Algorithmic and algebraic combinatorial aspects of polylogarithms with application on the computation of Drinfel’d associators

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Polylogarithms and polyzetas occur in

• Control theory (Lille).

• Analysis combinatorics (Flajolet, Labelle, Laforest, Salvy, Vallée, …).

• Vassiliev knot invariants & Drinfel’d associator (Kontsevich, González-Lorka, Lê, Murakami, Furusho, Racinet, …).

• Perturbative quantum field theory (Broadhurst, Kreimer, …).

• Chern classes of a manifold (Hoffman, …).

• K-theory (Gangl, Wojtkowiak, Zagier, …).

• Irrationality & transcendence of \(\zeta(2k + 1)\) (Borwein, Ecalle, Goncharov, Zagier, …).

Knizhnik–Zamolodchikov equation \(KZ_3\)

Drinfel’d constructed the solutions (80s) for the Knizhnik–Zamolodchikov equation \(KZ_3\)

\[
\frac{dG(z)}{dz} = \frac{1}{2i\pi} \left( \frac{A}{z} + \frac{B}{z-1} \right) G(z), \quad 0 < z < 1,
\]

where \(A, B\) are noncommuting symbols, and \(G(z)\) is a formal power series in \(A, B\) with coefficients that are analytic function of \(z\).

\[
G_1(z) \sim z^{A/2i\pi} = e^{A/2i\pi \log(z)}, \quad z \to 0,
G_2(z) \sim (1-z)^{B/2i\pi} = e^{B/2i\pi \log(1-z)}, \quad z \to 1,
\]

\[
\Phi_{KZ}(A,B) = G_2(z)^{-1}G_1(z).
\]

\[
G_1(1-z) \sim z^{B/2i\pi}\Phi_{KZ}(A,B), \quad z \to 0.
\]

Question 1 How to compute the associator \(\Phi_{KZ}(A,B)\)?
Drefel’d associators & $\Phi_{KZ}(A,B)$

Drefel’d defined associator $\Phi(A,B)$ as a Lie exponential satisfying 3 relations:

1. Duality: $\Phi(B,A) = \Phi^{-1}(A,B)$.
2. Hexagonal: $\ldots$
3. Pentagonal: $\ldots$

$\Phi_{KZ} \in \text{MZV} \langle A,B \rangle$ (Lê & Murakami), where

$$\text{MZV} = \left\{ \zeta(s_1, \ldots, s_k) = \sum_{n_1 > \ldots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \right\}.$$ 

Drefel’d proved also the existence of associators with rational coefficients.

**Question 2** How to compute the rational associators?

**Non-commutative formal power series**

$X^*$: the free monoid generated by an alphabet $X$ for the concatenation with $\epsilon$ (the empty word) as the neutral element.

A formal power series $S$ is an infinite sum

$$S = \sum_{w \in X^*} \langle S \rangle[w].$$

A finite FPS is called polynomial.

Let $x, y \in X, u, v \in X^*$, $x u \omega y v$ is the polynomial defined recursively as follows

$$x u \omega \epsilon = \epsilon \omega x u = x u,$$

$$x u \omega y v = y [x u \omega v] + x [u \omega (y v)].$$

Example $-$ $x_0 x_1 \omega x_0 x_1 = 4 x_0^2 x_1^2 + 2 x_0 x_1 x_0 x_1$.

$\mathbb{C} \langle X \rangle$, $\mathbb{C}(X)$ denote the sets of FPS and polynomials over $X$ and with coefficients in $\mathbb{C}$.

**Operations on formal power series**

For $S, T \in \mathbb{C} \langle X \rangle$, one defines

$$\forall w \in X^*, \langle S + T \rangle[w] = \langle S \rangle[w] + \langle T \rangle[w],$$

$$\forall w \in X^*, \langle ST \rangle[w] = \sum_{u, v \in X^*} \langle S \rangle[v] \langle T \rangle[u],$$

$$S \cup T = \sum_{u, v \in X^*} \langle S \rangle[u] \langle T \rangle[u \cup v].$$

$\text{Sh}_\mathbb{C}(X)$ denotes the polynomial algebra equipped the shuffle product $\cup$.

The exponentiel of $S$ is the sum

$$\text{exp}(S) = \sum_{k \geq 0} \frac{S^k}{k!}.$$

The logarithm of $1 + S$ is the sum

$$\log(1 + S) = \sum_{k \geq 0} (-1)^{k+1} \frac{S^k}{k}.$$
Theorem 1 (Radford) The $\mathbb{C}$-algebra $\text{Sh}_C(X)$ is the polynomial algebra generated by $\text{Lyn}(X)$.

The free Lie algebra

The free Lie algebra, noted by $\text{Lie}_C(X)$, is the $\mathbb{C}$-algebra of polynomials, over $X$, equipped the bracket $[,]$ defined as follows

$$\forall P, Q \in \text{Lie}_C(X), \quad [P, Q] = PQ - QP$$

and verifying the following properties

$$[P, P] = 0,$$

$$[P, [Q, R]] + [Q, [R, P]] + [R, [P, Q]] = 0.$$

An element of $\text{Lie}_C(X)$ is called Lie polynomial.

Let $S \in \mathcal{C}(X)$, $S$ is called Lie series if it can be written as follows

$$S = \sum_{k \geq 1} P_k,$$

where $P_k$ is a homogeneous Lie polynomial of degree $k$. $\text{Lie}_C(\mathcal{X})$ denotes the set of Lie series over $X$.

PBW basis and dual basis

The bracket form $P_l$ of a Lyndon word $l$ is defined recursively by

$$P_l = [P_u, P_v] = P_u P_v - P_v P_u \quad \text{for } u = \text{st}(l), \quad P_v = x \quad \text{for } x \in X,$$

The set $\{P_l; l \in \text{Lyn}(X)\}$ is a basis for the free Lie algebra $\text{Lie}_C(X)$.

The PBW basis $B = \{P_w; w \in X^*\}$ is obtained by setting

$$P_v = P_{l_1} P_{l_2} \ldots P_{l_k}, \quad v = l_{1}^{i_1} \ldots l_{k}^{i_k}, l_1 > \ldots > l_k,$$

and its dual basis $B^*$ is obtained by setting

$$S_l = x S_w, \quad \forall l \in \text{Lyn}(X), l = xw, x \in X, w \in X^*, \quad S_w = \frac{S_{l_1^{i_1} \cdots l_k^{i_k}}}{i_1! \ldots i_k!}, w = l_{1}^{i_1} \ldots l_{k}^{i_k}, l_1 > \ldots > l_k.$$

Example

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P_t$</th>
<th>$S_t = P_t^{-1}$</th>
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<tr>
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<td>$x_{01}$</td>
<td>$[x_0, x_1]$</td>
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<td>$x_{21}$</td>
<td>$[x_0, [x_0, x_1]]$</td>
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Factorization Hopf algebra

\( \mathbb{C}(X) \otimes \mathbb{C}(X) \) denotes the tensorial product of \( \mathbb{C}(X) \) with itself. The co-product \( \Phi \) of the concatenation is defined as follows

\[
\forall u, v \in X^*, \quad \langle \Phi(w|u \otimes v) \rangle = \langle w|u \otimes v \rangle
\]

\[
\iff \quad \Phi(w) = \sum_{u, v \in X^*, w = uv} u \otimes v.
\]

\( \Phi \) is an morphism for the shuffle algebra :

\[
\forall u, v \in X^*, \quad \Phi(u \cdot v) = \Phi(u) \cdot \Phi(v),
\]

A co-unity \( e \) is defined by :

\[
e: \mathbb{C}(X) \rightarrow \mathbb{C}(X),
\]

\[
P \mapsto e(P) = \langle P|e \rangle.
\]

For \( S \in \mathbb{C}(X) \), the antipode of \( S \) is the following FPS \( (\bar{w} \) denotes the mirror of \( w \))

\[
a(S) = \sum_{w \in X^*} (-1)^{|w|} \langle S|w \rangle \bar{w}.
\]

\((\mathbb{C}(X), \cdot, 1, \Phi, e, a)\) is the factorization Hopf algebra.

Decomposition Hopf algebra

The map \( \Gamma_2 \) is defined as follows

\[
\forall u, v, w \in X^*, \quad \langle \Gamma_2w|u \otimes v \rangle = \langle w|u \cdot v \rangle.
\]

In particular

\[
\forall x \in X, \quad \Gamma_2x = 1 \otimes x + x \otimes 1.
\]

It is extended to \( \mathbb{C}(\langle X \rangle) \) as follows

\[
\langle \Gamma_2S|u \otimes v \rangle = \sum_{w \in X^*} \langle S|w \rangle \Gamma_2w = \langle S|u \cdot v \rangle.
\]

\( \Gamma_2 \) is an morphism for the associative algebra :

\[
\forall u, v \in X^*, \quad \Gamma_2(uv) = \Gamma_2(u) \Gamma_2(v).
\]

And it is a co-associative coproduct

\((\mathbb{C}(\langle X \rangle), 1, \Gamma_2, e, a)\) is the decomposition Hopf algebra.

Primitive and group-like

Let \( S \in \mathbb{C}(\langle X \rangle) \), \( S \) is called primitive if

\[
\Gamma_2S = 1 \otimes S + S \otimes 1.
\]

\( S \) is called group-like if

\[
\Gamma_2S = S \otimes S.
\]

\( S \) verifies the Friedrichs criterion if

\[
\forall u, v \in X^*, \quad \langle S|u \cdot v \rangle = \langle S|u \rangle \langle S|v \rangle.
\]

Theorem 2 (Ree)

\[
S \in \text{Lie}_{\mathbb{C}}(\langle X \rangle)
\]

\[
\iff S \text{ is primitive}
\]

\[
\iff e^S \text{ is group-like}
\]

\[
\iff e^S \text{ verifies the Friedrichs criterion.}
\]

Diagonal series & Schützenberger factorization*

Let us consider, in the completed tensorial product \( \mathbb{C}(X) \hat{\otimes} \mathbb{C}(X) \), the following operation : the shuffle product for the left factor, the concatenation for right factor (for \( u_1, v_1, u_2, v_2 \in X^* \)):

\[
(u_1 \otimes v_1)(u_2 \otimes v_2) = (u_1 \cdot u_2) \otimes (v_1 \cdot v_2).
\]

By a Schützenberger factorization, the following diagonal series in \( \mathbb{C}(X) \hat{\otimes} \mathbb{C}(X) \)

\[
D = \sum_{w \in X^*} w \otimes w
\]

can be factorized in an infinite product, indexed by the Lyndon words :

\[
D = e^{x_1 \otimes x_1} \left[ \prod_{I \in \text{Ly}_X} e^{P_I} \otimes P_I \right] e^{x_0 \otimes x_0}
\]

\[
\iff \prod_{I \in \text{Ly}_X} e^{P_I} \otimes P_I = e^{-x_1 \otimes x_1} D e^{-x_0 \otimes x_0}.
\]

From iterated integrals to words

The iterated integrals over the differential forms \( \{ \omega_0(z), \ldots, \omega_n(z) \} \) can be encoded by the words \( w = x_{i_1} \ldots x_{i_k} \) over \( X = \{ x_0, \ldots, x_n \} \) (Fliess):

\[
\alpha_z^\omega(w) = \int_{t_0}^z \omega_i(t_1) \ldots \omega_i(t_k) = \begin{cases} 1 & \text{if } w = \epsilon, \\ \int_{t_0}^z \omega_i(t) \alpha_{z_0}^\omega(v) & \text{if } w = x_iv. \end{cases}
\]

Remark — In control theory, Fliess takes the differential forms \( \omega_i \) that are in the form \( a_i(t)dt \), where \( a_i(t) \) are real piecewise continuous.

\( \alpha \) is a \( C \)-algebra morphism for \( \omega \) (Chen):

\[
\alpha : Sh_C(X) \to \{ \text{comb. of iterated integrals, } +, \cdot, \}. 
\]

\[
\forall u, v \in X^* \setminus \{ \epsilon \}, \quad \alpha_z^\omega(u + v) = \alpha_z^\omega(u) + \alpha_z^\omega(v), \\
\forall \lambda \in C, u \in X^*, \quad \alpha_z^\omega(\lambda u) = \lambda \alpha_z^\omega(u), \\
\forall u, v \in X^*, \quad \alpha_z^\omega(u \cdot v) = \alpha_z^\omega(u) \alpha_z^\omega(v).
\]

Iterated integral and shuffle algebra

Let us consider the following differential forms

\[
\omega_0(z) = \frac{dz}{z} \quad \text{and} \quad \omega_2(z) = dz.
\]

Example — Note that

\[
x_0 \omega_2 = x_0x_2 + x_2x_0.
\]

But \( \alpha_0^\omega(x_0 \omega_2) = \alpha_0^\omega(x_0) \alpha_0^\omega(x_2) \) and \( \alpha_0^\omega(x_2 \omega_0) \) diverge while \( \alpha_0^\omega(x_2) = z. \)

Example — For any \( n \geq 0 \), one has

\[
\alpha_0^\omega(x_0^n x_2) = \alpha_0^\omega(x_2) = z.
\]

Theorem 3 (FPSAC98) For \( \omega_0 = dz/z, \omega_1 = dz/(1 - z) \), \( \alpha \) is injective from \( Sh_C(x_0, x_1) \) to the smallest algebra that contains \( C \) and that is stable under integration with respect to \( \omega_0, \omega_1. \)

Non-commutative g.s. of \( L_iw(z) \)

Definition 2 \( L(z) = \sum_{w \in X^*} L_iw(z) w. \)

Proposition 1 (FPSAC98) \( L(z) \) satisfies the differential equation (Drinfel’d equation):

\[
dL(z) = [x_0 \omega_0(z) + x_1 \omega_1(z)]L(z)
\]

with the boundary condition

\[
L(\varepsilon) = e^{x_0 \log \varepsilon} + o(\varepsilon) \quad \text{for} \quad \varepsilon \to 0^+. 
\]

Proof — (sketched) Observing that \( L(z) = 1 + \sum_{w \in X^*} L_iw(z) x_0 w + \sum_{w \in X^*} L_iw(z) x_1 w. \)

The exponential term \( e^{x_0 \log \varepsilon} \) comes from the definition of \( L_iw(z) x_0 \).

The coefficient of each other word \( w \) in \( L(\varepsilon) \) is easily seen to be bounded by \( o(\varepsilon^n \log^m \varepsilon) \), where \( n \) is the number of \( x_1 \)'s in \( w). \)
Solutions of Drinfel’d equation

Proposition 2 If \( G(z) \) and \( H(z) \) are solutions of Drinfel’d equation then
\[
d[H(z)^{-1}G(z)] = 0.
\]

Proof – Since \( H(z)H(z)^{-1} = 1 \) then
\[
[dH(z)]H(z)^{-1} = -H(z)[dH(z)^{-1}].
\]
Therefore if \( H(z) \) is solution then
\[
d[H(z)^{-1}] = -H(z)^{-1}[dH(z)]H(z)^{-1} = -H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)],
\]
\[
d[H(z)^{-1}G(z)] = [dH(z)^{-1}]G(z) + H(z)^{-1}[dG(z)]
   = -H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)]G(z) + H(z)^{-1}[x_0\omega_0(z) + x_1\omega_1(z)]G(z).
\]
We get then the expected result. □

Corollary 1 Let \( g_* \) be the substitution morphism defined by \( g_*x_0 = -x_1, g_*x_1 = -x_0 \). If \( H(z) \) is solution of Drinfel’d equation then
\[
d[H(z)^{-1}g_*H(1 - z)] = 0.
\]

\[L(z)\text{ is groupe like}\]

Theorem 4 (FPSAC98) \( \Delta L(z) = L(z) \otimes L(z) \).

Proof – (sketched) Intuitively speaking, it follows from the boundary condition and thus the limit at 0 of \( L(z) \) is a Lie exponential, and \( L(z) \) is a Lie exponential for any \( z \).

We have to prove \( T(z) = \Delta L(z) - L(z) \otimes L(z) \) vanishes for all \( z \). We claim that \( T \) satisfies
\[
dT(z) = (\Delta V(z)) T(z)dz,
\]
\[
\lim_{\varepsilon \to 0^+} T(z) = 0,
\]
where \( V(z) = [x_0\omega_0(z) + x_1\omega_1(z)] \). Thus we have a recursive formula for the coefficients of \( T(z) \) by means of differential equations with limit conditions in 0. Since these limits all vanish in 0, it follows by induction that the coefficients of \( T \) all vanish globally. □

Factorization of the g.s. \( L(z) \)

Corollary 2
\[\forall u, v \in X^*, \; Li_{u, v}(z) = Li_u(z) Li_v(z).\]

Proof – Use the Friedrichs criterion. □

Corollary 3
\[L(z) = e^{-\log(1-z)x_1}L_{\text{reg}}(z)e^{\log(z)x_0},\]
where
\[L_{\text{reg}}(z) = \prod_{i \in L_{\text{reg}}(X) \setminus \{x_0, x_1\}} \exp(L_i p_i(z) p_i).\]

Proof – Use the Schützenberger factorization. □

Asymptotic behaviour at \( z = 1 \)

Corollary 4 The asymptotic expansion of \( L(z) \) at \( z = 1 \) is given by:
\[L(1 - \varepsilon) \sim e^{-x_1 \log \varepsilon} L_{\text{reg}}(1) e^{x_0 \varepsilon}, \text{ for } \varepsilon \to 0^+.\]

Example – For \( \varepsilon \to 0^+ \), we have
\[Li_{x_0}(1 - \varepsilon) \sim \varepsilon \text{ and } Li_{x_1}(1 - \varepsilon) \sim -\log \varepsilon.\]
The Radford theorem gives
\[x_1^2 x_0 = x_0 x_1^2 - x_0 x_1 \omega x_1 + 1/2 x_0 \omega x_1^2.\]
Therefore
\[Li_{x_1^2 x_0}(1 - \varepsilon) \sim \zeta(2, 1) + \zeta(2) \log \varepsilon - \frac{1}{2} \varepsilon \log^2 \varepsilon + \ldots\]
\[\sim \zeta(3) + \zeta(2) \log \varepsilon - \frac{1}{2} \varepsilon \log^2 \varepsilon + \ldots\]
The last expression is obtained by use of the Euler’s identity \( \zeta(2, 1) = \zeta(3) \). □

In the other words, for any \( w \in X^*, \text{ for } \varepsilon \to 0^+, \)
\[Li_w(1 - \varepsilon) \sim \sum_{i \geq 1} Q_w(i)(\log \varepsilon)^i.\]
Non-commutative g.s. of polyzetas

Let \( \zeta_{\mu} = \zeta \circ \text{reg}_{\mu} \), where
\[
\text{reg}_{\mu} : \mathbb{C} \langle X \rangle \to \mathbb{C} \langle X \rangle,
\]
such that
\[
\forall w \in x_0 X^* x_1, \quad \text{reg}_{\mu} x_0 = \text{reg}_{\mu} x_1 = 0,
\]
\[
\forall u, v \in X^*, \quad \text{reg}_{\mu} u \text{ reg}_{\mu} v = \text{reg}_{\mu} u \text{ reg}_{\mu} v.
\]

Definition 3 \( Z = \sum_{w \in \{x_0, x_1\}} \zeta_{\mu}(w) w \).

Theorem 5 (FPSAC98)

\[
Z = L_{\text{reg}}(1) = \prod_{I \in \text{Ly}(X) \setminus \{x_0, x_1\}} \exp[\zeta(P_I) P_I].
\]

Proof — \( Z \) is the image by \( \zeta_{\mu} \otimes \text{id} \) of \( D \). \qed

Corollary 5 \( \forall u, v \in X^*, \zeta_{\mu}(u \text{ reg}_{\mu} v) = \zeta_{\mu}(u) \zeta_{\mu}(v) \).

Therefore, for any convergent words \( u \) and \( v \),
\( \zeta(u \text{ reg}_{\mu} v) = \zeta(u) \zeta(v) \).

Chen series & analytic continuation of \( L(z) \)

For a differentiable path \( \gamma : [0, 1] \to \mathbb{C} \setminus \{0, 1\} \) between \( a \) and \( b \), let \( S_\gamma \) be the evaluation at \( b \) of the solution of the differential equation
\[
\begin{aligned}
dS_\gamma(z) &= [x_0 \omega_0(z) + x_1 \omega_1(z)] S_\gamma(z), \\
nS_\gamma(\alpha) &= 1.
\end{aligned}
\]

\( S_\gamma \in \mathbb{C} \langle X \rangle \) is called the Chen series along \( \gamma \). \( S_\gamma \) is a Lie exponential and it depends only on the homotopy class of \( \gamma \) (Chen).

Proposition 3 (FPSAC98) Let \( x_0 \sim x \) be a differentiable path on \( \mathbb{C} \setminus \{0, 1\} \) s.t. \( L \) admits an analytic continuation. Then \( L(z) = S_{x_0 \sim z} L(x_0) \).

Proof — \( L(z) \) and \( S_{x_0 \sim z} L(x_0) \) satisfy the Drinfel'd equation taking the same value at \( x_0 \). \qed

Corollary 6
\[
S_{x \sim 1-\epsilon} \sim e^{-x_1 \log \epsilon} Z e^{-x_0 \log \epsilon} \quad \text{for} \quad \epsilon \to 0^+.
\]

Proof — \( S_{x \sim 1-\epsilon} = L(1-\epsilon)L(\epsilon)^{-1} \) and the behaviour of \( L \) lead to the expected result. \qed

\( Z \) and \( \log Z \) up to order 4 by computer

\[
\begin{aligned}
Z &= \ldots + \frac{2}{5} \zeta(2)^2 [x_0, [x_0, x_1]] + \zeta(3) [x_0, [x_0, x_1], x_1] + \ldots \\
\log Z &= \zeta(2) [x_0, x_1] + \zeta(3) [x_0, [x_0, x_1]] + [[x_0, x_1], x_1] + \ldots
\end{aligned}
\]

Monodromy of the g.s. \( L(z) \)

Paths of integration

Theorem 6 (FPSAC98) The monodromy of \( L(t) \) for \( t \in [0, 1] \) around 0 and 1 is given by
\[
\begin{aligned}
M_0 L(t) &= L(t) e^{2i\pi x_0}, \\
M_1 L(t) &= L(t) Z^{-1} e^{-2i\pi x_1} Z = L(t) e^{2i\pi m_1},
\end{aligned}
\]

where \( m_1 \) is a Lie series given by the formula
\[
m_1 = \prod_{I \in \text{Ly}(X) \setminus \{x_0, x_1\}} e^{-\zeta_I \text{ ad} P_I (-x_1)}.
\]

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Proof of the monodromy theorem

• Monodromy of \( L(z) \) around 0
  \[
  \mathcal{M}_0 L(t) = S_{t-\epsilon}S_{0}(z)S_{t+\epsilon}L(t),
  \]
  \[
  = L(t)L^{-1}(z)S_{0}(z)L(z),
  \]
  \[
  = L(t) \lim_{\epsilon \to 0^+} L^{-1}(z)S_{0}(z)L(z),
  \]
  \[
  = L(t) \lim_{\epsilon \to 0^+} e^{-\epsilon x_0 \log_e e^{2ix_0e^{2ix_0}}}.
  \]
  \[
  = L(t)e^{2ix_0}.
  \]

• Monodromy of \( L(z) \) around 1
  \[
  \mathcal{M}_1 L(t) = S_{1-\epsilon}S_{1}(z)S_{1+\epsilon}L(t),
  \]
  \[
  = L(t)L^{-1}(1-z)S_{1}(z)L(1-z),
  \]
  \[
  = L(t) \lim_{\epsilon \to 0^+} Z^{-1}e^{\epsilon x_1log_e e^{-2ix_1x_1e^{-x_1}}}.
  \]
  \[
  = L(t)Z^{-1}e^{-2ix_1x_1} Z.
  \]

Using the expression of \( Z \) and the formula
  \[
  e^{a}e^{-a} = e^{-a},
  \]
  we get finally the expression for \( m_1 \).

Monodromy around \( z = 1 \) (for \( p = 2i\pi \))

\[
\mathcal{M}_1 Lx_0 = Lx_0
\]
\[
\mathcal{M}_1 Lx_1 = Lx_1 - p
\]
\[
\mathcal{M}_1 Lx_0x_1 = Lx_0x_1 - pLx_0
\]
\[
\mathcal{M}_1 Li_2 x_0^2 = Li_2 x_0^2 - \frac{p^2}{2} Li_0^2
\]
\[
\mathcal{M}_1 Li_3 x_0^3 = Li_3 x_0^3 - \frac{p^3}{6} Li_0^3
\]
\[
\mathcal{M}_1 Li_2 x_0^2 = Li_2 x_0^2 - \frac{p^2}{4} Li_0^2 + p\zeta(x_0 x_1)
\]
\[
\mathcal{M}_1 Li_3 x_0^3 = Li_3 x_0^3 + \frac{p^3}{6} Li_0^3 - p\zeta(x_0 x_1)
\]
\[
\mathcal{M}_1 Li_4 x_0^4 = Li_4 x_0^4 + \frac{p^4}{24} Li_0^4
\]

The series \( m_1 \) up to order 6 by computer

\[
m_1 = -[x_1] + \zeta(x_0 x_1)[x_0 x_1^2] + \zeta(x_0 x_1)[x_0 x_1^2] +
\]
\[
\zeta(x_0 x_1)[x_0 x_1^2] + \zeta(x_0 x_1)[x_0 x_1^2] -
\]
\[
\zeta(x_0 x_1)[x_0 x_1^2] + \zeta(x_0 x_1)[x_0 x_1^2] +
\]
\[
\zeta(x_0 x_1)[x_0 x_1^2] + \zeta(x_0 x_1)[x_0 x_1^2] +
\]
\[
\zeta(x_0 x_1)[x_0 x_1^2] + \zeta(x_0 x_1)[x_0 x_1^2] +
\]
\[
\zeta(x_0 x_1)[x_0 x_1^2] + \zeta(x_0 x_1)[x_0 x_1^2] +
\]

Structure of the monodromy group

Corollary 7 Monodromy of \( Li_w \) is given by

\[
\forall w \in X^*, \quad \mathcal{M}_0 Li_{wx_0} = Li_{wx_0} + 2i\pi Li_w + \cdots
\]
\[
\mathcal{M}_1 Li_{wx_1} = Li_{wx_1} - 2i\pi Li_w + \cdots
\]

The remaining terms are combinations of polylogarithms encoded by words of length < \( |w| \).

Proof – The monodromy theorem implies

\[
M_0 = e^{2i\pi m_0} = 1 + 2i\pi x_0 + \text{words of length > 1}
\]
\[
M_1 = e^{2i\pi m_1} = 1 - 2i\pi x_1 + \text{words of length > 1}
\]

Corollary 8 The monodromy group of \( Li_w \) for \( |w| \leq n \) is nilpotent at order \( n + 1 \).

Proof – \( M_0 = e^{2i\pi x_0} \) and \( M_1 = e^{-2i\pi x_1} \). From \( e^{A}e^{B} = e^{A+B} \), it follows that the commutator \( M_0 M_1 M_0^{-1} M_1^{-1} \) does not contain any Lie brackets of length 1. Iterating this computation, the brackets of lengths 2, next 3, etc. until \( n \) disappear. \( \square \)
A structure theorem

Theorem 7 (FPSAC98) \(\text{The polylogarithms are linearly independent.}\)

Proof – This is trivial for \(n = 0\). Assume that we have proved our assertion for all \(k, 0 \leq k \leq n - 1\). For \(k = n\),

\[
\sum_{|w| \leq n} \lambda_w L_i w = 0
\]

\[
\iff \lambda_1 + \sum_{|u| = n} \lambda_{ux} L_i u_0 + \sum_{|u| < n} \lambda_{ux1} L_i u_1 = 0.
\]

(the \(\lambda_w\) are elements of \(\mathbb{C}\)). Applying \((\mathcal{M}_0 - \text{Id})\), we have

\[
\begin{cases}
2i\pi \sum_{|u| = n-1} \lambda_{ux0} L_i u + \sum_{|u| < n-1} \mu_u L_i u = 0, \\
2i\pi \sum_{|u| = n-1} \lambda_{ux1} L_i u + \sum_{|u| < n-1} \nu_u L_i u = 0.
\end{cases}
\]

By the induction hypothesis, we get the expected result. \(\square\)

\(L(1-t)\) by computer

\[
\begin{align*}
\text{Li}_1(1-t) &= -\log(t) \\
\text{Li}_2(1-t) &= -\text{Li}_2(t) + \log(t)\text{Li}_1(t) + \zeta(2) \\
\text{Li}_3(1-t) &= -\text{Li}_{2,1}(t) + \text{Li}_1(t)\text{Li}_2(t) \\
&\quad -\frac{1}{2}\log(t)\text{Li}_1(t)^2 \\
&\quad -\zeta(2)\text{Li}_1(t) + \zeta(2) \\
\text{Li}_{2,1}(1-t) &= -\text{Li}_3(t) + \log(t)\text{Li}_2(t) \\
&\quad -\frac{1}{2}\log(t)\text{Li}_1(t)^2 \\
&\quad -\zeta(2)\text{Li}_1(t) + \zeta(3) \\
\text{Li}_4(1-t) &= -\text{Li}_{2,1,1}(t) + \text{Li}_1(t)\text{Li}_{2,1}(t) \\
&\quad -\frac{1}{2}\text{Li}_1(t)^2\text{Li}_2(t) \\
&\quad +\frac{1}{6}\log(t)\text{Li}_1(t)^3 \\
&\quad +\frac{1}{4}\zeta(2)\text{Li}_1(t)^2 \\
&\quad -\zeta(3)\text{Li}_1(t) + \frac{1}{2}\zeta(2) \\
\end{align*}
\]

Duality relation

Proposition 5 Let \(\tau\) be the composition of the mirror morphism and of the involutive substitution morphism \(x_0 \to x_1\) and \(x_1 \to x_0\). Then

\[
Z = \tau(Z).
\]

Proof – For \(t \in ]0,1]\), one has

\[
S_{t \to 1-t}(x_0, x_1) = S_{1-t \to t}(-x_1, -x_0) \\
= S_{1-t \to t}(-x_1, -x_0) \\
= \tau[S_{1-t \to t}(x_0, x_1)].
\]

By the renormalisation

\[
S_{t \to 1-t} \sim e^{-x_1 \log t} e^{-x_0 \log t}, \quad \text{for} \ t \to 0^+, \quad \text{and then}
\]

\[
\tau(S_{t \to 1-t}) \sim e^{-x_1 \log t} \tau(Z) e^{-x_0 \log t}, \quad \text{for} \ t \to 0^+
\]

we get the expected result. \(\square\)
\[ L(1 - 1/t) \]

**Proposition 6 (FPSAC98)** For any \( t \in ]0, 1] \),
\[ L(1 - 1/t) = g_\ast (L(t)) g_\ast (Z^{-1}) e^{i \pi x_0}, \]
and \( g_\ast \) is defined by \( g_\ast x_0 = -x_0 + x_1, g_\ast x_1 = x_1 \).

Proof - \( L(1 - 1/t) = S_{\epsilon \to 1 - 1/t} S_{\epsilon \to \epsilon} L(\epsilon) = S_{\epsilon \to 1 - 1/t} e^{i \pi x_0} e^{i \log \epsilon} \). For \( g(t) = 1 - 1/t \) then \( g_\ast \omega_0 = -\omega_0 + \omega_1 \) and \( g_\ast \omega_1 = -\omega_0 \). This leads to \( g_\ast x_0 = -x_0 + x_1 \) and \( g_\ast x_1 = -x_0 \). Thus
\[ S_{\epsilon \to 1 - 1/t} = g_\ast S_{\epsilon \to \epsilon} = g_\ast (L(t) L^{-1}(1 - \epsilon)) = g_\ast (L(t) Z^{-1} e^{i \pi x_1 \log \epsilon}). \]

\[ \Box \]

**Corollary 10** For any \( w \in X^* \), for \( \epsilon \to 0^+ \),
\[ \text{Li}_w(-1/\epsilon) \sim \frac{(-1)^{\left| w \right|} e^{-i \pi}}{\left| w \right| !} \log^{| w |}(\epsilon). \]

**Hexagonal relation**

**Proposition 7** Let \( \rho \) be the substitution morphism \( x_0 \to x_1 \) and \( x_1 \to x_0 \). Then
\[ Ze^{i \pi x_0} \rho(Z) e^{i \pi (x_0 + x_1)} \rho^2(Z) e^{-i \pi x_1} = 1. \]

Proof - Let \( g(z) = 1 - 1/z \) permuting the singularities 0, 1 and \( \infty \). Then \( g_\ast \omega_0 = -\omega_0 + \omega_1 \) and \( g_\ast \omega_1 = -\omega_0 \). This leads to \( g_\ast x_0 = -x_0 + x_1 \) and \( g_\ast x_1 = -x_0 \). Thus
\[ S_{\epsilon \to 1 - \epsilon} e^{i \pi x_0} g_\ast (S_{\epsilon \to 1 - \epsilon} e^{i \pi x_0}) g_\ast^2 (S_{\epsilon \to 1 - \epsilon} e^{i \pi x_0}) = 1. \]

By the renormalisation
\[ S_{\epsilon \to 1 - \epsilon} \sim e^{-x_1 \log \epsilon} Z e^{-x_0 \log \epsilon}, \text{ for } \epsilon \to 0^+, \]
we get the expected result. \( \Box \)

By Campbell-Baker-Hausdorff formula, one has

**Corollary 11** \( \zeta(2) = \pi^2/6. \)

**Drinfel’d associator** \( \Phi_{KZ}(A, B) \)

and non-commutative g.s. of polyzetas

By changing
\[ x_0 := \frac{A}{2t\pi} \text{ and } x_1 := \frac{-B}{2t\pi}, \]
we have
\[ \Phi_{KZ}(A, B) \equiv Z(x_0, x_1) \]

Thus
\[ \log \Phi_{KZ}(A, B) = \frac{1}{24} \left(A, B \right) \]
\[ + \frac{\zeta(3)}{(2t\pi)^3} \left( [[A, B], B] - [A, [A, B]] \right) \]
\[ + \frac{1}{1440} \left( [[[A, B], B], B] - [A, [A, [A, B]]] \right) \]
\[ + \frac{1}{4} [A, [A, [A, B]]] + \cdots \]