)^k \sum_{n_1 > \cdots > n_k \ge 1} t_{n_1}^4 \cdots t_{n_k}^4.$$

Hence from the identity

$$\prod_{n=1}^{\infty} (1 + t_n t) = \sum_{k=0}^{\infty} t^k \sum_{n_1 > \cdots > n_k \ge 1} t_{n_1} \cdots t_{n_k} \tag{2.4}$$

one deduces

$$\phi(y_2^*) = \prod_{n=1}^{\infty} (1 + t_n^2), \quad \phi\big((-y_2)^*\big) = \prod_{n=1}^{\infty} (1 - t_n^2) \quad \text{et} \quad \phi\big((-y_4)^*\big) = \prod_{n=1}^{\infty} (1 - t_n^4),$$

which implies Lemma 2.3. □

We now prove the Zagier-Broadhurst formula.

Theorem 2.5. *For any $n \geq 1$,*

$$\zeta\big(\{3,1\}_n\big) = 4^{-n}\zeta\big(\{4\}_n\big).$$

This formula was originally conjectured by D. Zagier [Z 1994] and, according to [B² 1999], first proved by D. Broadhurst.

Remark. (See formulae (36) and (37) of [B³ 1997], (3) of [B² 1999], example 6.3 of [B³L 2001]) Since

$$\zeta\big(\{2\}_n\big) = \frac{\pi^{2n}}{(2n+1)!}$$

(see (2.6) below) and

$$\frac{1}{2n+1}\zeta\big(\{2\}_{2n}\big) = \frac{1}{2^{2n}}\zeta\big(\{4\}_n\big).$$

one deduces

$$\zeta\big(\{3,1\}_n\big) = 2 \cdot \frac{\pi^{4n}}{(4n+2)!}.$$

Proof Here is the proof by Hoang Ngoc Minh [M 2000] using syntaxic identities. Theorem 2.5 can be formulated as

$$y_4^n - (4y_3y_1)^n \in \ker \widehat{\zeta}.$$

From Lemma 2.3

$$y_2^* \star (-y_2)^* = (-y_4)^*$$

and identities 1.1 of fasc.3

$$y_2^* ш (-y_2)^* = (-4y_3y_1)^*$$

one deduces, for any $n \geq 1$,

$$\sum_{i+j=2n} (-1)^j y_2^{2i} \star y_2^{2j} = (-y_4)^n$$

and

$$\sum_{i+j=2n} (-1)^j y_2^{2i} ш y_2^{2j} = (-4y_3y_1)^n,$$

hence

$$y_4^n - (4y_3y_1)^n = \sum_{i+j=2n} (-1)^{n-j}\big(y_2^{2i} \star y_2^{2j} - y_2^{2i} ш y_2^{2j}\big) \in \ker \widehat{\zeta}.$$

□

Remark. From the proof just given one deduces

$$\zeta(\{4\}_n) = 4^n\zeta(\{3,1\}_n) = \sum_{i+j=2n} (-1)^{n-j}\zeta(\{2\}_{2i})\zeta(\{2\}_{2j}).$$

From

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right)$$

one deduces the generating series for the numbers $\zeta\bigl(\{2\}_k\bigr)$, namely

$$\sum_{k\geq 0}\zeta\bigl(\{2\}_k\bigr)(-z^2)^k = \frac{\sin(\pi z)}{\pi z}\cdot$$

This provides a closed formula for these numbers:

(2.6) $$\zeta\bigl(\{2\}_k\bigr) = \frac{\pi^{2k}}{(2k+1)!}\cdot$$

`Remark.` Other proofs of Theorem 2.5 are given in [B^{3}L 1998] and [B^{3}L 2001]§ 11.2). The modification of Broadhurst's proof which we give here is taken from [B^{3}L 2001]. We start with the right hand side. We introduce the generating function

$$F(t) = \sum_{n\geq 0} 2\cdot\frac{\pi^{4n}t^{4n}}{(4n+2)!}\cdot$$

Since

$$1+(-1)^k - i^k - (-i)^k = \begin{cases} 0 & \text{if } k\equiv 0,1,-1 \pmod 4\\ 4 & \text{if } k\equiv 2 \pmod 4,\end{cases}$$

we have

$$\begin{aligned} F(t) &= \frac{1}{2}\sum_{k\geq 0}\frac{\pi^{k-2}t^{k-2}}{k!}\cdot\bigl(1+(-1)^k - i^k-(-i)^k\bigr)\\ &= \frac{1}{2\pi^2t^2}\bigl(e^{\pi t}+e^{-\pi t}-e^{i\pi t}-e^{-i\pi t}\bigr)\\ &= \frac{1}{\pi^2t^2}\bigl(\cosh(\pi t)-\cos(\pi t)\bigr)\\ &= G(u)G(\overline{u}),\end{aligned}$$

where

$$G(u) = \frac{\sin(\pi u)}{\pi u} \quad \text{et} \quad u = \frac{1}{2}t(1+i), \quad \overline{u} = \frac{1}{2}t(1-i).$$

From *Gauss relation*:

$${}_2F_1\left(\begin{matrix}\alpha\ ,\ \beta\\ \gamma\end{matrix}\Bigm|1\right) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

if the real part of $\gamma-\alpha-\beta$ is positive, one deduces

$$G(u) = \frac{1}{\Gamma(1-u)\Gamma(1+u)} = {}_2F_1\left(\begin{matrix}u\ ,\ -u\\ 1\end{matrix}\Bigm|1\right).$$

Therefore the conclusion of Theorem 2.5 can be written

$$\sum_{n\geq 0}\zeta(\{3,1\}_n)t^{4n} = \left|{}_2F_1\left(\begin{matrix}u\,,\,-u\\1\end{matrix}\Big|1\right)\right|^2 = \frac{1}{\pi^2u^2}\big(\cosh(\pi u)-\cos(\pi u)\big) \tag{2.8}$$

with $u = t(1+i)/2$ as before. The relation (2.8) will follow, by specializing $z = 1$, from the more general formula ([B^{3}L 2001], Theorem 11.1)

$$\sum_{n\geq 0}\mathrm{Li}_{\{3,1\}_n}(z)t^{4n} = {}_2F_1\left(\begin{matrix}u\,,\,-u\\1\end{matrix}\Big|z\right)\cdot{}_2F_1\left(\begin{matrix}\overline{u}\,,\,-\overline{u}\\1\end{matrix}\Big|z\right) \tag{2.7}$$

which holds for $|z|\leq 1$. One checks (2.7) as follows: first one expands the two sides as series in z and see that they match up to order 4:

$$1+\frac{t^4}{8}z^2+\frac{t^4}{18}z^3+\frac{t^8+44t^4}{1536}z^4+\cdots$$

Finally one checks that both sides of (2.7) are annihilated by the differential operator

$$\left((1-z)\frac{d}{dz}\right)^2\cdot\left(z\frac{d}{dz}\right)^2 - t^4.$$

$\square$

Following [C 2001], we deduce from (2.6) the rationality of $\zeta(2k)/\pi^{2k}$, by means of the *Newton's formulae* which relate the symmetric series

$$M_s = M_s(\underline{t}) = \sum_{n\geq 1} t_n^s \quad (s\geq 1)$$

to the quasi-symmetric ones

$$\lambda_k(\underline{t}) = QM_{\{1\}_k}(\underline{t}) = \sum_{n_1>\cdots>n_k\geq 1} t_{n_1}\cdots t_{n_k},$$

namely:

Lemma 2.9. *For $k\geq 1$,*

$$M_k = \sum_{j=1}^{k-1}(-1)^{j+1}\lambda_j M_{k-j} + (-1)^{k+1}k\lambda_k.$$

Consider the morphism of algebras $\widetilde{\phi}:\mathrm{QSym}\to\mathbb{R}$ which maps t_n onto $1/n^2$. Clearly we have, for $k\geq 1$,

$$\widetilde{\phi}(\lambda_k) = \zeta(\{2\}_k) \quad \text{et} \quad \widetilde{\phi}(M_k) = \zeta(2k).$$

Hence Lemma 2.9 implies

$$\zeta(2k)=\sum_{j=1}^{k-1}(-1)^{j+1}\zeta(\{2\}_j)\zeta(2k-2j)+(-1)^{k+1}k\zeta(\{2\}_k).$$

Using (2.6) one deduces by induction

$$\zeta(2k)\pi^{-2k}\in\mathbb{Q}.$$

For instance from

$$M_2=\lambda_1M_1-2\lambda_2,\quad M_3=\lambda_1M_2-\lambda_2M_1+3\lambda_3,$$

$$M_4=\lambda_1M_3-\lambda_2M_2+\lambda_3M_1-4\lambda_4$$

we derive

$$\zeta(4)=\zeta(2)^2-2\zeta(2,2),\quad \zeta(6)=\zeta(2)\zeta(4)-\zeta(2,2)\zeta(2)+3\zeta(2,2,2)$$

and

$$\zeta(8)=\zeta(2)\zeta(6)-\zeta(2,2)\zeta(4)+\zeta(2,2,2)\zeta(2)-4\zeta(2,2,2,2),$$

which yields

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\quad \zeta(6)=\frac{\pi^6}{945},\quad \zeta(8)=\frac{\pi^8}{9450}.$$

Notice also the relations

$$M_{\{1\}_k}=\lambda_1^k\quad\text{et}\quad QM_{\{1\}_k}=\lambda_k.$$

3. The Harmonic Algebra of Multiple Polylogarithms

We shall use another case of the harmonic $\star$ product, on the free algebra $K<\mathcal{Y}>$ on the alphabet $\mathcal{Y}$ of pairs (s,z) with s a positive integer and z a complex number satisfying $|z|\le 1$. It will be convenient to write the elements in $\mathcal{Y}^*$ (the words) as $\binom{s_1,\dots,s_k}{z_1,\dots,z_k}$, or simply $\binom{\underline{s}}{\underline{z}}$, which means that the concatenation of $\binom{\underline{s}}{\underline{z}}$ and $\binom{\underline{s}'}{\underline{z}'}$ is denoted by $\binom{\underline{s},\underline{s}'}{\underline{z},\underline{z}'}$. For instance

$$\binom{s_1}{z_1}\binom{s_2}{z_2}=\binom{s_1,s_2}{z_1,z_2}.$$

The star product on the corresponding set of polynomials $K\langle\mathcal{Y}\rangle$ is defined inductively by

$$e\star w=w\star e=w$$

for any $w\in\mathcal{Y}^*$ and

$$(3.1)\quad (\binom{s}{z}u)\star(\binom{s'}{z'}v)=\binom{s}{z}(u\star\binom{s'}{z'}v)+\binom{s'}{z'}(\binom{s}{z}u\star v)+\binom{s+s'}{zz'}(u\star v)$$

for $u \in \mathcal{Y}^*$, $s \geq 1$ and $z \in \mathbb{C}$. This star product may be described as follows: start with $\binom{\underline{s}}{\underline{z}}$ and $\binom{\underline{s}'}{\underline{z}'}$ in $\mathcal{Y}^*$. Write

$$y_{\underline{s}} \star y_{\underline{s}'} = \sum_{\underline{s}''} y_{\underline{s}''},$$

as in (2.2). Then

$$\begin{pmatrix} \underline{s} \\ \underline{z} \end{pmatrix} \star \begin{pmatrix} \underline{s}' \\ \underline{z}' \end{pmatrix} = \sum_{\underline{s}''} \begin{pmatrix} \underline{s}'' \\ \underline{z}'' \end{pmatrix},$$

where the component z''_i is z_j if the corresponding s''_i is just a s_j (corresponding to a 0 in $\underline{s}'$), it is z'_ℓ if the corresponding s''_i is just a s'_ℓ (corresponding to a 0 in $\underline{s}$), and finally it is $z_j z'_\ell$ if the corresponding s''_i is a $s_j + s'_\ell$. Here is an example:

$$\begin{array}{cccccccc} \underline{s} & s_1 & s_2 & 0 & s_3 & s_4 & \cdots & 0 \\ \underline{s}' & 0 & s'_1 & s'_2 & 0 & s'_3 & \cdots & s'_{k'} \\ \underline{s}'' & s_1 & s_2 + s'_1 & s'_2 & s_3 & s_4 + s'_3 & \cdots & s'_{k'} \\ \underline{z}'' & z_1 & z_2 z'_1 & z'_2 & z_3 & z_4 z'_3 & \cdots & z'_{k'}. \end{array}$$

For instance

$$\begin{pmatrix} s \\ z \end{pmatrix} \star \begin{pmatrix} s' \\ z' \end{pmatrix} = \begin{pmatrix} s, s' \\ z, z' \end{pmatrix} + \begin{pmatrix} s + s' \\ zz' \end{pmatrix} + \begin{pmatrix} s', s \\ z', z \end{pmatrix}.$$

Also

$$\begin{pmatrix} s \\ z \end{pmatrix} \star \begin{pmatrix} s'_1, s'_2 \\ z'_1, z'_2 \end{pmatrix} = \begin{pmatrix} s, s'_1, s'_2 \\ z, z'_1, z'_2 \end{pmatrix} + \begin{pmatrix} s + s'_1, s'_2 \\ zz'_1, z'_2 \end{pmatrix} + \begin{pmatrix} s'_1, s, s'_2 \\ z'_1, z, z'_2 \end{pmatrix} + \begin{pmatrix} s'_1, s + s'_2 \\ z'_1, zz'_2 \end{pmatrix} + \begin{pmatrix} s'_1, s'_2, s \\ z'_1, z'_2, z \end{pmatrix}.$$

4. Multiple Polylogarithms in Several Variables and Stuffle

The functions of k complex variables (*)

$$\mathrm{Li}_{\underline{s}}(z_1, \ldots, z_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}$$

(*) Our notation for

$$\mathrm{Li}_{(s_1, \ldots, s_k)}(z_1, \ldots, z_k),$$

also used for instance in [C 2001], corresponds to Goncharov's notation [G 1997, G 1998] for

$$\mathrm{Li}_{(s_k, \ldots, s_1)}(z_k, \ldots, z_1).$$

have been considered as early as 1904 by N. Nielsen [N 1904], and rediscovered later by A.B. Goncharov [G 1997, G 1998]. Recently, J. Écalle [É 2000] used them for z_i roots of unity (in case $s_1 \geq 2$): these are the *decorated multiple polylogarithms*. Of course one recovers the one variable functions $\mathrm{Li}_{\underline{s}}(z)$ by specializing $z_2 = \cdots = z_k = 1$. For simplicity we write $\mathrm{Li}_{\underline{s}}(\underline{z})$, where $\underline{z}$ stands for $(z_1, \ldots, z_k)$. There is an integral formula for them which extends the relation (see fascicule 3)

$$\mathrm{Li}_{\underline{s}}(z) = \int_0^z \omega_{\underline{s}}.$$

To start with, in

$$\mathrm{Li}_s(z) = \int_0^z \omega_0^{s-1}\omega_1$$

we replace each integration variable t_i by $t_i' = t_i z$, which amounts to replace the differential $\omega_1(t) = dt/(1-t)$ by $zdt/(1-zt)$ and the Chen integration $\int_0^z$ by $\int_0^1$:

$$\mathrm{Li}_s(z) = \int_0^1 \omega_0^{s-1}\frac{zdt}{1-zt}.$$

It will be convenient to define

$$\omega_z(t) = \begin{cases} \dfrac{zdt}{1-zt} & \text{if } z \neq 0, \\ \dfrac{dt}{t} & \text{if } z = 0. \end{cases}$$

Hence, for $k = 1$ and $z \neq 0$,

$$\mathrm{Li}_s(z) = \int_0^1 \omega_0^{s-1}\omega_z.$$

We extend this formula to the multiple polylogarithms thanks to the differential equations

$$z_1 \frac{\partial}{\partial z_1}\mathrm{Li}_{\underline{s}}(\underline{z}) = \mathrm{Li}_{(s_1-1, s_2, \ldots, s_k)}(\underline{z})$$

for $s_1 \geq 2$, while for $s_1 = 1$

$$(1 - z_1)\frac{\partial}{\partial z_1}\mathrm{Li}_{(1, s_2, \ldots, s_k)}(\underline{z}) = \mathrm{Li}_{(s_2, \ldots, s_k)}(z_1 z_2, z_3, \ldots, z_k).$$

Hence

$$\mathrm{Li}_{\underline{s}}(\underline{z}) = \int_0^1 \omega_0^{s_1-1}\omega_{z_1}\omega_0^{s_2-1}\omega_{z_1 z_2}\cdots\omega_0^{s_k-1}\omega_{z_1\cdots z_k}. \tag{4.1}$$

Because of the occurrence of the products $z_1 \cdots z_j$ $(1 \leq j \leq k)$, Goncharov [G 1998] performs the change of variables

$$y_j = z_1^{-1}\cdots z_j^{-1} \quad (1 \leq j \leq k) \quad \text{et} \quad z_j = \frac{y_{j-1}}{y_j} \quad (1 \leq j \leq k)$$

with $y_0=1$. Set

$$\omega'_y(t)=-\omega_{y^{-1}}(t)=\frac{dt}{t-y},$$

so that $\omega'_0=\omega_0$ and $\omega'_1=-\omega_1$. Following the notation of [B³L 2001], we define

$$\begin{aligned}\lambda\binom{s_1,\dots,s_k}{y_1,\dots,y_k}&=\mathrm{Li}_{\underline{s}}(1/y_1,y_1/y_2,\dots,y_{k-1}/y_k)\\ &=\sum_{\nu_1\ge1}\cdots\sum_{\nu_k\ge1}\prod_{j=1}^{k}y_j^{-\nu_j}\left(\sum_{i=j}^{k}\nu_i\right)^{-s_j}.\\ &=(-1)^p\int_0^1\omega_0^{s_1-1}\omega'_{y_1}\cdots\omega_0^{s_k-1}\omega'_{y_k}.\end{aligned}\tag{4.2}$$

This is Theorem 2.1 of [G 1998] (see also [G 1997]). With this notation some formulae are simpler. For instance the shuffle relation is easier to write with λ: the shuffle is defined on words ω'_y ($y\in\mathbb{C}$, including $y=0$) by induction with (see § 1):

$$(\omega'_y u)ш(\omega'_{y'}v)=\omega'_y(uш\omega'_{y'}v)+\omega'_{y'}(\omega'_y uшv).$$

Hence the functions $\mathrm{Li}_{\underline{s}}(\underline{z})$ satisfy shuffle relations. Moreover they also satisfy *stuffle relations* arising from the product of two series. For this we use the star product defined in § 1 for the set $\mathcal{Y}$ of pairs (s,z) with $s\ge1$ and $|z|<1$, where the underlying field K is $\mathbb{C}$. It will be convenient to write $\mathrm{Li}\binom{\underline{s}}{\underline{z}}$ in place of $\mathrm{Li}_{\underline{s}}(\underline{z})$, and to extend the definition of Li by $\mathbb{C}$-linearity: for

$$S=\sum_{\binom{\underline{s}}{\underline{z}}\in\mathcal{Y}^*}\left(S\left|\binom{\underline{s}}{\underline{z}}\right.\right)\binom{\underline{s}}{\underline{z}}\in\mathbb{C}\langle\mathcal{Y}\rangle,$$

define

$$\mathrm{Li}(S)=\sum_{\binom{\underline{s}}{\underline{z}}\in\mathcal{Y}^*}\left(S\left|\binom{\underline{s}}{\underline{z}}\right.\right)\mathrm{Li}_{\underline{s}}(\underline{z}).$$

Then

$$\mathrm{Li}(u)\mathrm{Li}(v)=\mathrm{Li}(u\star v)\tag{4.3}$$

for any u and v in $\mathbb{C}\langle\mathcal{Y}\rangle$. These relations amount to

$$\mathrm{Li}\left(\binom{\underline{s}}{\underline{z}}\star\binom{\underline{s}'}{\underline{z}'}\right)=\mathrm{Li}\binom{\underline{s}}{\underline{z}}\,\mathrm{Li}\binom{\underline{s}'}{\underline{z}'}.$$

Example. For $k=1=k'=1$ we get

$$\mathrm{Li}_s(z)\mathrm{Li}_{s'}(z')=\mathrm{Li}_{(s,s')}(z,z')+\mathrm{Li}_{(s',s)}(z',z)+\mathrm{Li}_{s+s'}(zz').\tag{4.4}$$

For instance, for $s=1$, $s'=2$ and $z=z'$, we deduce

$$\mathrm{Li}_1(z)\mathrm{Li}_2(z)=\mathrm{Li}_{(1,2)}(z,z)+\mathrm{Li}_{(2,1)}(z,z)+\mathrm{Li}_3(z^2).$$

Here is another example with $k=1$ and $k'=2$:

$$\begin{aligned}\mathrm{Li}_s(z)\mathrm{Li}_{(s'_1,s'_2)}(z'_1,z'_2)=\mathrm{Li}_{(s,s'_1,s'_2)}(z,z'_1,z'_2)+\mathrm{Li}_{(s'_1,s,s'_2)}(z'_1,z,z'_2)+\mathrm{Li}_{(s'_1,s'_2,s)}(z'_1,z'_2,z)+\\ \mathrm{Li}_{(s+s'_1,s'_2)}(zz'_1,z'_2)+\mathrm{Li}_{(s'_1,s+s'_2)}(z'_1,zz'_2).\end{aligned} \tag{4.5}$$

We consider now the special case of the relations (4.3) when all coordinates of $\underline{z}$ and $\underline{z}'$ are set equal to 1. Recall the definition (§ 1) of the stuffle $\star$ on the set $\mathbb{Q}\langle x_0,x_1\rangle$ of polynomials in x_0,x_1.

The *second standard relations* between multiple zeta values are

$$\widehat{\zeta}(y_{\underline{s}}\star y_{\underline{s}'})=\widehat{\zeta}(y_{\underline{s}})\widehat{\zeta}(y_{\underline{s}'}) \tag{4.6}$$

whenever $s_1\ge 2$ and $s'_1\ge 2$.

For $k=k'=1$ this relation reduces to Nielsen Reflexion Formula

$$\zeta(s)\zeta(s')=\zeta(s,s')+\zeta(s',s)+\zeta(s+s').$$

In particular

$$\zeta(s)^2=2\zeta(s,s)+\zeta(2s)\quad\text{for}\quad s\ge 2;$$

for instance

$$\zeta(2,2)=\frac{1}{2}\zeta(2)^2-\frac{1}{2}\zeta(4)=\frac{\pi^2}{120}\cdot$$

Another example is given by (4.5) with $z=z'_1=z'_2=1$:

$$\zeta(s)\zeta(s'_1,s'_2)=\zeta(s,s'_1,s'_2)+\zeta(s'_1,s,s'_2)+\zeta(s'_1,s'_2,s)+\zeta(s+s'_1,s'_2)+\zeta(s'_1,s+s'_2)$$

for $s\ge 2$, $s'_1\ge 2$ and $s'_2\ge 1$.

Remark. The generating series for the multiple polylogarithms in several variables is the following

$$\sum_{s_1\ge 1}\cdots\sum_{s_1\ge 1}\mathrm{Li}_{\underline{s}}(\underline{z})t_1^{s_1-1}\cdots t_k^{s_k-1}=\sum_{n_1>\cdots>n_k\ge 1}\frac{z_1^{n_1}}{(n_1-t_1)}\cdots\frac{z_k^{n_k}}{(n_k-t_k)}\cdot$$

Compare with

$$\sum_{s_1\ge 1}\cdots\sum_{s_k\ge 1}\mathrm{Li}_{\underline{s}}(z)t_1^{s_1-1}\cdots t_k^{s_k-1}=\sum_{n_1>\cdots>n_k\ge 1}\frac{z^{n_1}}{(n_1-t_1)\cdots(n_k-t_k)}$$

for $k\ge 1$, $|z|<1$ and $|t_i|<1$ $(1\le i\le k)$.

A very general function worth to be considered is

$$\sum_{n_1>\cdots>n_k\geq 1} \frac{z_1^{n_1}}{(n_1-t_1)^{s_1}}\cdots\frac{z_k^{n_k}}{(n_k-t_k)^{s_1}}. \tag{4.7}$$

This function depends on complex variables $(z_1,\ldots,z_k)$, $(t_1,\ldots,t_k)$, and on positive integers $(s_1,\ldots,s_k)$ (one could even take complex numbers for $(s_1,\ldots,s_k)$). In the case $k=1$, this is Lerch function ([C 2001] formula (61)) which specializes to Hurwitz function ([C 2001] formula (56)) for $z_1=1$. For $k\geq 1$, if we specialize $t_1=\cdots=t_k=0$, we recover the multiple polylogarithms in several variables (hence also the multiple polylogarithms in only one variable, and therefore also the multiple zeta values). On the other hand if we specialize $z_1=\cdots=z_k=0$ in (4.7), we get Hurwitz multizeta functions which have been studied by Minh and Petitot, and have a double shuffle structure (shuffle products for series and for integrals).

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