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On Algebraic Numbers of Small Height: Linear Forms in One Logarithm

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We produce a lower bound for $|\alpha-1|$ when α is an algebraic number with relatively small height. The bound is rather sharp in the dependence on the degree of α. The proof rests on the transcendence method of Schneider, but with Siegel's lemma replaced by Laurent's interpolation determinant.

1. INTRODUCTION

Let α be an algebraic number of degree D, with complex conjugates $\alpha_1, ..., \alpha_D$ and minimal polynomial

$$a_0X^D + a_1X^{D-1} + \cdots + a_D = a_0(X-\alpha_1)\cdots(X-\alpha_D),$$

where $a_0, ..., a_D$ are relatively prime integers in $\mathbf{Z}$ and $a_0 > 0$. We denote, as usual, by $\mathrm{M}(\alpha)$ and $\mathrm{h}(\alpha)$ respectively Mahler's measure and Weil's height of α:

$$\mathrm{M}(\alpha) = a_0 \prod_{i=1}^{D} \max\{1, |\alpha_i|\}, \qquad \mathrm{h}(\alpha) = \frac{1}{D}\log \mathrm{M}(\alpha).$$

We recall Liouville's inequality (compare with Lemma 5 below): if

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$P \in \mathbf{Z}[X]$ is a polynomial of degree $\leqslant N$ and length[1] $\leqslant L$, which does not vanish at the point α, then

$$|P(\alpha)| \geqslant L^{-D+1}\mathrm{M}(\alpha)^{-N}.$$

For instance, when $P(X) = X - 1$, one gets

$$|\alpha - 1| \geqslant 2^{-D+1}\mathrm{M}(\alpha)^{-1},$$

provided that $\alpha \neq 1$. This inequality is sometimes sharp, as shown by the following example: choose two rational integers $a \geqslant 2$ and $D \geqslant 2$, and define

$$\alpha = \left(1 - \frac{1}{a}\right)^{1/D}.$$

Then α is of degree D with Mahler's measure $\mathrm{M}(\alpha) = a$, and $|\alpha - 1|$ is essentially $1/aD$:

$$0 < 1 - \alpha - \frac{1}{aD} < \frac{1}{a^2D}.$$

In this particular case Liouville's inequality reads $|\alpha - 1| \geqslant 2^{-D+1}a^{-1}$, which is sharp in term of $\mathrm{M}(\alpha) = a$, but not in terms of the degree D. Similar examples can be produced by replacing the polynomial $aX^D - a + 1$ (which is the minimal polynomial of α in the previous example), for instance, by $X^D - aX^{D-1} + a$ or by $X^D - aX + a$; in particular one gets algebraic integers close to 1.

Our purpose is to improve the dependence on D in Liouville's lower bound for $|\alpha - 1|$.

The first (and up to now single) result in this direction is given in [M] (and is reproduced in the book of Schmidt [S, Chap. VIII, Sect. 11]). The method of [M] was completely elementary: the auxiliary functions were polynomials. Here, we use the exponential function. Since a lower bound for $|\alpha - 1|$ is equivalent to a lower bound for $|\log \alpha|$, the present work can be considered as a variant of [MW], where Schneider's method was developed for estimating linear combinations of two logarithms of algebraic numbers.

Our main result (the theorem in Section 2) implies the following result.

PROPOSITION. *Let μ be a positive number and let α be a complex algebraic number of degree D. Assume* $\log \mathrm{M}(\alpha) \leqslant \mu$ *and* $\alpha \neq 1$. *Then*

$$|\alpha - 1| \geqslant \exp\{-(\tfrac{3}{2}\sqrt{D\mu \log_+(D/\mu)} + 2\mu + \log_+(D/\mu))\},$$

[1] The length is the sum of the absolute values of the coefficients.

where $\log_+ x = \max\{\log x, 0\}$. *Moreover, for each* ε, $0 < \varepsilon < 1$, *one can replace* $\frac{3}{2}$ *in this formula by* $1 + \varepsilon$ *provided that* $\log \mathrm{M}(\alpha) \leqslant \mu \leqslant \varepsilon' D$, *where* ε' *is sufficiently small depending on* ε.

For instance we deduce from the proposition above (see Section 2) that if α is an irrational algebraic number of degree $D \geqslant 2$ then

$$|\alpha - 1| \geqslant \max\{2, \mathrm{M}(\alpha)\}^{-3\sqrt{D \log D}}.$$

This improves one result of [M], where $3\sqrt{D \log D}$ was replaced by $4\sqrt{D}\log(4D)$ under the extra assumption $\mathrm{M}(\alpha) \leqslant 2$.

The present paper is organized as follows. In Section 2 we state the fundamental inequality and we deduce the proposition above. In Section 3 we formulate and prove several auxiliary lemmas. The proof of the fundamental inequality is given in Section 4. Finally, in Section 5 we derive further consequences of our main inequality.

2. The Fundamental Inequality

Let E be a number field of degree $D = [E : \mathbf{Q}]$ and G the set of complex embeddings of E into $\mathbf{C}$. Further let $\alpha_1, ..., \alpha_K$ be non-zero pairwise distinct elements of E with $K \geqslant 2$. Furthermore let G' be a subset of G with $|G'|$ elements,[2] for each $\sigma \in G'$, let $\varphi_\sigma \in i\mathbf{R}$ be a purely imaginary number, let $\log \sigma\alpha_k$ be any determination of the logarithm of $\sigma\alpha_k$ $(1 \leqslant k \leqslant K)$, and let $\mathscr{K}_\sigma$ be a non-empty subset of $\{1, ..., K\}$ with at least two elements; we put

$$\Lambda_\sigma = \frac{1}{|\mathscr{K}_\sigma|} \sum_{k \in \mathscr{K}_\sigma} |\log(\sigma\alpha_k) + \varphi_\sigma|.$$

Theorem. *We have*

$$\sum_{\sigma \in G'} \left(\frac{|\mathscr{K}_\sigma|}{K}\right)^2 \left(\left(1 - \frac{1}{|\mathscr{K}_\sigma|}\right) \log\left(\frac{|\mathscr{K}_\sigma|}{K\Lambda_\sigma}\right) - 1\right)$$
$$\leqslant \left(1 - \frac{1}{K}\right) \frac{2D}{K} \sum_{k=1}^{K} \mathrm{h}(\alpha_k) + \frac{D}{K}\left(2 + \log\left(\frac{2K+1}{4}\right)\right).$$

This theorem is very general and—we hope—this generality can be useful. But, we use only a special case of this result. We state this special case as

[2] We denote by $|X|$ the cardinality of a set X.

COROLLARY 1. *When $\mathcal{K}_\sigma = \{1, ..., K\}$, we get the inequality*

$$-\sum_{\sigma \in G'} \log \Lambda_\sigma \leqslant \frac{2D}{K} \sum_{k=1}^{K} \mathrm{h}(\alpha_k) + \frac{D}{K-1}\left(2 + \log\left(\frac{2K+1}{4}\right)\right) + \frac{K}{K-1}|G'|.$$

We now deduce the proposition of Section 1 from this inequality. The case $\alpha = 0$ is straightforward. Assume first that α is a root of unity of order $n \geqslant 2$. Then $|\alpha - 1| \geqslant 4/n$; in particular the result is obvious if $n \leqslant 4$. It is easy to check, for $n \geqslant 5$, that $9D^2 > n\log(n/4)$ and $2D > e\log(n/4)$ (see, for instance, [MW, III, Appendix]). If we choose $2\mu \geqslant \log(n/4)$ (resp. $\mu \leqslant 4D/n$), the result holds trivially because of the contribution of 2μ (resp. $\log_+(D/\mu)$). For $4D/n < \mu < D/e$, we have $D\mu\log(D/\mu) > (4/n)D^2\log(n/4)$ and once more the result follows.

Therefore we assume that α is neither 0 nor a root of unity. If $D = 1$, then the result is a trivial consequence of Liouville's estimate. Thus we assume $D \geqslant 2$. We also note that this result is trivially implied by Liouville's inequality for $\mu \geqslant D\log 2$. We assume $\mu \leqslant D\log 2$ and we put $\mu = D/t$. A short computation shows that

$$\frac{1}{t} + \frac{3}{2}\sqrt{\frac{\log t}{t}} \geqslant \log 2 \qquad \text{for} \quad \frac{1}{\log 2} \leqslant t \leqslant 15,$$

hence this proposition is still a consequence of Liouville's estimate on the range $\frac{1}{15} \leqslant \mu/D \leqslant \log 2$. Thus, we assume $\mu < D/15$. Since $\log(D/\mu) > 2$ for $0 < \mu \leqslant D/15$, we may also assume that $|\alpha - 1| \leqslant e^{-2}$.

Let us denote by log the principal determination of the logarithm in the disk $|z-1| < 1$. Since $|\alpha - 1| \leqslant e^{-2}$, we have $|\log\alpha| \leqslant 1.08\,|\alpha - 1|$. Indeed, for $|z| \leqslant R < 1$, using the Schwarz lemma together with the bound $|\log(1+z)| \leqslant -\log(1-R)$ we deduce $|\log(1+z)/z| \leqslant -\log(1-R)/R$; hence for $|z| \leqslant e^{-2}$ we have $|\log(1+z)| \leqslant 1.08\,|z|$.

Let $K \geqslant 2$ be an integer. Since the algebraic number α in the proposition is complex, we have an embedding, say σ_0, of $\mathbf{Q}(\alpha)$ into $\mathbf{C}$ such that $\sigma_0(\alpha) = \alpha$. We choose $G' = \{\sigma_0\}$, $\varphi_{\sigma_0} = 0$, $\mathcal{K}_{\sigma_0} = \{1, ..., K\}$, and $\alpha_k = \alpha^{r_k}$ $(1 \leqslant k \leqslant K)$, with

$$r_k = (-1)^{k+1}\left[\frac{k}{2}\right] \qquad (1 \leqslant k \leqslant K),$$

so that

$$\sum_{k=1}^{K} |r_k| = \begin{cases} K^2/4 & \text{if } K \text{ is even,} \\ (K^2-1)/4 & \text{if } K \text{ is odd,} \end{cases}$$

$$\Lambda_{\sigma_0} \leqslant \frac{K}{4}|\log\alpha| \leqslant \frac{1.08K}{4}|\alpha - 1|,$$

and

$$\frac{D}{K}\sum_{k=1}^{K} \mathrm{h}(\alpha_k) = \frac{D}{K}\sum_{k=1}^{K} \frac{|r_k| \log \mathrm{M}(\alpha)}{D} \leqslant \frac{K}{4} \log \mathrm{M}(\alpha) \leqslant \frac{K\mu}{4}.$$

By Corollary 1 above,

$$-\log|\alpha-1| \leqslant \frac{D}{K-1}\left(2+\log\left(\frac{2K+1}{4}\right)\right) + \frac{K}{2}\mu + \frac{K}{K-1} + \log\left(\frac{1.08K}{4}\right).$$

This inequality holds for every rational integer $K \geqslant 2$. The coefficient of μ is at least 1, which means that the result improves Liouville's inequality only if the coefficient of D is less than $\log 2$. In particular we need to choose $K \geqslant 6$. For $K = 6$ the inequality is not trivial; but the first good value is $K = 11$, which gives

$$\begin{aligned} -\log|\alpha-1| &\leqslant \frac{D}{10}(2+\log(5.75)) + 5.5\mu + 1.1 + \log(2.97) \\ &< 0.375D + 5.5\mu + 2.19, \end{aligned}$$

which improves Liouville's inequality as soon as $D \geqslant 14.2\mu + 9.1$.

The above result obtained for $K = 11$ shows that our proposition is also true for $15 \leqslant t \leqslant 36$; indeed

$$\frac{3}{2}\sqrt{\frac{\log t}{t}} \geqslant 0.375 + \frac{3.5}{t} \qquad \text{for} \quad 15 \leqslant t \leqslant 36.$$

Thus, we assume $t \geqslant 36$ and choose $K = 2 + [\sqrt{t \log t}]$, hence $K \geqslant 13$ and

$$\frac{D}{K-1}\left(2+\log\left(\frac{2K+1}{4}\right)\right) \leqslant \frac{D}{\sqrt{t\log t}}\left(2+\log\left(0.75+\frac{1}{2}\sqrt{t\log t}\right)\right).$$

We get

$$-\log|\alpha-1| \leqslant \frac{D}{\sqrt{t\log t}}\left(2+\log\left(0.75+\frac{1}{2}\sqrt{t\log t}\right)\right) + \frac{2+\sqrt{t\log t}}{2}\mu + 1.1 + \log\left(\frac{1.08}{4}\times(2+\sqrt{t\log t})\right).$$

Then, it is easy to verify that the condition $t > 36$ implies

$$2 + \log\left(0.75 + \frac{1}{2}\sqrt{t \log t}\right) < \log t + \sqrt{\frac{\log t}{t}}$$

and

$$1.1 + \log\left(\frac{1.08}{4} \times (2 + \sqrt{t \log t})\right) < \log t.$$

This ends the proof of the first half of the proposition. Finally, for sufficiently large t, the right hand side of the two last inequalities above can be replaced by $(\frac{1}{2} + \varepsilon) \log t$; the second half of the proposition follows. ∎

We conclude this section with a proof of the remark following the proposition. Since

$$D < 3\sqrt{D \log D} \qquad \text{for} \quad 2 \leqslant D \leqslant 30,$$

Liouville's estimate implies our claim on this range. Thus, we suppose $D \geqslant 31$ and we take $\mu = \max\{\log(\mathrm{M}(\alpha)), 1\}$. Then, $\log(D/\mu) \leqslant \log D$; since, for $D \geqslant 31$, we have

$$\frac{1}{\log 2}\left(\frac{3}{2} + \frac{2}{\sqrt{D \log D}} + \sqrt{\frac{\log D}{D}}\right) < 3,$$

we deduce from our proposition

$$|\alpha - 1| > \max\{M(\alpha), e\}^{\,3 \log 2 \sqrt{D \log D}} \qquad \text{for} \quad D \geqslant 31.$$

This implies our claim.

3. Preliminary Lemmas

The first lemma gives an upper bound for the number of consecutive integral zeroes of an exponential polynomial. The second lemma, due to Michel Laurent [L1, L2], provides an upper bound for the absolute value of an interpolation determinant. The next result is an estimate for some Feldman-like polynomials. Finally, we state a variant of Liouville's inequality. For the sake of completeness we give proofs for all these lemmas, even when they are well known.

LEMMA 1. *Let Ω be a field $\alpha_1, ..., \alpha_K$ non-zero elements of Ω which are pairwise distinct, and $A_1, ..., A_K$ non-zero polynomials in $\Omega[X]$, of degrees, say, $L_1, ..., L_K$. Then the function $\mathbf{Z} \to \Omega$, which is defined by*

$$F(m) = \sum_{k=1}^{K} A_k(m)\,\alpha_k^m,$$

cannot vanish on a set of $L_1 + \cdots + L_K + K$ consecutive integers.

Proof. We prove the result by induction on $L = L_1 + \cdots + L_K + K$. For $K = 1$ the result is obvious. Assume $K \geqslant 2$. There is no loss of generality in assuming $\alpha_K = 1$ [just replace $F(m)$ by $\alpha_K^{-m} F(m)$]. Define

$$\tilde{F}(m) = F(m+1) - F(m) = \sum_{k=1}^{K} B_k(m)\,\alpha_k^m,$$

where

$$B_k(X) = A_k(X+1)\,\alpha_k - A_k(X) \qquad (1 \leqslant k \leqslant K).$$

In particular B_K is either 0 or a polynomial of degree $< L_K$, while, for $1 \leqslant k < K$, B_k is a non-zero polynomial of degree L_k. The induction hypothesis shows that $\tilde{F}$ cannot vanish on a set of $L - 1$ consecutive integers. The result follows. ∎

The next lemma involves functions of one complex variable; when f is such a function which is analytic in a disk $|z| \leqslant R$ (namely f is continuous on the closed disk and analytic inside), we write $|f|_R$ for $\sup\{|f(z)|;\, |z| = R\}$.

LEMMA 2. *Let $N \leqslant M$ be two positive integers, let $f_1, ..., f_N$ be analytic functions in a disk $|z| \leqslant R$ of $\mathbf{C}$, let $\alpha_1, ..., \alpha_M$ be points in a smaller disk $|z| \leqslant r$, with $r \leqslant R$, and let $\delta_{\nu\mu}$ be complex numbers, $N < \nu \leqslant M$, $1 \leqslant \mu \leqslant M$. For $1 \leqslant \nu \leqslant N$* and *$1 \leqslant \mu \leqslant M$, suppose that $\delta_{\nu\mu} = f_\nu(\alpha_\mu)$. Let Δ be the determinant of the $M \times M$ matrix $(\delta_{\nu\mu})_{1 \leqslant \nu, \mu \leqslant M}$. Then*

$$|\Delta| \leqslant \left(\frac{R}{r}\right)^{-N(N-1)/2} M!\left(\prod_{\nu=1}^{N} |f_\nu|_R\right) \max_{\{\mu\}} \prod_{\nu=N+1}^{M} |\delta_{\nu,\mu_\nu}|,$$

where $\{\mu\}$ denotes the set of $(M-N)$-tuples $(\mu_{N+1}, ..., \mu_M)$ of pairwise distinct elements of $\{1, ..., M\}$.

Proof. The idea is due to Michel Laurent [L1, L2]. For $1 \leqslant \mu \leqslant M$ we define

$$d_{\nu\mu}(z) = \begin{cases} f_\nu(\alpha_\mu z) & \text{for} \quad 1 \leqslant \nu \leqslant N, \\ \delta_{\nu\mu} & \text{for} \quad N < \nu \leqslant M. \end{cases}$$

The function of one complex variable

$$D(z) = \det(d_{\nu\mu}(z))_{1 \leqslant \nu, \mu \leqslant M}$$

is analytic in the disk $|z| \leqslant R/r$; we claim that this function has a zero of multiplicity $\geqslant N(N-1)/2$ at the origin.

By linearity of the determinant, the proof of this claim reduces to the case where $f_\nu(z) = z^{\kappa_\nu}$, with $1 \leqslant \nu \leqslant N$, for some rational non-negative integers $\kappa_1, ..., \kappa_N$; if two of the κ_ν are equal, then $D(z) = 0$; otherwise $\kappa_1 + \cdots + \kappa_N \geqslant N(N-1)/2$ while

$$D(z) = z^{\kappa_1 + \cdots + \kappa_N} D(1),$$

which proves our claim.

From the classical Schwarz lemma we deduce

$$|D(1)| \leqslant \left(\frac{R}{r}\right)^{-N(N-1)/2} |D|_{R/r}.$$

But clearly $D(1) = \Delta$, and

$$|D|_{R/r} \leqslant M! \max_{\{\varphi\}} \prod_{\nu=1}^{M} |d_{\nu,\varphi(\nu)}|_{R/r},$$

where φ runs over the set of bijective maps of $\{1, ..., M\}$ onto itself. The desired result readily follows. ∎

LEMMA 3. *Let J be a positive integer; for each integer j in the range $0 \leqslant j \leqslant J$ we define a polynomial $\Delta_j \in \mathbf{C}[z]$ of degree j by $\Delta_0 = 1$ and*

$$\Delta_j(z) = \frac{1}{j!} z(z-1)(z+1)(z-2) \cdots (z + (-1)^{j+1} [j/2]) \qquad (1 \leqslant j \leqslant J).$$

Then for all $j = 0, ..., J$ and all $z \in \mathbf{C}$,

$$\log |\Delta_j(z)| \leqslant J + j \log\left(\frac{1}{4} + \frac{|z|}{J}\right).$$

Proof. For $j = 0$ and for $z = 0$ the result is obvious (with $\log 0 = -\infty$). Let us assume $1 \leqslant j \leqslant J$ and $z \neq 0$. In the definition of $\Delta_j(z)$ we bound the modulus of each factor $z + a$ by $|z| + |a|$, then—using the inequality of the arithmetical–geometrical mean—we get

$$j! \, |\Delta_j(z)| \leqslant \left(|z| + \frac{j}{4}\right)^j.$$

[Consider the two cases j odd and j even.] Since $j! \geqslant j^j e^{-j}$ for $j \geqslant 1$, we deduce

$$|\Delta_j(z)| \leqslant \left(\frac{1}{4}+\frac{|z|}{j}\right)^j e^j.$$

For $T \geqslant t > 0$ we have

$$t + t\log\left(\frac{1}{4}+\frac{1}{t}\right) \leqslant T + t\log\left(\frac{1}{4}+\frac{1}{T}\right);$$

indeed the right hand side is an increasing function of T in the range $T \geqslant t$. We apply this estimate with $t = j/|z|$ and $T = J/|z|$:

$$\left(\frac{1}{4}+\frac{|z|}{j}\right)^j e^j \leqslant \left(\frac{1}{4}+\frac{|z|}{J}\right)^j e^J.$$

This completes the proof of Lemma 3. ▮

LEMMA 4. *Let E be a number field of degree D, G the set of embeddings of E into $\mathbf{C}$, and $\alpha_1, ..., \alpha_s$ elements in E. For each $i = 1, ..., s$, we denote by $a_0(\alpha_i)$ the leading coefficient of the minimal polynomial of α_i and by d_i the degree of α_i. Further let $P \in \mathbf{Z}[X_1, ..., X_s]$ be a polynomial of degree at most L_i in X_i $(1 \leqslant i \leqslant s)$. Then*

$$\left(\prod_{i=1}^{s} a_0(\alpha_i)^{L_i D/d_i}\right) \prod_{\sigma\in G} P(\sigma\alpha_1, ..., \sigma\alpha_s) \in \mathbf{Z}.$$

Proof. The number considered is clearly rational, since it is invariant under action of the automorphisms of $\mathbf{C}$ (it is a norm). We must prove that it is also an algebraic integer; for this, it suffices to prove that if $\gamma_1, ..., \gamma_k$ are distinct conjugates of an algebraic number γ with minimal polynomial having leading coefficient a_0, then $a_0\gamma_1, ..., \gamma_k$ is an algebraic integer.

Fact. Let α be an algebraic number; if $G = (X-\alpha)\,H(X)$ is a polynomial whose coefficients are algebraic integers, then the same property holds for the polynomial H.

This fact is proved by induction on the degree of the polynomial G. If this degree is 1, then the conclusion is trivial. If this degree is $s+1 > 1$, then write $G(X) = bX^s(X-\alpha) + F(X)$ with F of degree $\leqslant s$. Since b is the leading coefficient of G and α is a root of G, the number $b\alpha$ is an algebraic integer; if follows that the coefficients of the polynomial F are algebraic integers. Since α is a root of F, the inductive argument leads to the conclusion.

Now we complete the proof of Lemma 4. The fact above shows that the coefficients of the polynomial $H(X)=a_0(X-\gamma_1)\cdots(X-\gamma_k)$ are algebraic integers. Thus $H(0)=\pm a_0\gamma_1, ..., \gamma_k$ is an algebraic integer. ▮

Lemma 4 leads at once to the following variant of Liouville's inequality.

LEMMA 5. *Let E be a number field of degree D, and $\alpha_1, ..., \alpha_s$ elements in E. For each $i=1, ..., s$, we denote by d_i the degree of α_i. Further let $P \in \mathbf{Z}[X_1, ..., X_s]$ be a polynomial of degree at most L_i in X_i $(1 \leqslant i \leqslant s)$ such that $P(\alpha_1, ..., \alpha_s) \neq 0$. Then*

$$|P(\alpha_1, ..., \alpha_s)| \geqslant L(P)^{-D+1} \prod_{i=1}^{s} \mathrm{M}(\alpha_i)^{-L_iD/d_i}.$$

4. Proof of the Theorem

This section is devoted to the proof of our fundamental inequality. Let us first remark that for each $\sigma \in G'$ we have $\Lambda_\sigma \neq 0$: from the assumption $\alpha_1 \neq \alpha_2$ we deduce $\log \sigma\alpha_1 \neq \log \sigma\alpha_2$, which shows that the relations $\log \sigma\alpha_k + \varphi_\sigma = 0$ cannot hold for two distinct values of k.

There is no loss of generality in assuming $G' \neq \emptyset$, and also

$$|\mathscr{K}_\sigma| \geqslant 2 \qquad \text{and} \qquad \log \frac{|\mathscr{K}_\sigma|}{K\Lambda_\sigma} \geqslant \frac{|\mathscr{K}_\sigma|}{|\mathscr{K}_\sigma|-1}$$

for all $\sigma \in G'$. In particular $|\mathscr{K}_\sigma| > K\Lambda_\sigma$.

For each $\sigma \in G'$ we define a real number $E_\sigma > 1$ by setting $E_\sigma = |\mathscr{K}_\sigma|/K\Lambda_\sigma$. Next we choose a positive odd integer J (which will tend to infinity later) and we define $M=K(J+1)$, $M'=M/2$, $M_\sigma = |\mathscr{K}_\sigma|\,(J+1)$ for $\sigma \in G'$. Note that we have $E_\sigma = M_\sigma/M\Lambda_\sigma$.

We divide the proof into several steps: we define a non-zero element γ of E, we bound $|\sigma\gamma|$ from above, first for $\sigma \in G$, next for $\sigma \in G'$, we write Liouville's inequality, and we let J tend to infinity.

First Step. Definition of $\gamma \in E$, $\gamma \neq 0$. We recall the notation Δ_j, which was introduced in Lemma 3. We consider the square $M \times M$ matrix[3]

$$(\Delta_j\,(m)\,\alpha_k^m)_{(j,k;m)},$$

[3] The ordering of the pairs (j, k) is irrelevant: we are interested only in the absolute value of the determinant.

where (j, k) is the index of row, say, $0 \leqslant j \leqslant J$, $1 \leqslant k \leqslant K$, while m is the index of columns, with $-M' < m \leqslant M'$. Lemma 1 shows that this matrix is regular. We denote by γ its determinant.

Second Step. Upper bound for $|\sigma\gamma|$, $\sigma \in G$. Let us denote by $\mathscr{S}$ the set of bijective maps from $[0, J] \times [1, K]$ onto $[-M'+1, M']$ (where, for rational integers $A < B$, we denote by $[A, B]$ the set $\{A, A+1, ..., B\}$). Then

$$\sigma\gamma = \sum_{\psi \in \mathscr{S}} \varepsilon_\psi \prod_{j=0}^{J} \prod_{k=1}^{K} \Delta_j(\psi(j, k))(\sigma\alpha_k)^{\psi(j,k)}$$

with $\varepsilon_\psi \in \{1, -1\}$. The right hand side is a polynomial in $\sigma\alpha_1, ..., \sigma\alpha_k$, $\sigma\alpha_1^{-1}, ..., \sigma\alpha_k^{-1}$, with rational integer coefficients. The degree of this polynomial in $\sigma\alpha_k$ (resp. in $\sigma\alpha_k^{-1}$) is

$$\max\left\{0, \max_{\psi \in \mathscr{S}} \sum_{j=0}^{J} \psi(j, k)\right\} \qquad \left(\text{resp. } \max\left\{0, \max_{\psi \in \mathscr{S}} \sum_{j=0}^{J} -\psi(j, k)\right\}\right).$$

For $\psi \in \mathscr{S}$ and $0 \leqslant j \leqslant J$, let us define

$$\psi_j = \max\{0, \max_{1 \leqslant k \leqslant K} \psi(j, k)\};$$

then $\psi_0, ..., \psi_J$ are pairwise distinct and $\leqslant M/2$. Hence

$$\psi_0 + \cdots + \psi_J \leqslant \frac{M}{2} + \left(\frac{M}{2} - 1\right) + \cdots + \left(\frac{M}{2} - J\right).$$

Denote by N the sum on the right hand side:

$$N = \frac{M}{2}(J+1) - \frac{J(J+1)}{2} = \frac{M}{2}(J+1)\left(1 - \frac{J}{M}\right).$$

We deduce

$$|\sigma\gamma| \leqslant M!\left(\prod_{j=0}^{J} \prod_{k=1}^{K} \max_{|m| \leqslant M/2} |\Delta_j(m)|\right) \times \left(\prod_{k=1}^{K} (\max\{1, |\sigma\alpha_k|\} \max\{1, |\sigma\alpha_k|^{-1}\})^N\right).$$

Thanks to Lemma 3 we have

$$\log \max_{|m| \leqslant M/2} |\Delta_j(m)| \leqslant J + j \log\left(\frac{1}{4} + \frac{M}{2J}\right);$$

since $\sum_{j=0}^{J}\sum_{k=1}^{K} j = MJ/2$, we deduce

$$\sum_{j=0}^{J}\sum_{k=1}^{K} \log \max_{|m| \leqslant M/2} |\Delta_j(m)| \leqslant MJ + \frac{1}{2} MJ \log\left(\frac{1}{4}+\frac{M}{2J}\right).$$

We conclude that for all $\sigma \in G$, we have

$$\frac{1}{M}\log|\sigma\gamma| \leqslant \log M + J + \frac{1}{2} J \log\left(\frac{1}{4}+\frac{M}{2J}\right)$$
$$+\frac{N}{M}\sum_{k=1}^{K}\left(\log_+|\sigma\alpha_k| + \log_+(|\sigma\alpha_k|^{-1})\right).$$

Third Step. Upper bound for $|\sigma\gamma|$, $\sigma \in G'$. The conclusion of Step 2 is not sufficient: we need a non-trivial estimate for $\sigma \in G'$; this is the only part of the proof which is not completely elementary.

Let us fix $\sigma \in G'$. We define

$$\lambda_{\sigma k} = \log \sigma\alpha_k + \varphi_\sigma \qquad (1 \leqslant k \leqslant K),$$

and, for $k \in \mathscr{K}_\sigma$ (and only for these values), we introduce the analytic functions

$$f_{jk}(z) = \Delta_j(z)\, e^{\lambda_{\sigma k} z} \qquad (0 \leqslant j \leqslant J).$$

Since $|e^{\varphi_\sigma m}| = 1$ for all $m \in \mathbf{Z}$, we readily see that

$$|\sigma\gamma| = |\det(\delta_{jk}(m))_{(j,k;m)}|,$$

where, for $0 \leqslant j \leqslant J$, $1 \leqslant k \leqslant K$, and $|m| \leqslant M/2$,

$$\delta_{jk}(m) = \begin{cases} f_{jk}(m) & \text{if } k \in \mathscr{K}_\sigma, \\ e^{\varphi_\sigma m}\Delta_j(m)\,\sigma\alpha_k^m & \text{if } k \notin \mathscr{K}_\sigma. \end{cases}$$

We now use Lemma 2 with N replaced by M_σ: for each real number $R > M/2$,

$$|\sigma\gamma| \leqslant \left(\frac{2R}{M}\right)^{-M_\sigma(M_\sigma-1)/2} M!\left(\prod_{j=0}^{J}\prod_{k\in\mathscr{K}_\sigma}|f_{jk}|_R\right)$$
$$\times\left(\max_{\underline{m}}\prod_{j=0}^{J}\prod_{k\notin\mathscr{K}_\sigma}|\Delta_j(m_{jk})\,\sigma\alpha_k^{m_{jk}}|\right),$$

where $\underline{m} = (m_{jk})$, $0 \leqslant j \leqslant J$, $k \notin \mathscr{K}_\sigma$, with $|m_{jk}| \leqslant M$ and m_{jk} pairwise distinct. We use Lemma 3 exactly as in the second step: we have

$$\log |\Delta_j|_R \leqslant J + j \log \left(\frac{1}{4} + \frac{R}{J} \right)$$

and

$$\sum_{j=0}^{J} \sum_{k \in \mathscr{K}_\sigma} \log |\Delta_j|_R \leqslant M_\sigma J + \frac{1}{2} M_\sigma J \log \left(\frac{1}{4} + \frac{R}{J} \right),$$

$$\sum_{j=0}^{J} \sum_{k \notin \mathscr{K}_\sigma} \log |\Delta_j (m_{jk})| \leqslant (M - M_\sigma) J + \frac{1}{2} (M - M_\sigma) J \log \left(\frac{1}{4} + \frac{M}{2J} \right).$$

For $|z| = R$ we bound $|e^{\lambda_{\sigma k} z}|$ by $\exp\{|\lambda_{\sigma k} R|\}$; using the definitions of Λ_σ and $\lambda_{\sigma k}$ we can write

$$\Lambda_\sigma = \frac{1}{|\mathscr{K}_\sigma|} \sum_{k \in \mathscr{K}_\sigma} |\lambda_{\sigma k}|,$$

which gives

$$\sum_{j=0}^{J} \sum_{k \in \mathscr{K}_\sigma} |\lambda_{\sigma k}| \, R = (J+1) \, |\mathscr{K}_\sigma| \, \Lambda_\sigma R = M_\sigma \Lambda_\sigma R.$$

On the other hand we have (we again use the parameter $N = M(J+1) \times (1 - J/M)/2$ introduced in Step 2)

$$\max_{\underline{m}} \sum_{j=0}^{J} \sum_{k \notin \mathscr{K}_\sigma} m_{jk} \log |\sigma\alpha_k| \leqslant N \sum_{k \notin \mathscr{K}_\sigma} \left(\log_+ |\sigma\alpha_k| + \log_+ (|\sigma\alpha_k|^{-1}) \right).$$

Thus, we get

$$\begin{aligned} \log |\sigma\gamma| \leqslant & - \frac{M_\sigma (M_\sigma - 1)}{2} \log \frac{2R}{M} + M \log M + \Lambda_\sigma M_\sigma R \\ & + JM \left(1 + \frac{1}{2} \log \left(\frac{1}{4} + \frac{M}{2J} \right) \right) \\ & + N \sum_{k \notin \mathscr{K}_\sigma} \left(\log_+ |\sigma\alpha_k| + \log_+ (|\sigma\alpha_k|^{-1}) \right) \\ & + \frac{J}{2} M_\sigma \log \left(\left(\frac{1}{4} + \frac{R}{J} \right) \Big/ \left(\frac{1}{4} + \frac{M}{2J} \right) \right). \end{aligned}$$

Now, we choose $R = M_\sigma/2\Lambda_\sigma = E_\sigma M/2$ (recall that E_σ was introduced at the beginning of the proof and satisfies $E_\sigma = |\mathscr{K}_\sigma|/K\Lambda_\sigma > 1$). Finally, for each $\sigma \in G'$, we conclude

$$\begin{aligned}\frac{1}{M}\log|\sigma\gamma| \leqslant {} & -\frac{M_\sigma(M_\sigma-1)}{2M}\log E_\sigma + \log M + \frac{M_\sigma^2}{2M} \\ & + J\left(1+\frac{1}{2}\log\left(\frac{1}{4}+\frac{M}{2J}\right)\right) \\ & + \frac{N}{M}\sum_{k\notin\mathscr{K}_\sigma}(\log_+|\sigma\alpha_k| + \log_+(|\sigma\alpha_k|^{-1})) \\ & + \frac{JM_\sigma}{2M}\log\left(\frac{J+2E_\sigma M}{J+2M}\right).\end{aligned}$$

Fourth Step. Liouville inequality. For $1 \leqslant k \leqslant K$, we denote by d_k the degree of α_k over $\mathbf{Q}$, and by $a_0(\alpha_k)$ (resp. $a_0(\alpha_k^{-1})$) the leading coefficient of the minimal polynomial of α_k (resp. of α_k^{-1}). Using Lemma 4, we see that the number

$$\prod_{k=1}^{K}(a_0(\alpha_k)\,a_0(\alpha_k^{-1}))^{ND/d_k}\prod_{\sigma\in G}|\sigma\gamma|$$

is a positive integer, hence is $\geqslant 1$. Since $\mathrm{M}(\alpha) = \mathrm{M}(\alpha^{-1})$ when α is a non-zero algebraic number, we obtain

$$\begin{aligned}\mathrm{h}(\alpha_k) &= \frac{1}{d_k}\log a_0(\alpha_k) + \frac{1}{D}\sum_{\sigma\in G}\log_+|\sigma\alpha_k| \\ &= \frac{1}{d_k}\log a_0(\alpha_k^{-1}) + \frac{1}{D}\sum_{\sigma\in G}\log_+|\sigma\alpha_k^{-1}|.\end{aligned}$$

We now bound $|\sigma\gamma|$ thanks to Step 3 for $\sigma \in G'$, and thanks to Step 2 for $\sigma \notin G'$; we conclude

$$\begin{aligned}0 \leqslant {} & \frac{2ND}{M}\sum_{k=1}^{K}\mathrm{h}(\alpha_k) + D\log M + DJ + \frac{1}{2}DJ\log\left(\frac{2M+J}{4J}\right) \\ & + \frac{1}{2}\sum_{\sigma\in G'}\frac{M_\sigma}{M}\left(J\log\left(\frac{J+2E_\sigma M}{J+2M}\right) + M_\sigma - (M_\sigma-1)\log E_\sigma\right).\end{aligned}$$

Fifth Step. Conclusion. We divide the previous relation by $M/2$:

$$\sum_{\sigma\in G'}\left(\frac{M_\sigma(M_\sigma-1)}{M^2}\log E_\sigma-\frac{M_\sigma^2}{M^2}-\frac{M_\sigma^2}{M^2}\log\left(\frac{J+2E_\sigma M}{J+2M}\right)\right)$$
$$\leqslant\frac{4ND}{M^2}\sum_{k=1}^{K}\mathrm{h}(\alpha_k)+\frac{2D\log M}{M}+\frac{2DJ}{M}+\frac{DJ}{M}\log\left(\frac{2M+J}{4J}\right).$$

Next we let J (hence M) tend to infinity: $M/J\to K$, $M_\sigma/J\to K_\sigma$, $N/M^2\to(K-1)/2K^2$; therefore

$$\sum_{\sigma\in G'}\left(\left(\frac{|\mathscr{K}_\sigma|}{K}\right)^2\log(E_\sigma/e)-\frac{|\mathscr{K}_\sigma|}{K^2}\log\left(\frac{1+2E_\sigma K}{1+2K}\right)\right)$$
$$\leqslant\left(1-\frac{1}{K}\right)\frac{2D}{K}\sum_{k=1}^{K}\mathrm{h}(\alpha_k)+\frac{2D}{K}+\frac{D}{K}\log\left(\frac{2K+1}{4}\right).$$

To simplify, we use the upper bound

$$\frac{1+2E_\sigma K}{1+2K}\leqslant E_\sigma.$$

We obtain the estimate

$$\sum_{\sigma\in G'}\left(\frac{|\mathscr{K}_\sigma|}{K}\right)^2\log\left(\frac{|\mathscr{K}_\sigma|}{eKA_\sigma}\right)-\frac{|\mathscr{K}_\sigma|}{K^2}\log\left(\frac{|\mathscr{K}_\sigma|}{KA_\sigma}\right)$$
$$\leqslant\left(1-\frac{1}{K}\right)\frac{2D}{K}\sum_{k=1}^{K}\mathrm{h}(\alpha_k)+\frac{2D}{K}+\frac{D}{K}\log\left(\frac{2K+1}{4}\right).$$

This completes the proof of the theorem. ▮

5. Further Corollaries

We first show a connection between our result and Lehmer's problem, as well as the conjecture of Schinzel and Zassenhaus [SZ]. The best known result so far is Dobrowolski's in [D]; an alternative proof has been given by Cantor and Straus [CS], involving a determinant in place of the auxiliary function. Our approach plays a similar role with respect to Stewart's paper [St].

COROLLARY 2. *Let α be a non-zero algebraic number of degree $D \geqslant 2$; if α is not a root of unity, then*

$$\mathrm{h}(\alpha) \geqslant \frac{1}{500D^2 \log D}.$$

Proof. The proof consists of two steps: in the first step we deduce an estimate from Corollary 1; in the second one we choose the parameters.

First Step. A consequence of Corollary 1. Let α be a complex algebraic number of degree $D \geqslant 2$ such that $\mathrm{M}(\alpha) < 1.3$ and $|\alpha| \geqslant 1$. Let $K \geqslant 2$ and $A \geqslant 1$ be two positive integers. We define $C > 0$ by

$$(1/C^2) = (\pi/A)^2 + (AK \log |\alpha|)^2$$

and we assume

$$\log C \geqslant AKD\,\mathrm{h}(\alpha) + \frac{K}{K-1} + \frac{D}{2K-2}\left(2 + \log\left(\frac{2K+1}{4}\right)\right).$$

Then α is a root of unity.

By a well-known result of Smyth [Sm], the condition $\mathrm{M}(\alpha) < \theta_0$, where $\theta_0 = 1.3247...$ is the real root of $X^3 - X - 1$, implies that α must be reciprocal (which means that α^{-1} is a conjugate of α). Let σ_0 be the natural embedding of $\mathbf{Q}(\alpha)$ into $\mathbf{C}$ (α is a complex algebraic number) and let σ_1 be defined by $\sigma_1(\alpha) = \alpha^{-1}$. We take $G' = \{\sigma_0, \sigma_1\}$. Using Dirichlet's box principle, exactly like Stewart in [St], we can find integers $1 \leqslant r_1 < r_2 < \cdots < r_K \leqslant AK$ together with a purely imaginary number $\varphi \in i\mathbf{R}$ satisfying

$$|\mathrm{Im}\,\log(\alpha^{r_k}) - \varphi| \leqslant \frac{\pi}{A} \qquad \text{for} \quad 1 \leqslant k \leqslant K,$$

where log denotes the principal value of the logarithm. Then we also have

$$|\mathrm{Im}\,\log(\alpha^{-r_k}) + \varphi| \leqslant \frac{\pi}{A} \qquad \text{for} \quad 1 \leqslant k \leqslant K.$$

From the definition of C we deduce

$$|\log(\alpha^{r_k}) - \varphi| \leqslant 1/C$$

and

$$|\log(\alpha^{-r_k}) + \varphi| \leqslant 1/C.$$

We now take $\mathscr{K}_{\sigma_0} = \{1, 2, ..., K\}$ and $\alpha_k = \alpha^{r_k}$ $(1 \leqslant k \leqslant K)$. Therefore $A_{\sigma_0} \leqslant 1/C$. Since

$$\sum_{k=1}^{K} \mathrm{h}(\alpha_k) = \mathrm{h}(\alpha) \sum_{k=1}^{K} r_k < AK^2 \mathrm{h}(\alpha),$$

Corollary 1 shows that the numbers α^{r_k} are not pairwise distinct, hence α is a root of unity.

Second Step. Choice of the parameters A, K and C. We first note that the assumption $\mathrm{M}(\alpha) < 2$ implies that α in an algebraic integer (and even an algebraic unit). According to [SZ], for a non-zero algebraic integer α which is not a root of unity, when $\overline{|\alpha|}$ denotes the maximum of the absolute values of the conjugates of α, we have

$$\overline{|\alpha|} \geqslant 1 + 2^{-D-4}.$$

For $2 \leqslant D \leqslant 8$ we have

$$2^{D+4} < 250 D \log D;$$

therefore we may assume $D \geqslant 10$ (a reciprocal algebraic number is of even degree).

If $\mathrm{h}(\alpha) \leqslant 1/(500 D^2 \log D)$ then $\log |\alpha| \leqslant 1/(500 D \log D)$. We take $A = 40$ and $K = [D \log D]$ and we use the bounds

$$K \geqslant 23, \qquad AK/(500 D \log D) \leqslant 0.08, \qquad C \geqslant 8.919, \qquad \log C > 2.188,$$

$$K/(K-1) < 1.045, \qquad 2 + \log \frac{2K+1}{4} < 1.9393 \log D,$$

$$1/(2K-2) < 0.5476/(D \log D),$$

and finally

$$0.08 + 1.045 + 1.9393 \times 0.5476 < 2.187 < \log C.$$

This completes the proof of Corollary 2. ∎

COROLLARY 3. *Let α be a an algebraic number of degree D. Let μ and ε be real numbers such that* $\log \mathrm{M}(\alpha) \leqslant \mu \leqslant D/11$ *and* $0 < \varepsilon \leqslant \mu/D$. *Then the number s of conjugates $\sigma\alpha$ which satisfy $|\sigma\alpha - 1| \leqslant \varepsilon$ is bounded by*

$$s < 9 \frac{\sqrt{D\mu \log(D/\mu)}}{\log(\varepsilon^{-1})}.$$

Proof. We take $G'=\{\sigma; \log|\sigma\alpha-1|<\varepsilon\}$, so that $|G'|=s$. For all $\sigma\in G'$, we choose $\varphi_\sigma=0$. We define $K=[\sqrt{(D/\mu)\log(D/\mu)}]$. Hence we have $K\geqslant 5$. We follow the proof of the proposition. We consider K numbers $\alpha_k=\alpha^{r_k}$, with $|r_k|\leqslant K/2$. From our assumption $\varepsilon\leqslant\mu/D\leqslant 1/11$ and from the inequality $11\log(1.1)<1.05$ we deduce $\Lambda_\sigma<(1.05K/4)\,|\sigma\alpha-1|$ for $\sigma\in G'$.

Corollary 1 leads to the inequality

$$s\left(\log\left(\frac{4\varepsilon^{-1}}{1.05K}\right)-\frac{K}{K-1}\right)\leqslant\frac{K}{2}\mu+\frac{D}{K-1}\left(2+\log\frac{2K+1}{4}\right).$$

We use the estimates

$$\log(1.05K/4)+K/(K-1)<0.69\log(\varepsilon^{-1}),$$
$$K-1>0.64\sqrt{(D/\mu)\log(D/\mu)},$$
$$\left(2+\log\frac{2K+1}{4}\right)\leqslant 1.4\log(D/\mu)\quad\text{and}\quad\left(\frac{1}{2}+\frac{1.4}{0.64}\right)\frac{1}{0.31}<9.$$

This completes the proof of Corollary 3. ∎

COROLLARY 4. *Let E be a number field of degree D,* $\sigma_1,\ldots,\sigma_r$ *be distinct embeddings of E into* **C**, *and* $\alpha_1,\ldots,\alpha_s$ *be multiplicatively independent elements in* E^*. *Suppose that* μ *is a real number satisfying* $D\mathrm{h}(\alpha_i)\leqslant\mu$ *for* $1\leqslant i\leqslant s$ *and* $D/\mu\geqslant 6$. *Then*

$$\max_{1\leqslant i\leqslant s}\ \max_{1\leqslant j\leqslant r}|\sigma_j\alpha_i-1|$$
$$\geqslant\min\left\{\exp\left(-\frac{9s\mu}{r}\left(\frac{D}{\mu}\log\frac{D}{\mu}\right)^{1/(s+1)}\right),\frac{1}{16s^2}\left(\frac{D}{\mu}\log\frac{D}{\mu}\right)^{-2/(s+1)}\right\}.$$

Proof. We take $G'=\{\sigma_1,\ldots,\sigma_r\}$, so that $|G'|=r$. For all $\sigma\in G'$, we choose $\varphi_\sigma=0$. Let t be a positive integer which will be chosen later. We consider that $K=(2t+1)^s$ numbers

$$\alpha^{(k)}=\alpha_1^{j_1}\cdots\alpha_s^{j_s}\qquad\text{with}\quad|j_h|\leqslant t\quad\text{for}\quad 1\leqslant h\leqslant s.$$

Put $\varepsilon=\max_{1\leqslant i\leqslant s}\max_{1\leqslant j\leqslant r}|\sigma_j\alpha_i-1|$ and suppose that

$$\varepsilon<\min\left\{\exp\left(-\frac{9s\mu}{r}\left(\frac{D}{\mu}\log\frac{D}{\mu}\right)^{1/(s+1)}\right),\frac{1}{16s^2}\left(\frac{D}{\mu}\log\frac{D}{\mu}\right)^{-2/(s+1)}\right\}.$$

Since $\varepsilon<\frac{1}{16}$ and $16\log(\frac{16}{15})<1.04$, for $\sigma\in G'$ we have

$$\Lambda_\sigma\leqslant\max_k|\log(\sigma\alpha^{(k)})|\leqslant 1.04\varepsilon st.$$

We choose

$$t = \left[\tfrac{1}{2}((D/\mu)\log(D/\mu))^{1/(s+1)} + \tfrac{1}{2}\right].$$

Therefore we have $K \geqslant 4$; the upper bound $1.04 \times e^{4/3} < 4$ implies $\log \Lambda_\sigma + K/(K-1) \leqslant \log(4\varepsilon st)$. Thanks to the estimate

$$\sum_{|j| \leqslant t} j = t(t+1) < (t + \tfrac{1}{2})(2t+1)$$

we deduce from Corollary 1 the inequality

$$r \log\left(\frac{1}{4\varepsilon st}\right) \leqslant s\mu\left(t + \frac{1}{2}\right) + \frac{D}{K-1}\left(2 + \log\frac{2K+1}{4}\right).$$

We use the estimates

$$t \leqslant ((D/\mu)\log(D/\mu))^{1/(s+1)}, \qquad K \leqslant 3^s t,$$

$$2 + \log\frac{2K+1}{4} < 2.12 s \log(D/\mu),$$

$$\frac{D}{K-1}\left(2 + \log\frac{2K+1}{4}\right) \leqslant 2.83 s\mu((D/\mu)\log(D/\mu))^{1/(s+1)},$$

and

$$s\mu(t + \tfrac{1}{2}) \leqslant \tfrac{3}{2} s\mu((D/\mu)\log(D/\mu))^{1/(s+1)}.$$

We conclude

$$r \log\left(\frac{1}{4\varepsilon st}\right) < 4.5 s\mu((D/\mu)\log(D/\mu))^{1/(s+1)}.$$

Therefore either $\sqrt{\varepsilon} < 4st\varepsilon$ or

$$r \log \varepsilon < 9 s\mu((D/\mu)\log(D/\mu))^{2/(s+1)}.$$

In both cases we derive a contradiction. ∎

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